On Hadamard Matrices of Order $2^t pq$:I

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Abstract

We prove a new result for orthogonal designs showing if all full orthogonal designs, $OD(r; a, b, r-a-b)$, exist, where $gcd(a, b, r-a-b) = 2^t$, then all full orthogonal designs, $OD(s; c, d, s-c-d)$, exist, where $gcd(c, d, s-c-d) = 2^{t+u}$, $u \geq 0$.

It is known that, for infinitely many numbers $r = 2^w p$, such $OD(r; a, b, r-a-b)$ exist. In particular we show $OD(4; x, y, 4-x-y)$, $OD(24; x, y, 24-x-y)$ (and thus $OD(24; 2x, 2y, 24-2x-2y)$), $OD(40; 2x, 2y, 40-2x-2y)$, $OD(4.28; 4x, 4y, 4.28-4x-4y)$, $OD(8.36; 8x, 8y, 8.36-8x-8y)$ and $OD(16.44; 16x, 16y, 16.44-16x-16y)$ exist.

These orthogonal designs are used to show that Hadamard matrices can be constructed, for any odd q, in all these cases of order $2^w pq$, where $w > w_0$. In all cases this bound $w_0 < [2\log_2(q-3)]$ (or its more precise counterparts given in the paper) is an improvement on previously known results.

Moreover it is established that if $p = 3, 5, 7, 9, 11, 15, 21, 27$ then for the first 600 (or less for higher p) odd numbers pq, more than 97% of Hadamard matrices exist of order $2^y pq$ with $y \leq 4$. For example if $p = 15, 21, 27$ and q is an odd number < 500 then an Hadamard matrix of order $60q$, $120q$, $84q$, $168q$, $108q$ or $216q$ is known.

1. Introduction.

Let $A = [a_{ij}]$ be a matrix of order with $a_{ij} \in \{0, 1, -1\}$. A is called a *weighing matrix* of weight p and order n, if $AA^T = A^TA = pI_n$, where I_n denotes the identity matrix of order n. Such a matrix is denoted by $W(n, p)$. If squaring all its entries gives an incidence matrix of a SBIBD then W is called a balanced weighing matrix.

An *orthogonal design* (OD) A, of order n, type $(s_1, s_2, ..., s_t)$ on the commuting variables $(\pm x_1, \cdots, \pm x_t, 0)$ is a square matrix of order n with entries $\pm x_k$ or 0 and with $|x_k|$ occuring s_k times in each row and column and such that the rows are pairwise orthogonal.

In other words

$$AA^T = (s_1 x_1^2 + \cdots + s_t x_t^2) I_n.$$

This is denoted $OD(n; s_1, s_2, ..., s_t)$.

An *Hadamard matrix*, $A = [a_{ij}]$, is either an $OD(n;n)$ or a $W(n,n)$, *i.e.* it is a square matrix of order n with entries $a_{ij} \in \{1,-1\}$ which satisfies

$$AA^T = A^TA = nI_n.$$

ARS COMBINATORIA 23B(1987), pp. 195-213.

2. Full Orthogonal Designs.

Seberry (Wallis), [Geramita and Seberry, Lemma 7.1], showed

LEMMA 2.1. *Suppose all orthogonal designs* $OD(n; a, b, n-a-b)$ *exist*, $0 \le a+b \le n$. *Then all orthogonal designs* $OD(2n; x, y, 2n-x-y)$ *exist*, $0 \le x+y \le 2n$.

She uses this result to establish

COROLLARY 2.2.

(i) *All orthogonal designs* $OD(2^t; a, b, 2^t-a-b)$ *exist*, $0 \le a+b \le 2^t$, $t \ge 1$.

(ii) *All orthogonal designs* $OD(2^t.3; a, b, 2^t.3-a-b)$ *exist*, $0 \le a+b \le 2^t.3$, $t \ge 3$.

Cooper and Seberry [1975] were not able to establish the existence of either all $OD(40; a, b, 40-a-b)$, $0 \le a+b \le 40$ or all $OD(80; x, y, 80-x-y)$, $0 \le x+y \le 80$ but were able to establish

LEMMA 2.3. *All* $OD(40; 2a, 2b, 40-2a-2b)$, *exist*, $0 \le a+b \le 20$.

We now show that results such as this are sufficient to obtain a useful result similar to Lemma 2.1:

LEMMA 2.4. *Suppose all* $OD(n; a, b, n-a-b)$ *exist*, $0 \le a+b \le n$, $gcd(a, b, n) \ge 2^t$, *some* t. *Then all* $OD(2n; x, y, 2n-x-y)$ *exist*, $0 \le x+y \le 2n$, $gcd(x, y, 2n) \ge 2^t$.

PROOF. Consider the 3-tuple $(x, y, 2n-x-y)$. Either $gcd(x, y, 2n-x-y) > 2^t$ or $gcd(x, y, 2n-x-y) = 2^t$.

Suppose $gcd(x, y, 2n-x-y) > 2^t = 2^{t+u}$ say. Consider the $OD(n; ½x, ½y, n-½(x+y))$ in order n, which exists by hypothesis since $½x+½y+(n-½(x+y)) = n$ and $gcd(½x, ½y, n-½(x+y)) = 2^{t+u-1} \ge 2^t$. We apply the doubling theorem to this design obtaining the required $OD(2n; x, y, n-x-y)$.

Now suppose $gcd(x, y, 2n-x-y) = 2^t$. Without loss of generality we suppose $gcd(x, y) = 2^t$. If $x = y$ we use the $OD(n; x, 0, n-x)$ in the doubling theorem to obtain an $OD(2n; x, x, 2n-2x)$ where $gcd(x, x, 2n-2x) = 2^t$.

Otherwise we assume $x < y$. Now the $OD(n; ½(y-x), x, n-½(x+y))$ exists and $gcd(½(y-x), x, n-½(x+y)) = 2^t$. We use the doubling theorem to obtain the $OD(2n; y-x, x, 2n-x-y)$ which gives (equating variables) the $OD(2n; x, y, 2n-x-y)$ where $gcd(x, y, 2n-x-y) = 2^t$. □

This has the following important consequence.

COROLLARY 2.5. *Suppose all* $OD(n; a, b, n-a-b)$ *exist*, $0 \le a+b \le n$, $gcd(a, b, n) \ge 2^t$, *some* t. *Then all* $OD(2^u n; x, y, 2^u n-x-y)$ *exist*, $0 \le x+y \le 2^u n$, $u \ge 0$, $gcd(x, y, 2^u n-x-y) \ge 2^t$.

PROOF. Use Lemma 2.4 repeatedly. □

COROLLARY 2.6. *All* $OD(2^u.40; x, y, 2^u.40-x-y)$ *exist*, $gcd(x, y) \ge 2$.

PROOF. Lemma 2.3 starts the induction for Lemma 2.4. □

COROLLARY 2.7. *All* $OD(2^{s+2}.28; 4x, 4y, 2^{s+2}-4x-4y)$, $OD(2^{s+3}.36; 8x, 8y, 2^{s+3}.36-8x-8y)$ *and* $OD(2^{s+4}.44; 16x, 16y, 2^{s+4}.44-16x-16y)$ *exist*, $s \ge 2$.

PROOF. See Appendices A, B, C, D. □

1. Hadamard Matrices Whose Orders Have Factors.

Suppose now we wish to form an Hadamard matrix of order $2^t.p.q$, $t \geq 2$, p, q odd.

We use the notation and method outlined in Germita and Seberry, pp. 302-304.

We recall Sylvester's theorem

THEOREM 3.1 (Sylvester). *Given any two relatively prime integers x and y, every integer* $N \geq (x-1)(y-1)$ *can be written in the form* $ax + by$ *for some non-negative integers a and b.*

COROLLARY 3.2. *Given* $x = q+1$ *and* $y = q-3$, *where q is odd and* $q \geq 11$, *there exist non-negative integers a and b such that*

$$2^u a(q+1) + 2^u b(q-3) = n = 2^t p \quad \text{for some } t,\ p \text{ odd given.}$$

PROOF. Let g be the greatest common divisor of $2^u(q+1)$ and $2^u(q-3)$. Then $g = 2^{u+1}$ or 2^{u+2}. Let m be the smallest power of 2 such $2^m p \geq N = \left[\frac{2^u(q+1)}{g} - 1\right]\left[\frac{2^u(q-3)}{g} - 1\right]$. Then by the theorem there exist integers a and b so that

$$\frac{2^u a(q+1)}{g} + \frac{2^u b(q-3)}{g} = 2^m p, \quad 2^t p = 2^m gp.$$

Since g is a power of 2 we have the result. □

THEOREM 3.3. *Let* $q \equiv 3 \pmod 4$ *be a prime* ≥ 9. *Suppose all* $OD(2^r p; x, y, 2^r p - x - y)$ *exist,* $r \geq r_0$, $0 \leq x+y \leq 2^r p$, $gcd(x, y, 2^r p) \geq 2^u$, *some u. Then there exists an* s_0 *given by*

$$2^{s_0} p \geq 2^{u-2}(q-7)(q-3),$$

such that an Hadamard matrix exists for every order $2^s pq$, $s \geq s_0$.

PROOF. Let $x = q+1$ and $y = q-3$; then by the previous corollary there exists an a and b such that

$$2^u a(q+1) + 2^u b(q-3) = n = 2^{s_0} p$$

$$\text{where } 2^{s_0} p = 2^m.gp \geq g\left[\frac{q+1}{4} - 1\right]\left[\frac{q-3}{4} - 1\right]$$

$$= 2^{u-2}(q-3)(q-7)$$

with $g = gcd(2^u(q+1), 2^u(q-3)) = 2^{u+2}$.

By the hypothesis of the theorem the orthogonal design $OD(2^r p;\ 2^u a,\ 2^u b,\ 2^r p - 2^u a - 2^u b)$ exists on the variables x_1, x_2, x_3.

Replace each variable x by the matrix J_q, each variable x_2 by $J_q - 2I_q$ and each variable x_3 by the back-circulant matrix B (see Geramita and Seberry, p. 303), which satisfies

$$B^T = B,\ BJ = J,\ B(J-2I) = (J-2I)B,\ BB^T = (q+1)I_q - J_q,$$

to form a matrix E of order $2^r pq$.

Now

$$DD^T = (2^u a x_1^2 + 2^u b x_2^2 + (2^r p - 2^u a - 2^u b) x_3^2) I$$

and

$$EE^T = [2^u a J^2 + 2^u b (J - 2I)^2 + (2^r p - 2^u a - 2^u b) B^2] \times I_{2^r p}$$
$$= 2^r pq I_{2^r pq}.$$

Thus E is the required Hadamard matrix.

Higher powers of 2 may be constructed via the Kronecker product with $\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$. □

Corollary 3.4. *Let $q \equiv 3 \ (mod\ 4)$ be a prime ≥ 11. Now all $OD(2^r.5;\ 2a,\ 2b,\ 2^r.5 - 2a - 2b)$ exist for $r \geq 3$. Thus there exists a t given by*

$$2^t .5 \geq \tfrac{1}{2}(q-7)(q-3)$$

such that an Hadamard matrix exists in every order $2^s.5q$, $s \geq t$. Hadamard matrices of order $20q$, $q = 3$ or 7 exist by other constructions.

Remark. My theorem (Lemma 7.9 of Geramita and Seberry) would have ensured the existence of an Hadamard matrix of order $2^n q$, where $2^{n+2} \geq (q-7)(q-3)$ and the existence of a conference matrix of order 10 gives the Hadamard matrix of order $2^{n+1}.5q$.

To see where we have an improvement let us tabulate a few values.

q	$\frac{1}{2}(q-7)(q-3)$	Old result gave	New result gives	Comment
11	16		$2^2.5.11$	$OD(20;1,1,18)$ exists
19	96	$2^6.5.19$	$2^5.5.19$	$OD(20;1,0,19)$ gives $2^2.5.19$
23	160	$2^7.5.23$	$2^5.5.23$	$OD(20;0,1,19)$ gives $2^2.5.23$
31	$2^4.21$	$2^8.5.31$	$2^7.5.31$	
67	$2^7.15$	$2^{10}.5.67$	$2^9.5.67$	

We now consider the analogous result for primes $\equiv 1 \ (mod\ 4)$.

Theorem 3.5. *Let $q \equiv 1 \ (mod\ 4)$ be a prime > 9. Suppose all $OD(2^r p;\ a,\ b,\ 2^r p - a - b)$ exist, $r \geq r_0$, $0 \leq a + b \leq 2^r p$, $gcd(a,\ b,\ 2^r p) \geq 2^u$, some u. Then there exists a t given by*

$$2^t p \geq 2^{u-1}(q-1)(q-5)$$

such that an Hadamard matrix exists in every order $2^s pq$, for $s \geq t+1$, that is $2^s p \geq 2^u (q-1)(q-5)$.

Proof. Choose x, y, a, b, t and D as in the previous theorem so

$$2^u a(q+1) + 2^u b(q-3) = n = 2^t p$$

where

$$2^t p = 2^m gp \geq g \left[\frac{(q+1)}{2} - 1 \right] \left[\frac{(q-3)}{2} - 1 \right]$$

$$= 2^{u-1}(q-1)(q-5)$$

with $g = gcd(2^u(q+1),\ 2^u(q-3)) = 2^{u+1}$.

Now by the hypothesis of the theorem, using the "doubling construction", an orthogonal design $OD(2^{r+1}p;\ 2a,\ 2b,\ 2^r p - a - b,\ 2^r p - a - b)$, F, exists for $t \geq r_0$ on variables x_1, x_2, x_3, x_4.

Form the matrix E by replacing each variable x_1 of F by J_q, each variable x_2 of F by $J_q - 2I_q$, and the variables x_3 and x_4 by the two circulant $(1, -1)$ incidence matrices $X = I + Q$, and $Y = I - Q$ (see Geramita and Seberry, p. 303). Now

$$X^T = X,\ Y^T = Y,\ XY = YX,\ X^2 + Y^2 = 2(q+1)I_q - 2J_q.$$

$$XJ = YJ = J,\ X(J-2I) = (J-2I)X,\ Y(J-2I) = (J-2I)Y,$$

$$FF^T = (2ax_1^2 + 2bx_2^2 + (2^r p - a - b)x_3^2 + (2^r p - a - b)x_4^2) \times I_{2^r p}$$

and

$$EE^T = (2aJ^2 + 2b(J-2I)^2 + (2^r p - a - b)(X^2 + Y^2)) \times I_{2^{r+1} p}$$
$$= 2^{r+1} pqI.$$

Now E is an Hadamard matrix and we define $s = t+1$. Higher powers of two are obtained by taking the Kronecker product with the matrix $\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$ as required. □

COROLLARY 3.6. *Let* $q \equiv 1 \pmod 4$ *be a prime* $q \geq 13$. *Now all* $OD(2^r.5; 2a, 2b, 2^r.5 - 2a - 2b)$ *exist for* $r \geq 3$. *Thus there exists a u given by*

$$2^u.5 \geq 2(q-1)(q-5)$$

such that an Hadamard matrix exists in every order $2^s.5q$, $s \geq u$. *Hadamard matrices of order* $20q$, $q = 1, 5, 9$, *exist by other constructions.*

REMARK. My theorem (Lemma 7.10 of Geramita and Seberry) would have ensured the existence of an Hadamard matrix of order $2^n q$, where

$$2^n \geq (q-1)(q-5)$$

and the existence of a conference matrix of order 10 gives the Hadamard matrix of order $2^{n+1}.5q$.

We tabulate a few values:

q	$(q-1)(q-5)$	Old result	New result	Comment
13	84	$2^8 \cdot 5 \cdot 13$	$2^6 \cdot 5 \cdot 13$	an $OD(80;5,1,74)$ gives H-mx
17	$2^6 \cdot 3$	$2^9 \cdot 5 \cdot 17$	$2^7 \cdot 5 \cdot 17$	
29	$2^5 \cdot 21$	$2^{11} \cdot 5 \cdot 29$	$2^9 \cdot 5 \cdot 29$	an $OD(160;1,5,154)$ is known giving H-mx $2^5 \cdot 5 \cdot 29$
37	$2^7 \cdot 9$	$2^{12} \cdot 5 \cdot 37$	$2^9 \cdot 5 \cdot 37$	

Further we note:

THEOREM 3.7. *Let* $P = p \cdot Q$ *where* p *is odd and* $Q = \Pi q_i^{\alpha_i}$ *is a decomposition into odd primes. Suppose all* $OD(2^r p; a, b, 2^r p - a - b)$ *exist,* $r \geq r_0, 0 \leq a+b \leq 2^r p$, $gcd(a, b, 2^r p) \geq 2^u$, *some* u. *Then there exists an Hadamard matrix of order*

$$2^s \cdot p \cdot Q, \text{ for } s \geq \sum_{q_i \varepsilon A} \alpha_i [log_2(q_i - 3)(q_i - 7) - 2]$$
$$+ \sum_{q_j \varepsilon B} \alpha_j [log_2(q_j - 1)(q_j - 5)] + u - \log_2 p$$
$$Q = \prod q_i^{\alpha_i}, q_i \varepsilon A, q \equiv 3 (mod 4), q_j \varepsilon B, q \equiv 1 (mod 4).$$

A cruder bound which is easy to use is $s \geq 2\log_2(Q-3)$.

PROOF. (i) Let $q \equiv 1(mod 4)$. Then by Theorem 3.5 there is an Hadamard matrix of order

$$2^t \cdot pq, \text{ for } t \geq \log_2(q-1)(q-5)+u-\log_2 p.$$

By my theorem (Lemma 7.10 of Geramita and Seberry) there is an Hadamard matrix of order $2^n q$, $n \geq \log_2(q-1)(q-5)$. Hence using the Kronecker product of Hadamard matrices there is an Hadamard matrix of order

$$2^s pq^m, \; s \geq m\log_2(q-1)(q-5)+u-\log_2 p.$$

In order for this method to be used $u-\log_2 p<0$ (otherwise my existing theorem is used). Thus we suppose $u-\log_2 p \leq 0$. Hence

$$\begin{aligned} m\log_2(q-1)(q-5)+u-\log_2 p &\leq m\log_2(q-1)(q-5) \\ &\leq m\log_2(q-3)^2 \\ &= 2\log_2(q-3)^m \\ &< 2\log_2(q^m-3) \\ &= 2\log_2(Q-3), Q=q^m \\ &= 2\log_2(pQ-3), \text{ (the previously known result).} \end{aligned}$$

(ii) Let $Qq^m \cdot w^n$, q, w primes, q, $w \equiv 1(mod 4)$. Then as before there is an Hadamard matrix of order

$$2^{s_1} \cdot pq^m, \text{ for } s_1 \geq mlog_2(q-1)(q-5)+u-\log_2 p.$$

and an Hadamard matrix of order

$$2^{s_2} \cdot w^n, \text{ for } s_2 \geq nlog_2(w-1)(w-5).$$

Using the Kronecker product of Hadamard matrices there is an Hadamard matrix of order

$$2^s pq^m w^n, \; s \geq m\log_2(q-1)(q-5)+n\log_2(w-1)(w-5)+u-\log_2 p.$$

As before

$$\begin{aligned} s \leq 2\log_2(q-3)^m+2\log_2(w-3)^n &= 2\log_2(q-3)^m(w-3)^n \\ &< 2\log_2(q^m w^n-3) \\ &= 2\log_2(Q-3) \\ &< 2\log_2(pQ-3). \end{aligned}$$

By induction we have the result for $Q=q_1^{\alpha_1} q_2^{\alpha_2} \cdots$, a decomposition of Q into prime powers.

(iii) Let $q \equiv 3(mod 4)$. Then by Theorem 3.3 there is an Hadamard matrix of order

$$2^t pq, t \geq \log_2(q-7)(q-3)+u-2-\log_2 p.$$

By my theorem there is an Hadamard matrix of order

$$2^n \cdot q, \text{ for } n \geq \log_2(q-7)(q-3)-2.$$

Hence using the Kronecker product of Hadamard matrices there is an Hadamard matrix of order

$$2^s pq^m, s \geq m\log_2(q-7)(q-3)-2m+u-\log_2 p.$$

As before we assume $u-\log_2 p<0$. Hence

$$\begin{aligned} m\log_2(q-7)(q-3)-2m+u-\log_2 p &\le m\log_2(q-5)^2-2m \\ &< 2\log_2(q-5)^m \\ &< 2\log_2(q-3)^m \\ &< 2\log_2(q^m-3) \\ &= 2\log_2(Q-3) \\ &< 2\log_2(pQ-3), \text{ (the previously known result).} \end{aligned}$$

and by induction, proceeding as before we have the result for $Q=\prod q_i^{\alpha_i}, q_i$ any odd prime.

4. Orthogonal Designs Required for the Result

A theorem of Eades, Robinson, Seberry (=Wallis) and Williams ensures that the ODs needed for the theorem can be obtained.

THEOREM 4.1. (see Theorem7.32, Geramita and Seberry), *Suppose* B *is a binary expansion of* s, *and suppose there is an orthogonal design of type* B *and order* t. *Let* u *and* k *be integers. Then there is an integer* q *such that every orthogonal design of type* $OD(2^{q+k}t;2^q a_1,2^q a_2,\cdots,2^q a_u)$, *for all tuples* $(a_1,a_2,\cdots,a_u)$ *such that* $a_1+a_2+\cdots+a_u=2^k s$.

REMARK. Golay sequences and complementary sequences, as described in Geramita and Seberry,of length $2^a, a\ge 1$ can be used to make may such ODs. For example

OD(20;2,2,8,8)
OD(28;4,8,8,8) or OD(28;2,2,8,16)
OD(36;2,2,16,16) or OD(36;4,8,8,16)
OD(44;4,8,16,16)
OD(52;2,2,16,32)

Whether the power of two is always small enough to improve the bound in the theorem needs further testing. Theorem 4.1 was used extensively in Appendices C and D to obtain the results:

THEOREM 4.2. *All* $OD(2^{s+2}\cdot 9;8a,8b,2^{s+2}\cdot 9-8a-8b)$ *exist,* $s\ge 3$. *All* $OD(2^{s+2}\cdot 11;16a,16b,2^{s+2}\cdot 11-16a-16b)$ *exist,* $s\ge 4$

Other knowledge of ODs was used in Appendices A and B to show:

THEOREM 4.3. *All* $OD(2^{s+1}\cdot 9;2a,2b,2^{s+1}\cdot 9-2a-2b)$ *exist,* $s\ge 1$, $q = 1, 3, 5$. *All* $OD(2^{s+2}\cdot 7;4a,4b,2^{s+2}\cdot 7-4a-4b)$ *exist* $s\ge 2$.

Applying these to Theorem 3.7 shows the theorem can be expected, given known results, to always yield an improvement. If more were known about ODs the results would improve further.

p	u	$u-\log_2 p$	Comment
3	1	-0.6	Some improvement
5	1	-1.3	Always an improvement
7	2	-0.8	Some improvement
7	1	-1.8	Always an improvement -(u=1 can be used in 65% cases)
9	3	-0.2	Some improvement
11	4	+0.5	No improvement
11	3	-0.5	Some improvement -(u=3 can be used in all but 3 cases)

5. Numerical Results.

We used these results to study the existence of Hadamard matrices of order $2^t pq$ for $p = 3, 5, 7, 9, 11, 15, 21, 27$ and the first 600, 500 or 250 odd q and found that in 97% of the cases studies $t = 2, 3$, or 4.

6. References.

Joan Cooper and Jennifer Seberry Wallis, A note on orthogonal designs in order eighty, (1976), *Ars Combinatoria*, **1**, 267-274.

Peter Eades, Peter J Robinson, Jennifer Seberry Wallis, Ian S Williams, An algorithm for orthogonal designs,(1975), *Congressus Numerantium*, **16** 279-292.

A.V.Geramita and Jennifer Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

Jennifer Seberry Wallis, On the existence of Hadamard matrices, *J. Combinatorial Th. (Ser A)*, **21** (1976),188-195.

APPENDIX A

All $OD(2^{s+1}\cdot q; 2a, 2b, 2^{s+1}\cdot q-2a-2b)$ exist, $s\geq 1$, q = 1, 3, 5.

The results may be obtained from Appendix G of Geramita and Seberry.

APPENDIX B

All $OD(2^{s+2}\cdot 7; 4a, 4b, 2^{s+2}\cdot 7-4a-4b)$ exist $s\geq 2$.

The results of Appendix C of Geramita and Seberry give the following ODs in order 28 :

OD(28;1,3,24)
OD(28;1,1,1,25)
OD(28;1,1,8,18)
OD(28;1,1,13,13)
OD(28;4,4,4,16)
OD(28;4,4,10,10)
OD(28;7,7,7,7).

Use these ODs as indicated below to obtain all $OD(4\cdot 28; 4x, 4y, 4(28-x-y))$. Hence all $OD(2^{s+2}\cdot 7; 4x, 4y, 2^{s+2}\cdot 7-4x-4y)$, $s\geq 2$, exist.

Reference #	Tuple in 28	Tuple in 56	Tuple in 128
A	1,3,24	2,6,24,24	4,12,24,24,48
B	7,7,7,7		
C	1,1,1,25	2,2,2,50	
D	1,1,8,18	2,2,8,8,36	2,2,4,16,16,72
D2			4,4,8,8,16,72
D3		2,2,16,18,18	4,4,16,16,36,36
E	1,1,13,13	2,2,26,26	
F	4,4,4,16	8,8,8,16,16	
F2		4,4,8,8,32	4,4,8,16,16,64
F3		12,12,16,16	12,12,24,16,16,32
G	4,4,10,10	4,4,8,20,20	4,4,8,16,40,40
G2		8,8,10,10,20	

The precise details appear in Table B.

Triple in 28	Exists in Order	OD Used	Triple in 28	Exists in Order	OD Used
0,1,27	28	A	3,3,22	112	D2
0,2,26	28	C	3,4,21	112	D2
0,3,25	28	C	3,5,20	112	D2
0,4,24	28	F	3,6,19	112	D2
0,5,23	56	D	3,7,18	112	D2
0,6,22	56	D	3,8,17	112	F2
0,7,21	28	B	3,9,16	112	F2
0,8,20	28	F	3,10,15	112	G
0,9,19	28	D	3,11,14	112	G
0,10,18	28	G	3,12,13	56	A
0,11,17	56	D3			
0,12,16	28	F	4,4,20	28	F
0,13,15	28	E	4,5,19	56	D
0,14,14	28	B	4,6,18	56	D
			4,7,17	112	F3
1,1,26	28	C	4,8,16	28	F
1,2,25	28	C	4,9,15	56	G2
1,3,24	28	A	4,10,14	28	G
1,4,23	56	D	4,11,13	112	D3
1,5,22	56	D	4,12,12	56	A
1,6,21	112	A			
1,7,20	112	D2	5,5,18	56	D
1,8,19	28	D	5,6,17	112	F2
1,9,18	28	D	5,7,16	112	F2
1,10,17	56	D3	5,8,15	56	G2
1,11,16	112	G	5,9,14	112	D3
1,12,15	56	A	5,10,13	56	G2
1,13,14	28	E	5,11,12	112	G
2,2,24	56	G	6,6,16	56	F3
2,3,23	112	D2	6,7,15	112	F3
2,4,22	56	D	6,8,14	56	F3
2,5,21	112	G	6,9,13	112	D3
2,6,20	56	G	6,10,12	56	G
2,7,19	112	D2	6,11,11	112	F3
2,8,18	28	D			
2,9,17	56	D3	7,7,14	28	B
2,10,16	56	G	7,8,13	112	F3
2,11,15	112	G	7,9,12	112	F3
2,12,14	56	G	7,10,11	112	F3
2,13,13	28	E			
			8,8,12	56	F
			8,9,11	56	D3
			8,10,10	28	G
			9,9,10	56	D3

Table B.

All $OD(2^{s+2}\cdot 9; 8a, 8b, 2^{s+2}\cdot 9-8a-8b)$ exist, $s \geq 3$

The results of the Appendices of Geramita and Seberry and those following may be used to construct all $OD(8\cdot 36; 8x, 8y, 8(36-x-y))$. Hence all $OD(2^{s+2}\cdot 9; 8x, 8y, 2^{s+2}\cdot 9-8x-8y), s \geq 3$, exist.

Use the text of Chapter 8 and the results of Appendix A of Geramita and Seberry which gives the following ODs :

OD(12;1,1,1,9)	OD(12;1,1,2,8)
OD(12;1,2,3,6)	OD(24;1,6,6,11)
OD(12;1,1,3,5,5,9)	OD(36;1,1,34)
OD(36;9,9,9,9)	OD(36;2,2,16,16)
OD(72;4,16,26,26)	OD(72;10,10,26,26).

These may be used to form the following :

OD(36;2,8,26)	OD(36;2,7,27)
OD(36;1,8,27)	OD(36;3,6,9,18)
OD(72;11,12,49)	OD(72;1,12,59)
OD(72;6,12,54)	OD(72;12,15,45)
OD(72;10,12,50)	OD(72;5,12,55)
OD(72;9,12,51)	OD(72;14,12,46).

In many case these designs yield $OD(2^{s+2}\cdot 9; 8x, 8y, 2^{s+2}\cdot 9-8x-8y)$ for $s<3$ but the result of the title can be obtained from the designs in Table C.

Reference #	Tuple in 36	Tuple in 72	Tuple in 144	Tuple in 288	Comment
C1	2,2,16,16	4,4,16,16,32	8,8,16,16,32,64	8,8,16,32,32,64,128 =8(1,1,2,4,4,8,16)	Doubling Construction
C2			8,16,24,32,64	16,32,24,24,64,128 =8(2,3,3,4,8,16)	Doubling Construction
C22			8,16,56,64	16,32,56,56,128 =8(2,4,7,7,16)	Doubling Construction
C23			8,16,32,88	16,32,64,88,88 =8(2,4,8,11,11)	Doubling Construction
C24				16,16,32,32,64,64,64 =8(2,2,4,4,8,8,8)	Doubling Construction
C3		16,16,20,20	32,32,20,20,40 =4(8,8,5,5,10)	32,32,64,40,40,80 =8(4,4,8,5,5,10)	Doubling Construction
C4	16,20			16,32,32,48,40,120 =8(2,4,4,6,5,15)	AOD((1,2,2,3):(2,6))
C5	2,16,18	4,16,16,36	8,16,16,32,72	16,32,32,64,72,72 =8(2,4,4,8,9,9)	Doubling Construction

Tuples in $2^t\cdot 9$ of type $(2^t\cdot a, 2^t\cdot b, 2^t\cdot(36-a-b))$

Tuple	t	Reference	Tuple	t	Reference	Tuple	t	Reference
0,a,36-a	3	B	3,3,30	5	C2	6,6,24	4	C1
			3,4,29	5	C1	6,7,23	5	C22
1,1,34	5	C1	3,5,28	5	C1	6,8,22	4	C1
1,2,33	5	C1	3,6,27	5	C2	6,9,21	5	C1
1,3,32	5	C1	3,7,26	5	C2	6,10,20	4	C1
1,4,31	5	C1	3,8,25	5	C1	6,11,19	5	C2
1,5,30	5	C1	3,9,24	5	C1	6,12,18	4	C1
1,6,29	5	C1	3,10,23	5	C2	6,13,17	5	C1
1,7,28	5	C1	3,11,22	5	C2	6,14,16	4	C1
1,8,27	5	C1	3,12,21	5	C1	6,15,15	5	C4
1,9,26	5	C1	3,13,20	5	C2			
1,10,25	5	C1	3,14,19	5	C2	7,7,22	5	C22
1,11,24	5	C1	3,15,18	5	C2	7,8,21	5	C2
1,12,23	5	C1	3,16,17	5	C1	7,9,20	5	C22
1,13,22	5	C1				7,10,19	5	C2
1,14,21	5	C1	4,4,28	4	C1	7,11,18	5	C22
1,15,20	5	C1	4,5,27	5	C1	7,12,17	5	C1
1,16,19	5	C1	4,6,26	4	C1	7,13,16	5	C22
1,17,18	5	C1	4,7,25	5	C1	7,14,15	5	C4
			4,8,24	3	C1			
2,2,32	2	C1	4,9,23	5	C1	8,8,20	3	C1
2,3,31	5	C2	4,10,22	4	C1	8,9,19	5	C1
2,4,30	4	C1	4,11,21	5	C1	8,10,18	3	C3
2,5,29	5	C1	4,12,20	4	C1	8,11,17	5	C1
2,6,28	4	C1	4,13,19	5	C1	8,12,16	5	C1
2,7,27	5	C2	4,14,18	4	C1	8,13,15	4	C3
2,8,26	3	C1	4,15,17	5	C1	8,14,14	5	C1
2,9,25	5	C1	4,16,16	2	C1			
2,10,24	3	C1				9,9,18	5	C5
2,11,23	5	C2	5,5,26	4	C3	9,10,17	5	C5
2,12,22	4	C1	5,6,25	5	C1	9,11,16	5	C5
2,13,21	5	C1	5,7,24	5	C1	9,12,15	5	C5
2,14,20	4	C1	5,8,23	4	C3	9,13,14	5	C5
2,15,19	5	C2	5,9,22	5	C1			
2,16,18	2	C5	5,10,21	4	C3	10,10,16	3	C3
2,17,17	5	C1	5,11,20	5	C1	10,11,15	5	C4
			5,12,19	5	C1	10,12,14	5	C3
			5,13,18	4	C3	10,13,13	4	C3
			5,14,17	5	C1			
			5,15,16	4	C3	11,11,14	4	C23
						11,12,13	5	C5
						12,12,12,	5	C24

Table C.

APPENDIX D

$$\text{All OD}\,(2^{s+2}\cdot 11; 16x, 16y, 2^{s+2}\cdot 11-16x-16y)\,\text{exist}, s\geq 4$$

The orthogonal designs OD(44;2,2,20,20) and OD(44;2,2,8,32) in order 44 can be used in a similar fashion to that used in Appendix C for order 36, to obtain all $OD(16\cdot 44; 16x, 16y, 16(44-x-y))$. Hence all $OD(2^{s+2}\cdot 11; 16x, 16y, 2^{s+2}\cdot 11-16x-16y), s\geq 4$ exist.

If

OD(8·44;8·4,8·13,8·27)
OD(8·44;8·4,8·14,8·26)
OD(8·44;8·7,8·18,8·19)

exist then the main result would change from $s \geq 4$ to $s \geq 3$.

APPENDIX E

HADAMARD MATRICES OF ORDER $2^t pq$ ODD q

We consider the first 500 or so Hadamard matrices of order $2^t pq$ for odd q and find the percentage that are known for each t.

p	sample size	$t=2$	$t \leq 3$	$t \leq 4$	$t \leq 5$	$t \leq 6$	$t \leq 10$	$t \leq 14$
3	600	87.5	96.0	97.5	98.5	99.0	100.0	
5	600	85.2	95.2	98.2	98.8		99.7	100.0
7	600	79.5	93.7	96.3	97.3	97.8	99.3	99.8
9	556	92.3	98.9	99.8	100.0			
11	500	79.2	92.2	95.4	98.2	96.6	99.6	100
15	250	98.6	99.6	100.0				
21	250	96.6	100.0					
27	250	93.4	99.8	100.0				

ORDERS DIVISIBLE BY 3

Orders, q, for which Hadamard matrices of order $2^3 \cdot 3 \cdot p$ are known but for which $2^2 \cdot 3 \cdot p$ are not yet known:

71·3=213	73·3=219	83·3=249	89·3=267	173·3=519	191·3=573
193·3=579	219·3=657	223·3=669	233·3=699	241·3=723	263·3=789
293·3=879	349·3=1047	353·3=1059	409·3=1227	433·3=1299	445·3=1335
447·3=1341	453·3=1359	491·3=1473	497·3=1491	503·3=1509	519·3=1557
557·3=1671	563·3=1689	573·3=1719	623·3=1869	673·3=2019	701·3=2103
761·3=2283	797·3=2391	831·3=2493	843·3=2529	881·3=2643	887·3=2661
911·3=2733	923·3=2769	933·3=2799	947·3=2841	979·3=2937	1013·3=3039
1047·3=3141	1049·3=3147	1061·3=3183	1091·3=3273	1093·3=3279	1129·3=3387
1151·3=3453	1181·3=3543	1193·3=3579			

Orders, q, for which Hadamard matrices of order $2^4 \cdot 3 \cdot p$ are known but for which $2^3 \cdot 3 \cdot p$ are not yet known:

179·3=537	571·3=1713	643·3=1929	853·3=2559	955·3=2865	991·3=2973
1039·3=3117	1063·3=3189	1165·3=3495			

Orders, q, for which Hadamard matrices of order $2^5 \cdot 3 \cdot p$ are known but for which $2^4 \cdot 3 \cdot p$ are not yet known:

$419 \cdot 3 = 1257$ $757 \cdot 3 = 2271$ $419 \cdot 3 = 1257$ $757 \cdot 3 = 2271$ $857 \cdot 3 = 2571$ $863 \cdot 3 = 2589$

Orders, q, for which Hadamard matrices of order $2^6 \cdot 3 \cdot p$ are known but for which $2^5 \cdot 3 \cdot p$ are not yet known:

$479 \cdot 3 = 1437$ $631 \cdot 3 = 1893$ $809 \cdot 3 = 2427$

Orders, q, for which Hadamard matrices of order $2^7 \cdot 3 \cdot p$ are known but for which $2^6 \cdot 3 \cdot p$ are not yet known:

$941 \cdot 3 = 2823$ $971 \cdot 3 = 2913$ $1031 \cdot 3 = 3093$

Orders, q, for which Hadamard matrices of order $2^t \cdot 3 \cdot p$ are known but for which $2^t \cdot 3 \cdot p$ are not yet known for any smaller power of t:

$2^8 \cdot 1103 \cdot 3 = 2^8 \cdot 3309$ $2^9 \cdot 509 \cdot 3 = 2^9 \cdot 1527$ $2^{10} \cdot 311 \cdot 3 = 2^{10} \cdot 933$

t	frequency	cum.freq.	cumulative %
2	525	525	87.5
3	51	576	96.0
4	9	585	97.5
5	6	591	98.5
6	3	594	99.0
7	3	597	99.5
8	1	598	99.7
9	1	599	99.8
10	1	600	100.0

ORDERS DIVISIBLE BY 5

Hadamard matrices of order $2^t \cdot 5 \cdot q$ are not yet known for the following q:

$2^3 \cdot 67 \cdot 5$ $2^3 \cdot 87 \cdot 5$ $2^3 \cdot 133 \cdot 5$ $2^6 \cdot 269 \cdot 5$ $2^{12} \cdot 347 \cdot 5$
$2^5 \cdot 359 \cdot 5$ $2^6 \cdot 443 \cdot 5$ $2^{10} \cdot 491 \cdot 5$ $2^9 \cdot 599 \cdot 5$ $2^9 \cdot 631 \cdot 5$
$2^8 \cdot 647 \cdot 5$ $2^{11} \cdot 653 \cdot 5$ $2^7 \cdot 739 \cdot 5$ $2^5 \cdot 749 \cdot 5$ $2^6 \cdot 773 \cdot 5$
$2^5 \cdot 857 \cdot 5$ $2^5 \cdot 863 \cdot 5$ $2^7 \cdot 947 \cdot 5$.

Orders, q, for which $2^4 \cdot q$ are known:

1115	2315	2855	2865	3065	3215
3755	4115	4265	4295	4535	4585
4775	5195	5315	5615	5825	6065
6155	6275	6365	6515	6755	6905
7115	7355	7415	8315	8375	8495
8615	9205	9935			

Orders, q, for which $2^3 \cdot q$ are known:

335	445	515	655	865	955
965	985	1165	1205	1255	1315
1335	1385	1465	1675	1745	1915
1985	2095	2155	2165	2215	2225
2285	2305	2335	2515	2545	2575
2815	2965	3085	3175	3365	3505
3545	3595	3715	3805	3845	3945
3985	4075	4405	4435	4645	4735
4765	4855	4895	5065	5095	5105
5165	5245	5485	5515	5585	5645
5755	5785	5815	5905	6085	6465
6495	6535	6575	6605	6685	6805
6925	6985	7045	7205	7305	7325
7405	7445	7545	7555	7615	7705
7745	7765	7985	8105	8155	8305
8345	8455	8485	8545	8885	8935
8945	9005	9055	9265	9325	9365
9465	9515	9565	9665	9815	9865
9885	9895	9925	9985.		

t	frequency	cum.freq.	cumulative %
2	511	511	85.2
3	60	571	95.2
4	18	589	98.2
5	4	593	98.8
7	3	596	99.3
9	2	598	99.7
11	1	599	99.8
12	1	600	100.0

ORDERS DIVISIBLE BY 7

Orders, $2^t \cdot q$, for which $q = 2^t \cdot 7 \cdot p$ are not known for small t (ie < 4) :

$2^8 \cdot 107 \cdot 7$	$2^5 \cdot 167 \cdot 7$	$2^5 \cdot 233 \cdot 7$	$2^6 \cdot 269 \cdot 7$	$2^{12} \cdot 347 \cdot 7$
$2^5 \cdot 359 \cdot 7$	$2^6 \cdot 443 \cdot 7$	$2^{10} \cdot 491 \cdot 7$	$2^9 \cdot 599 \cdot 7$	$2^9 \cdot 631 \cdot 7$
$2^8 \cdot 647 \cdot 7$	$2^{11} \cdot 653 \cdot 7$	$2^7 \cdot 739 \cdot 7$	$2^5 \cdot 749 \cdot 7$	$2^6 \cdot 773 \cdot 7$
$2^5 \cdot 857 \cdot 7$	$2^5 \cdot 863 \cdot 7$	$2^7 \cdot 947 \cdot 7$	$2^8 \cdot 1019 \cdot 7$	$2^{18} \cdot 1103 \cdot 7$
$2^8 \cdot 1123 \cdot 7$	$2^7 \cdot 1187 \cdot 7$.			

New proofs have been devised to give the results for $2^8 \cdot 107 \cdot 7$ and $2^{12} \cdot 347 \cdot 7$.

Orders, q, for which $2^4 \cdot 7 \cdot p$ are known but for which $2^3 \cdot 7 \cdot p$ are not known:

721=7.103	1981=7.283	4585=7.655	5033=7.719	5761=7.823	5873=7.839
6013=7.859	6181=7.883	6433=7.919	6755=7.965	6797=7.971	6853=7.979
7273=7.1039	7441=7.1063	8183=7.1169	8393=7.1199		

Orders, q, for which $2^3 \cdot 7 \cdot p$ are known but for which $2^2 \cdot 7 \cdot p$ are not known:

581=7.83	623=7.89	763=7.109	917=7.131	959=7.137	1043=7.149
1211=7.173	1351=7.193	1379=7.197	1491=7.213	1589=7.227	1841=7.263
1869=7.267	2051=7.293	2219=7.317	2233=7.319	2527=7.361	2611=7.373
2849=7.407	2863=7.409	3017=7.431	3031=7.433	3143=7.449	3157=7.451
3269=7.467	3409=7.487	3451=7.493	3479=7.497	3633=7.519	3899=7.557
3941=7.563	3983=7.569	4011=7.573	4053=7.579	4109=7.587	4319=7.617
4403=7.629	4781=7.683	4879=7.697	4907=7.701	4963=7.709	4977=7.711
5131=7.733	5271=7.753	5327=7.761	5341=7.763	5383=7.769	5453=7.779
5467=7.781	5509=7.787	5579=7.797	5747=7.821	5789=7.827	6083=7.869
6167=7.881	6209=7.887	6251=7.893	6377=7.911	6419=7.917	6461=7.923
6503=7.929	6531=7.933	6685=7.955	6713=7.959	6839=7.977	6881=7.983
7021=7.1003	7049=7.1007	7091=7.1013	7301=7.1043	7343=7.1049	7413=7.1059
7427=7.1061	7469=7.1067	7511=7.1073	7637=7.1091	7651=7.1093	7679=7.1097

t	frequency	cum.freq.	cumulative %
2	477	477	79.5
3	85	562	93.7
4	16	578	96.3
5	6	584	97.3
6	3	587	97.8
7	3	590	98.3
8	4	594	98.8
9	2	596	99.2
10	1	597	99.3
11	1	598	99.7
12	1	599	99.8
18	1	600	100.0

ORDERS DIVISIBLE BY 9

Orders, q, for which $q=2^3 \cdot 9 \cdot p$ are known but $q=2^2 \cdot 9 \cdot p$ is not yet known:

73·9=657	149·9=1341	151·9=1359	173·9=1557	191·9=1719	277·9=2493
281·9=2529	311·9=2799	349·9=3141	453·9=4077	479·9=4311	487·9=4401
501·9=4509	503·9=4527	541·9=4869	553·9=4977	579·9=5211	587·9=5283
657·9=5913	659·9=5931	701·9=6309	739·9=6651	751·9=6759	757·9=6813
797·9=7173	821·9=7389	827·9=7443	851·9=7659	881·9=7929	887·9=7983
911·9=8199	1021·9=9189	1031·9=9279	1049·9=9441	1061·9=9549	1091·9=9819
1093·9=9837					

Orders, q, for which $q=2^4 \cdot 9 \cdot p$ are known but $q=2^3 \cdot 9 \cdot p$ is not yet known:

571·9=5139 631·9=5679 643·9=5787 947·9=8523 1063·9=9567

Orders, q, for which $q=2^5 \cdot 9 \cdot p$ are known but $q=2^4 \cdot 9 \cdot p$ is not yet known: $2^4 \cdot 863 \cdot 9 = 2^4 \cdot 7767$

t	frequency	cum.freq.	cumulative %
2	513	513	92.3
3	37	550	98.9
4	5	555	99.8
5	1	556	100.0

ORDERS DIVISIBLE BY 11

Orders, q, for which $q=2^3 \cdot 11 \cdot p$ are known but for which $q=2^2 \cdot 11 \cdot p$ are not known:

71·11=781	73·11=803	83·11=913	89·11=979	109·11=1199	113·11=1243
131·11=1441	137·11=1507	149·11=1639	173·11=1903	193·11=2123	197·11=2167
203·11=2233	239·11=2629	259·11=2849	263·11=2893	267·11=2937	269·11=2959
277·11=3047	281·11=3091	287·11=3157	293·11=3223	317·11=3487	349·11=3839
353·11=3883	383·11=4213	389·11=4279	403·11=4433	409·11=4499	419·11=4609
431·11=4741	445·11=4895	449·11=4939	453·11=4983	491·11=5401	497·11=5467
503·11=5533	519·11=5709	529·11=5819	533·11=5863	541·11=5951	553·11=6083
563·11=6193	587·11=6457	593·11=6523	613·11=6743	617·11=6787	635·11=6985
655·11=7205	659·11=7249	679·11=7469	683·11=7513	689·11=7579	701·11=7711
709·11=7799	713·11=7843	723·11=7953	743·11=8173	755·11=8305	769·11=8459
773·11=8503	781·11=8591	787·11=8657	789·11=8679	803·11=8833	809·11=8899
797·11=8767	827·11=9097	843·11=9273	851·11=9361	865·11=9515	879·11=9669
887·11=9757	985·11=10835	995·11=10945			

Orders, q, for which $q=2^4 \cdot 11 \cdot p$ are known but for which $q=2^3 \cdot 11 \cdot p$ are not known:

67·11=737	103·11=1133	127·11=1337	151·11=1661	163·11=1793	179·11=1989
463·11=5093	487·11=5357	571·11=6281	623·11=6853	643·11=7073	751·11=8261
763·11=8393	823·11=9053	853·11=9383	859·11=9449	907·11=9977	991·11=10901

Orders, q, for which $q=2^5 \cdot 11 \cdot p$ are known but for which $q=2^4 \cdot 11 \cdot p$ are not known:

107·11=1177	233·11=2563	359·11=3949	647·11=7117	857·11=9427	107·11=1177
233·11=2563	359·11=3949	647·11=7117	857·11=9427	863·11=9493	911·11=10021
983·11=10813					

Orders, q, for which $q=2^6 \cdot 11 \cdot p$ are known but for which $q=2^5 \cdot 11 \cdot p$ are not known:

$2^5 \cdot 737 \cdot 11 = 8107$ 757·11=8327

Orders, q, for which $q=2^7\cdot 11\cdot p$ are known but for which $q=2^6\cdot 11\cdot p$ are not known:

443·11=4873 479·11=5269 653·11=7183 739·11=8129

Orders, $2^t\cdot q$, for which $q=2^t\cdot 11\cdot p$ are known but which are not known for any smaller t:

$2^{11}\cdot 347\cdot 11=2^{11}\cdot 3817$ $2^{14}\cdot 793\cdot 11=2^{14}\cdot 8723$ $2^8\cdot 971\cdot 11=2^8\cdot 10681$

t	frequency	cum.freq.	cumulative %
2	396	396	79.2
3	75	461	92.2
4	16	477	95.4
5	14	491	98.2
6	2	493	98.6
7	4	497	99.4
8	1	498	99.6
11	1	499	99.8
14	1	400	100.0

ORDERS DIVISIBLE BY 15

Hadamard matrices of order $2^2\cdot 15\cdot q$ are not yet known for the following q:

89 2.191 191 2.233 233 263 431 433 487.

t	frequency	cum.freq.	cumulative %
2	493	493	98.6
3	5	498	99.6
4	2	0.4	100.0

ORDERS DIVISIBLE BY 21

Hadamard matrices of order $2^2 \cdot 21 \cdot q$ are not yet known for the following q:

71	89	173	191	193	237
251	311	353	409	431	433
477	483	487	489	499.	

t	frequency	cum.freq.	cumulative %
2	483	483	96.6
3	17	3.4	100.0

ORDERS DIVISIBLE BY 27

Hadamard matrices of order $2^2 \cdot 27 \cdot q$ are not yet known for the following q:

371	379	381	389	395	399
403	413	419	421	423	425
429	431	433	437	439	451
453	463	467	475	477	2.479
479	481	487	489	493	497.

t	frequency	cum.freq.	cumulative %
2	467	467	93.4
3	32	499	99.8
4	1	500	100.0