

GENERALIZED HADAMARD MATRICES AND COLOURABLE DESIGNS IN THE CONSTRUCTION OF REGULAR GDDs WITH TWO AND THREE ASSOCIATION CLASSES

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Abstract: Generalized Hadamard matrices and colourable designs are used to construct many new group divisible designs.

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1. Introduction and preliminaries

Recently the $(0, 1)$ -matrices of order q^2 (sometimes called auxiliary matrices) used by a number of authors (Wallis (1971), Glynn (1978), Drake (1979), Jungnickel (1979), Mavron (1981) and Rajkundlia (1979)) in various guises, have been used in the construction of new families of Williamson type matrices (Seberry (1986)), the construction of new conference matrices (Seberry and Whiteman (1987)), the construction of new generalized Hadamard matrices (de Launey (1984)), and of new GDDs (Rodger, Sarvate and Seberry (1987)).

Auxiliary matrices, B_i , obey a number of equations including $B_i B_j^T = J$, $i \neq j$. This paper studies and applies matrices, A_i , analogous to auxiliary matrices, which satisfy $A_i A_j^T = hJ$, $i \neq j$, $h > 1$. These new matrices are obtained from generalized Hadamard matrices $\text{GH}(hg, G)$, $|G| = g$, $h > 1$. Let G be a finite group of order g . Let e be the identity element of G and let $R(G)$ denote the group ring of G . Let $J_{n,s}$ denote the $n \times s$ matrix whose entries are all 1. When $s = n$ we abbreviate $J_{n,s}$ to J_n .

A generalized Hadamard matrix $\text{GH}(gh, G) = (g_{ij}) = H$ over the group G of order g is a $gh \times gh$ matrix such that

- (i) $g_{ij} \in G$ for all $1 \leq i, j \leq gh$, and

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(ii) $\sum_{k=1}^{gh} g_{ik} g_{jk}^{-1} = \sum_{a \in G} ha$ whenever $i \neq j$ where the summation is in the group ring $R(G)$. We also write this as

$$HH^\dagger = hG.$$

Several families of generalized Hadamard matrices have been found by Butson (1962, 63), Drake (1979), Seberry (1978), Jungnickel (1979), Seiden (1954), Street (1979), Dawson (1985). Drake (1978) has also given various ways of combining known generalized Hadamard matrices to yield others. Non-existence of generalized Hadamard matrices has been explored by de Launey (1984).

Generalized Hadamard matrix existence theorem. Let $EA(p^i)$ be the elementary abelian group of order p^i , where p is prime. Then the following generalized Hadamard matrices exist:

- (i) (a) $\text{GH}(p^i, EA(p^i))$ for all $i \geq 1$;
- (b) $\text{GH}(p^{i+j}, EA(p^i))$ for all $i \geq 1, j \geq 0$;
- (ii) $\text{GH}(2^m p^i, EA(p))$ for all $0 \leq m \leq k, k \geq 1$;
- (iii) (a) $\text{GH}(2p^i, EA(p^i))$ for all $i \geq 1$;
- (b) $\text{GH}(4p^i, EA(p^i))$ for all $i \geq 1$;
- (iv) if $p^r - 1 = q^s$ for some prime q then there exist
 - (a) $\text{GH}(p^r(p^r - 1), EA(p^r))$ for all $r \geq 1$;
 - (b) $\text{GH}(p^{rk+h}(p^r - 1)^j, EA(p^i))$ for all $1 \leq i \leq r, 1 \leq j \leq k, h \geq 0$;
 - (c) $\text{GH}(2^m p^{k+ri+h}(p^r - 1)^j, EA(p))$ for all $0 \leq m \leq k, k \geq 1, i \geq j \geq 1, h \geq 0$.

Let $\text{GH}(gh, G) = (g_{ij})$ be a generalized Hadamard matrix with elements from the abelian group G of order $|G| = g$. From gh matrices Y_i of order gh by choosing

$$Y_i = \begin{bmatrix} g_{1i} \\ g_{2i} \\ \vdots \\ g_{gh,i} \end{bmatrix} (g_{1i}, g_{2i}, \dots, g_{gh,i}).$$

Then over $R(g)$ the matrices $Y_i, i = 1, \dots, gh$, have the following properties (where Y_j^\dagger means Y^T with the elements replaced by their inverses):

$$Y_i Y_j^\dagger = hG \quad \text{whenever } i \neq j, \tag{1.1}$$

$$\sum_{i=1}^{hg} Y_i Y_i^\dagger = h^2 g G (J_{hg} - I_{hg}) + h^2 g^2 I_{hg}. \tag{1.2}$$

Hence if we replace the entries of Y_i by the usual permutation matrices we obtain $hg^2 \times hg^2$ matrices, $A_i, i = 1, \dots, hg$, satisfying the following equations:

$$A_i A_j^T = h J_{hg^2}, \tag{1.3}$$

$$A_i J_{hg^2} = hg J_{hg^2}, \tag{1.4}$$

$$A_i (I_{hg} \times J_g) = J_{hg^2}, \tag{1.5}$$

$$\sum_{i=1}^{hg} A_i A_i^T = h^2 g ((J_{hg} - I_{hg}) \times J_g) + h^2 g^2 I_{hg^2}. \tag{1.6}$$

Example 1. $X = \text{GH}(6, Z_3)$, writing the elements of Z_3 as 0, 1, 2,

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

The 6 matrices of order 6, $Y_i, i = 1, \dots, 6$, are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 0 & 0 & 2 \\ 2 & 2 & 0 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 2 & 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 & 2 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix},$$

which satisfy

$$Y_i Y_j^\dagger = 2Z_3(J_6), \quad i \neq j$$

(that is two copies of Z_3 as each element) and

$$\sum_{i=1}^6 Y_i Y_i^\dagger = 36I_6 + 12Z_3(J_6 - I_6).$$

We now replace the elements of Z_3 by the corresponding 3×3 permutation matrices to get six matrices, A_1, \dots, A_6 , of order 18 which satisfy

$$A_i A_j^T = 2J_{18}, \quad \sum_{i=1}^6 A_i A_i^T = 36I_{18} + 12(J_6 - I_6) \times I_3,$$

$$A_i J_{18} = 6J_{18}, \quad A_i (I_6 \times J_3) = J_{18}.$$

The block cyclic matrices with first rows

- (i) $A_1 A_2 \dots A_{gh}$,
- (ii) $I \times J_g A_1 A_2 \dots A_{gh}$,
- (iii) $J_{gh} A_1 A_2 \dots A_{gh}$,
- (iv) $0_{gh} A_1 A_2 \dots A_{gh}$ (0_{gh} the zero matrix),

give the required incidence matrices for the next theorem:

Theorem 1. Suppose there exists a generalized Hadamard matrix $\text{GH}(gh, G)$. Then there exist regular group divisible designs with parameters

- (i) $v = b = g^2h^2, r = k = g^2h^2, \lambda_1 = 0, \lambda_2 = gh^2, m = g, n = g^2h^2,$
- (ii) $v = b = g^2h(gh + 1), r = k = g^2h^2 + g, \lambda_1 = g, \lambda_2 = gh^2, \lambda_3 = gh^2 - h + 2,$
- (iii) $v = b = g^2h(gh + 1), r = k = g^2h(h + 1), \lambda_1 = g^2h, \lambda_2 = gh(g + \lambda), \lambda_3 = h(gh + 2h - 1),$
- (iv) $v = b = g^2h(gh + 1), r = k = g^2h^2, \lambda_1 = 0, \lambda_2 = gh^2, \lambda_3 = h(gh - 1).$

Example 1 (continued). Using six matrices A_1, \dots, A_6 of order 18 with $g = 3$ and $h = 2$ in the theorem we obtain regular group divisible designs with parameters:

- (i) $v = b = 108, r = k = 36, \lambda_1 = 0, \lambda_2 = 12, m = 3, n = 36;$
- (ii) $v = b = 126, r = k = 39, \lambda_1 = 3, \lambda_2 = 12, \lambda_3 = 12;$
- (iii) $v = b = 126, r = k = 54, \lambda_1 = 18, \lambda_2 = 30, \lambda_3 = 22;$
- (iv) $v = b = 126, r = k = 36, \lambda_1 = 0, \lambda_2 = 12, \lambda_3 = 10.$

Some further examples are given in Table 1. All these designs appear to be new.

Table 1

Generalized Hadamard matrix	PBIBD									
	g	h	Association classes	v	b	r	k	λ_1	λ_2	λ_3
$\text{GH}(10, Z_5)$	5	2	2	500	500	100	100	0	20	
$\text{GH}(10, Z_5)$	5	2	3	550	550	105	105	5	20	20
$\text{GH}(12, Z_3)$	3	4	2	432	432	144	144	0	48	
$\text{GH}(12, Z_3)$	3	4	3	468	468	147	147	3	48	46
$\text{GH}(12, Z_2 \times Z_2)$	4	3	2	576	576	144	144	0	36	
$\text{GH}(12, Z_2 \times Z_2)$	4	3	3	624	624	148	148	4	36	35
$\text{GH}(20, Z_5)$	5	4	2	2000	2000	400	400	0	80	
$\text{GH}(20, Z_5)$	5	4	3	2100	2100	405	405	5	80	77

Theorem 2. Let C be the incidence matrix of a BIBD (gh, b, r, h, λ) where $\lambda = h(h - 1)/(g - 1)$ and $r = \lambda + h^2$. Further let A_1, A_2, \dots, A_{gh} of order g^2h^2 be the matrices constructed above from a $\text{GH}(gh, G)$, where $|G| = g$. Then

$$D = [C \times J_g A_1 A_2 \dots A_{gh}]$$

is the incidence matrix of a BIBD $(g^2h, gr^1, gh(g^2h - 1)/(g - 1), gh, gh(gh - 1)/(g - 1))$.

Proof. Follows from

$$\begin{aligned} DD^T &= ((r - \lambda)I + \lambda J) \times gJ + h^2g^2I_{hg^2} + h^2gJ_{hg^2} - h^2gI_{hg} \times J_g \\ &= h^2g^2I + gh(gh - 1)/(g - 1)J, \end{aligned}$$

$$JD = ghJ.$$

Example 2. See the designs in Table 2.

Table 2

$\text{GH}(gh, \text{EA}(g))$	$C = \text{BIBD}(gh, b, r, h, \lambda)$	Constructed BIBD $(g^2h, gr^1, r^1, gh, \lambda^1)$
$\text{GH}(6, \text{EA}(3))$	(6, 15, 5, 2, 1)	(18, 153, 51, 6, 15)
$\text{GH}(12, \text{EA}(3))$	(12, 66, 22, 4, 6)	(36, 630, 210, 12, 66)
$\text{GH}(12, \text{EA}(4))$	(12, 44, 11, 3, 2)	(48, 752, 188, 12, 44)
$\text{GH}(9, \text{EA}(3))$	(9, 36, 12, 3, 3)	(27, 351, 117, 9, 36)

2. Constructions using colourable designs

As in Rodger, Sarvate and Seberry (1987), a BIBD or PBIBD with r applications is called *colourable* if one of a set of r (latin) symbols can be associated with each 1 in the incidence matrix of the design so that each symbol occurs *exactly* once per row and no symbol occurs more than once per column.

Theorem 3. *Suppose there exists a $\text{GH}(gh, G)$ and a BIBD (v, b, gh, k, λ) which is colourable. Then there exists a*

$$\text{GDD}(g^2hv, g^2hb, g^2h^2, ghk, \lambda_1 = 0, \lambda_2 = h^2g, \lambda_3 = h\lambda).$$

Proof. Replace the latin symbols by A_1, \dots, A_{gh} of order g^2h and the zeros by the zero matrix of order g^2h .

Example 3. To illustrate we use the $\text{GH}(6, Z_3)$ and note that the BIBDs with parameters (6, 6, 6), (7, 6, 5), (4, 8, 6, 3, 2), (7, 14, 6, 3, 2), (7, 21, 6, 2, 1), (11, 6, 3) and (13, 6, 1) are all colourable. Thus we have the designs in Table 3. All these PBIBD appear to be new.

Table 3

BIBD parameters					PBIBD parameters						
v	b	r	k	λ	v	b	r	k	λ_1	λ_2	λ_3
4	8	6	3	2	72	144	36	18	0	12	4
6	6	6	6	6	108	108	36	36	0	12	12
7	7	6	6	5	126	126	36	36	0	12	10
7	14	6	3	2	126	252	36	18	0	12	4
7	21	6	2	1	126	378	36	12	0	12	2
11	11	6	6	3	198	198	36	36	0	12	6
31	31	6	6	1	558	558	36	36	0	12	2

3. Constructions using semi-colourable designs

A design (BIBD or PBIBD) is called *semi-colourable* if latin symbols can be associated with the incidence matrix of the design so that one distinguished symbol A , say, occurs a fixed number of times (>1) in each row and every other symbol occurs exactly once in each row while in each column the symbols are all different. In addition the replacement of the distinguished symbol by zero must leave a new colourable design.

Example 4. The designs in the following cases are obtained by cycling the first rows of the initial block or blocks under the given modulus.

- (i) (Aabc), (Adef) mod 4;
- (ii) (Aab), (Acd), (Aef) mod 3;
- (iii) (Aab0c00), (Ade0f00) mod 7;
- (iv) (Aa00b), (A0cd0) mod 5;
- (v) (0Aa0b00), (0cAd00), (0ef0A00) mod 7.

Construction Lemma 1. A semi-colourable design on $(s+1)t$ symbols A, a_1, \dots, a_{st} , where the distinguished element occurs t times, can be formed by cycling the t initial blocks,

$$(Aa_1, a_2 \dots a_s), (Aa_{s+1} a_{s+2} \dots a_{2s}), \dots, (Aa_{st-s+1} \dots a_{st}) \pmod{s+1}.$$

The first block design has parameters

$$(s+1, (s+1)t, (s+1)t, s+1, (s+1)t);$$

the second block design (with the distinguished element removed) has parameters

$$(s+1, (s+1)t, st, s, (s-1)t).$$

See Examples 4(i), (ii).

Construction Lemma 2. Let $D_i = \{d_{i1}, \dots, d_{is}\}$, $i = 1, \dots, t$, be initial blocks mod n of a design (n, tn, st, s, λ) with the property that $d_{ij} \neq 0$ for any i, j , $d_{ij} \in D_i \rightarrow -d_{ij} \notin D_i$. Then a semi-colourable design on $(s+1)t$ symbols A, a_{11}, \dots, a_{st} , where the distinguished element occurs t times can be formed by associating A with the zero element of each initial block D_i and a_{ij} with d_{ij} .

Now cycle the blocks. The new design has parameters

$$(n, tn, (s+1)t, s+1, \lambda+t).$$

See Example 4(iii).

Construction Lemma 3. Let $X+Y$ be the incidence matrix of a design, where Y is also the incidence matrix of a design that is colourable and X is a design where the

intersections between rows of X is zero then $AX + Y$ (coloured) is a semi-colourable design.

See Example 4(iv).

Construction Lemma 4. Let Y be the incidence matrix of a design with parameters (v, b, r, k, λ) which is colourable with the symbols a_1, \dots, a_r and in which each symbol also occurs once per column. Take r copies of Y now coloured with the symbols a_{i1}, \dots, a_{ir} , $i = 1, \dots, r$. Replace a_{ii} , $i = 1, \dots, r$, by the distinguished symbol A to obtain a semi-colourable design with parameters $(v, br, r^2, k, r\lambda)$ with the underlying second design $(v, br, r(r-1), k, \lambda(r-1))$.

See Example 4(v).

These lemmas give us the general results:

Theorem 4. Suppose there exists a $\text{GH}(gh, G)$, $|G| = g$, and a $\text{BIBD}(v, b, gh + f, k, \lambda)$ which is semi-colourable (the distinguished element occurring f times per row with underlying design having parameters $(v, b, gh, k - 1, \mu)$). Then there exists a

$$\text{GDD}(g^2hv, g^2hb, g^2h^2 + gf, gh(k - 1) + g, \lambda_1 = fg, \lambda_2 = h^2g, \lambda_3 = \lambda + \mu(h - 1)).$$

Proof. Replace the distinguished element by $I_{gh} \times J_g$ and the other latin symbols by A_1, \dots, A_{gh} of order g^2h . Replace the zero elements by the zero matrix of order g^2h .

Corollary 5. Suppose there exists a $\text{GH}(gh, G)$, $|G| = g$. Then there exists a

$$\text{GGD}(g^2h(gh + 1), g^2h(gh + 1), g^2h^2 + g, g^2h^2 + g, \lambda_1 = g, \lambda_2 = h^2g, \lambda_3 = gh^2 - h + 2).$$

Proof. Use the cyclic design with first row

$$I_{gh} \times J_g A_1 A_2 \dots A_{gh}.$$

Corollary 6. Suppose there exists a $\text{GH}(gh, G)$, $|G| = g$. Then there exist

- (i) $\text{GDD}(2g^2h, 2g^3h^2, gh(gh + 1), g(h + 1), \lambda_1 = g^2h, \lambda_2 = h^2g, \lambda_3 = 2gh)$;
- (ii) $\text{GDD}(g^2h(g + 1), g^2h^2(g + 1), gh(gh + 1), \lambda_1 = gh, \lambda_2 = h^2g, \lambda_3 = h(gh - h + 2))$;
- (iii) $\text{GDD}(g^2h(h + 1), g^3h(h + 1), g^2(h^2 + 1), \lambda_1 = g^2, \lambda_2 = h^2g, \lambda_3 = g(h^2 - h + 2))$.

Proof. For (i) use the $\text{BIBD}(2, 2gh, 2gh, 2, 2gh)$ as

$$\begin{bmatrix} AA_1AA_2 \dots AA_{gh} \\ A_1AA_2A \dots A_{gh}A \end{bmatrix}.$$

For (ii) use the $\text{BIBD}(g + 1, h(g + 1), h(g + 1), g + 1, h(g + 1))$ with blocks which cycle

the first rows

$$(AA_1 \cdots A_g), (AA_{g+1} \cdots A_{2g}), \dots, (AA_{gh-g} \cdots A_{gh}).$$

For (iii) we use the BIBD $(h+1, g(h+1), g(h+1), h+1, g(h+1))$, with blocks which cycle the first rows

$$(AA_1 \cdots A_h), (AA_{h+1} \cdots A_{2h}), \dots, (AA_{gh-h} \cdots A_{gh}).$$

Example 5. See Table 4.

Table 4

Semicolourable design	Underlying colourable design	GH exist?	New design
$(v, b, gh+f, k, \lambda)$	$(v, b, gh, k-1, \mu)$	g, h	$(g^2hv, g^2hb, g^2h^2 + gf, gh(k-1) + g, \lambda_1, \lambda_2, \lambda_3)$
(7, 7, 7, 7, 7)	(7, 7, 6, 6, 5)	3, 2	(126, 126, 39, 3, 12, 12)
(4, 8, 8, 4, 8)	(4, 8, 6, 3, 2)	3, 2	(72, 144, 42, 21, 6, 12, 10)
(3, 9, 9, 3, 9)	(3, 9, 6, 2, 3)	3, 2	(54, 162, 45, 15, 9, 12, 12)
(2, 12, 12, 2, 12)	(2, 12, 6, 1, 0)	3, 2	(36, 176, 54, 15, 18, 12, 12)
(7, 14, 8, 4, 4)	(7, 14, 6, 3, 2)	3, 2	(126, 252, 42, 21, 6, 12, 6)
(7, 21, 9, 3, 3)	(7, 21, 6, 2, 1)	3, 2	(126, 378, 45, 15, 9, 12, 4)
(2, 24, 24, 2, 24)	(2, 24, 12, 1, 0)	3, 4	(72, 864, 180, 15, 36, 48, 24)
(2, 24, 24, 2, 24)	(2, 24, 12, 1, 0)	4, 3	(96, 1152, 192, 16, 48, 36, 24)
(3, 18, 18, 3, 18)	(3, 18, 12, 2, 6)	3, 4	(108, 648, 162, 27, 18, 48, 36)
(3, 18, 18, 3, 18)	(3, 18, 12, 2, 6)	4, 3	(144, 864, 168, 28, 24, 36, 30)
(5, 15, 15, 3, 15)	(5, 15, 12, 4, 9)	3, 4	(180, 540, 153, 51, 9, 48, 42)
(5, 15, 15, 3, 15)	(5, 15, 12, 4, 9)	4, 3	(240, 720, 156, 52, 28, 12, 36, 33)
(5, 30, 30, 3, 6)	(5, 30, 12, 2, 3)	3, 4	(180, 1080, 198, 27, 54, 48, 15)
(5, 30, 30, 3, 6)	(5, 30, 12, 2, 3)	4, 3	(240, 1140, 216, 28, 72, 36, 12)
(7, 28, 16, 4, 6)	(7, 28, 12, 3, 3)	3, 4	(252, 1008, 156, 39, 12, 48, 15)
(7, 28, 16, 4, 6)	(7, 28, 12, 3, 3)	4, 3	(336, 1344, 160, 40, 16, 36, 12)

4. New designs from old

Most of our construction has involved constructions starting from a BIBD (v, b, gh, k, λ) to get new designs with parameters

$$V = g^2hv, B, R, K, \lambda_1, \lambda_2, \lambda_3.$$

We now note that if a BIBD $(v, b^1, r^1, K/g, \lambda^1)$, C say, exists then

$$E = I_{gh} \times J_g \times C$$

gives a

$$\text{GDD}(g^2hv, B^1, R^1, K, \lambda_1 = r^1g, \lambda_2 = g, \lambda_3 = 0)$$

with the same association classes as if it has essentially been constructed via a similar Kronecker product design. This design may sometimes be usefully combined with another design to reduce the number of different values of λ_i . For instance:

Construction Lemma 5. *Suppose there exists a GH(gh, G), $|G|=g$. Further suppose there exists a BIBD ($g^2h^2, B', R', gh^2, \lambda$) = C . Then with F the $g \times f$ matrix of ones,*

$$[F \times C \mid \text{design in Theorem 1(i)}]$$

is a

$$\text{GDD} (g^3h^2, g^3h^2 + B'f, R'f + g^2h^2, g^2h^2, \lambda_1 = R'f, \lambda_2 = \lambda f + gh^2).$$

Example 6. Let $g = 3, h = 2$ and $C = \text{BIBD} (36, 105, 35, 12, 11)$. Then the lemma gives a

$$\text{GDD} (108, 105f + 108, 35f + 36, 36, \lambda_1 = 35f, \lambda_2 = 11f + 12)$$

where $f \geq 1$ is integer.

Construction Lemma 6. *Suppose there exists a generalized Hadamard matrix GH(gh, G), $|G|=g$. Further suppose there exists a BIBD (gh, b, r, h, λ) = C . Then there exists a*

$$\text{GDD} (v' = g^2h, b', rg + g^2h^2, gh, \lambda_1 = rg, \lambda_2 = \lambda g + gh^2).$$

Proof. The incidence matrix is

$$[C \times J_g A_1 A_2 \cdots A_{gh}].$$

Example 7. There is a BIBD (6, 15, 5, 2, 1) and a GH(6, Z_3). Hence there exists a BIBD (18, 153, 51, 6, 15).

Corollary 7. *Let q be a prime power. Then there exists a GH($2q, \text{EA}(q)$). Further there exists a BIBD ($2q, q(2q - 1), 2q - 1, 2, 1$). Hence there exists a*

$$\text{GDD} (2q^2, 6q^3 - q^2, 6q^2 - q, 2q, \lambda_1 = q(2q - 1), \lambda_2 = 5q).$$

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