# ON INEQUIVALENT WEIGHING MATRICES

H.C. CHAN, C.A. RODGER AND JENNIFER SEBERRY

ABSTRACT. A weighing matrix $W = W(n,k)$ of order $n$ and weight $k$ is a square matrix of order $n$, with entries 0, +1 and −1 which satisfies $WW^T = kI_n$.

Tools such as Smith Normal Form, profile, maximum integer and some group theoretic and coding theory methods are used to classify some matrices for $1 \leq k \leq n \leq 20$, into equivalence classes under pre-and post-multiplication by monomial matrices (permutation matrices where the non-zero elements are +1 or −1).

The inequivalent weighing matrices are classified for $k \leq 5$.

## 1. Introduction.

A *weighing matrix* $W = W(n,k)$ of *order* $n$ and *weight* $k$ is an orthogonal $n \times n$ matrix with entries 0,1 and −1 and $k$ non-zero entries in each row and in each column. Clearly

$$WW_n = kI.$$

Weighing matrices have long been studied by statisticians and combinatorialists because of their use in efficient weighing experiments, in optics and in the construction of error-correcting codes (see for example [**2**]). Two weighing matrices $A$ and $B$, both of order $n$ and weight $k$, are said to be equivalent iff one can be transformed into the other by using the following operations:

(i) multiply any row or column by −1, and
(ii) interchange two rows or two columns.

In this paper we consider the question of determining the number of inequivalent $W(n,k)$. In the process we classify the inequivalent weighing matrices with $k \leq 5$.

A weighing matrix for which $n = k$ is called a *Hadamard matrix*. These are conjectured to exist for $n + 1, 2$ and $4t$ [**17**] and Seberry has shown that for any $q$ there exists a Hadamard matrix of order $2^s q$ for every $s \geq [2\log_2(q-3)]$. Little is known by way of general theorems on the number of inequivalent Hadamard matrices, though they are known to be unique up to equivalence for $n = 1, 2, 4, 8$ and 12. It is also known that there are

5 inequivalent Hadamard matrices of order 16 [8],
at least 2 inequivalent skew Hadamard matrices of order 16 [11],
3 inequivalent Hadamard matrices of order 20 [8],
59 inequivalent Hadamard matrices of order 24 [9],
for inequivalent Hadamard matrices of order 28, see [9],
at least 11 inequivalent Hadamard matrices of order 32 and
at least 110 inequivalent Hadamard matrices of order 36 [1].

Mullin [14] has studied the existence of balanced weighing designs and Schellenberg [16] has shown that although SBIBD(16, 6, 2)'s exist there is no BW(16, 6, 2). Only a few results have been published on inequivalent weighing designs; at least two BW(13, 9)'s exist [5] (which are inequivalent as they generate linear ternary codes with differing minimum distances) and a BW(19, 4) exists [13].

Finally it is relevant to note the following existence theorems for weighing matrices.

1. If $n$ is odd then a $W(n, k)$ only exists if
   (i) $k$ is a square and
   (ii) $(n-k)^2 + (n-k) + 1 \geq n$.
2. If $n \equiv 2 \pmod 4$ then for a $W(n, k)$ to exist,
   (i) $k \leq n-1$ and
   (ii) $k$ is the sum of two squares.
3. A skew symmetric $W(n, k)$ only exists if $k$ is the sum of three squares.
4. No symmetric $W(16, 15)$ exists.

(See [2] for 1, 2 and 3 and [15] for 4).
Also, it has been conjectured that

1. (Seberry) if $n \equiv 0 \pmod 4$ then a $W(n, k)$ exists for all $1 \leq k \leq n$,
2. (Geramita and Seberry) if $n \equiv 4 \pmod 8$ then a skew symmetric $W(n, k)$ exists iff $k$ is the sum of 3 squares,
3. (Geramita and Seberry) if $n \equiv 0 \pmod 8$, a skew symmetric $W(n, k)$ exists for all $k = 0, 1, \ldots, n-1$.

## 2. Inequivalent Weighing Matrices.

We shall define several techniques which in turn will be used to distinguish between inequivalent weighing matrices.

*2.1. Intersection Patterns of Rows.*

Given any row, say row $j$ of a specific weighing matrix, $W(n,k)$, we say that $x_{2i}$ rows of $W(n,k)$ *intersect* row $j$ in $2i$ places if there are $x_{2i}$ rows, each of which has exactly $2i$ non-zero elements occurring in columns containing non-zero elements in row $j$. Then define $W(n,k)$ to have an *intersection pattern* $(x_0, x_2, x_4, \ldots)$ (corresponding to row $j$). Then clearly we have the Intersection Pattern Conditions (IPC):

1. $\sum_{j=0} x_{2j} = n-1$ and
2. $\sum_{j=0} jx_{2j} = k(k-1)/2$.

IPC1 follows from the definition and IPC2 follows by counting non-zero elements in rows and then in columns. Unless otherwise stated we shall assume that the distinguished row is the first row.

Sometimes the intersection pattern allows us to obtain considerable information about the structure of the weighing matrix.

EXAMPLE 1: In any intersection pattern of any $W(8,6)$, $x_6 \le 1$; for if two rows intersect the first row in 6 columns, say the first 6 then the last two rows cannot contain 6 non-zero elements as required.

EXAMPLE 2: Let $k=4$ and suppose that only $x_0$ and $x_2$ are non-zero. Then IPC1 and IPC2 give

$$x_0 + x_2 = n - 1$$

and

$$x_2 = 6.$$

Attempting to construct a $W(n,4)$ with intersection pattern $(n-7,6)$ row by row leads to a choice for the $(5,5)$ element of 1 or 0 only and no further choice giving the following ($-1$ is denoted here and on all matrices with $-$):

$$B(8,4) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 & 0 \\ 1 & 0 & 0 & - & 0 & - & - & 0 \\ 0 & 1 & - & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & - & 0 & 1 & 0 & - \\ 0 & 0 & 1 & - & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & - & 1 & - \end{bmatrix}$$

$$W(7,4) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 1 & - & - & 1 & 0 \end{bmatrix}$$

***All weighing designs with $k = 4$ and intersections of 0 and 2 only.***

Again considering a $W(n,4)$ and supposing that $x_4 = 1$ and otherwise only $x_0$ and $x_2$ are non-zero gives

$$x_0 + x_2 = n - 2$$

and

$$x_2 = 4.$$

Constructing $W(n,4)$ with intersection pattern $(n-6,4,1)$ gives:

$$C(2n,4) = \left[\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & & & & & & & & \\ 1 & 1 & - & - & & & & & & & & \\ 1 & - & 0 & 0 & 1 & 1 & & & & & & \\ 1 & - & 0 & 0 & - & - & & & & & & \\ & & 1 & - & & & 1 & 1 & & & & \\ & & 1 & - & & & - & - & & & & \\ & & & & 1 & - & & & & & & \\ & & & & 1 & - & & & & & & \\ & & & & & \ddots & & & & \ddots & & \\ & & & & & & & & & & 1 & 1 \\ & & & & & & & & & & - & - \\ & & & & & & & & 1 & - & - & 1 \\ & & & & & & & & 1 & - & 1 & - \end{array}\right]$$

$W(2n,4)$, $n \geq 3$ *with one intersection* 4 *and all others* 0 *and* 2 (all unspecified positions are 0).

The classification of $W(n,k)$ for $k \leq 3$ is straightforward.

(i) Any $W(n,1)$ is equivalent to $I_n$.

(ii) Any $W(n,2)$ is equivalent to $\underset{n/2}{\oplus} W(2,2)$.

(iii) Any $W(n,3)$ is equivalent to $\underset{n/4}{\oplus} W(4,3)$.

So for any $n$ and any $k \leq 3$, the $W(n,k)$ all lie in a single equivalence class.

We now use intersecting patterns to obtain a decomposition theorem for weighing matrices with $k = 4$. ($B(8,4)$ and $C(2n,4)$ are defined in Example 2). This essentially classifies all inequivalent $W(n,4)$'s.

THEOREM 3. *Any $W(n,4)$ is equivalent to*

$$\underset{a \text{ copies}}{\oplus} W(4,4) \quad \underset{b \text{ copies}}{\oplus} B(8,4) \quad \underset{d \text{ copies}}{\oplus} W(7,4) \quad \underset{i \text{ copies}}{\oplus} \Big(\underset{t_i \text{ copies}}{\oplus} C(6+2t_i,4)\Big)$$

PROOF: Suppose we wish to construct a $W(n,4)$. Then under the four 1's of the first row we can have:

a) three other rows containing four ones and all other rows with zero ones, that is we have $W(n,4) + W(n-4,4)$; or
b) one other row containing four ones and four rows containing two ones; that is we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & & & \\ 1 & 1 & - & - & & & \\ 1 & - & 0 & 0 & 1 & 1 & \\ 1 & - & 0 & 0 & - & - & \\ & & 1 & - & \multicolumn{3}{|c}{} \\ & & 1 & - & \multicolumn{3}{|c}{X} \\ & & & & \multicolumn{3}{|c}{} \end{bmatrix}$$

This pattern can only be completed to a $C(6+2t,4)$ for any $t \geq 0$, when $n$ is even and cannot be completed otherwise, or

c) six other rows containing two ones, that is we have

$$\left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 1 & & & \\ 1 & - & & & 1 & 1 & \\ 1 & & - & & - & & 1 \\ 1 & & & - & & - & - \\ \hline & 1 & - & & & & \\ & 1 & & - & & Y & \\ & & 1 & - & & & \end{array}\right].$$

This case can only be completed to a $B(8,4)$ or a $W(7,4)$. So its occurrence gives either

(i) $W(n,4) = B(8,4) \oplus W(n-8,4)$ or
(ii) $W(n,4) = W(7,4) \oplus W(n-7,4)$.

In Appendix B the $W(n,4)$ are classified for $n \leq 20$ and ennumerated for $n \leq 4$.

EXAMPLE 4: To characterize the $W(2n, 5)$ (since 5 is not a perfect square, no $W(2n+1, 5)$ exist) we define the following families of $W(2n, 5)$'s (all unspecified positions are 0).

$$E(4t+2,5) = \left[\begin{array}{*{23}{c}}
1&1&1&1&1&0 \\
1&1&-&-&0&1 \\
1&-&0&0&0&0&1&1&1&0 \\
1&-&0&0&0&0&-&-&0&1 \\
1&0&0&0&-&-&0&0&-&- \\
1&0&0&0&-&-&0&0&1&1 \\
&&1&-&0&0&&&&&1&1&1&0 \\
&&1&-&0&0&&&&&-&-&0&1 \\
&&1&0&-&1&&&&&0&0&-&- \\
&&0&1&-&1&&&&&0&0&1&1 \\
&&&&&&1&-&0&0&&&&&\ddots \\
&&&&&&1&-&0&0 \\
&&&&&&1&0&-&1 \\
&&&&&&0&1&-&1 \\
&&&&&&&&&&\ddots&&&&&\ddots \\
&&&&&&&&&&&&&&&&&&&1&1&1&0 \\
&&&&&&&&&&&&&&&&&&&-&-&0&1 \\
&&&&&&&&&&&&&&&&&&&0&0&-&- \\
&&&&&&&&&&&&&&&&&&&0&0&1&1 \\
&&&&&&&&&&&&&&&1&-&0&0&1&0&-&1 \\
&&&&&&&&&&&&&&&1&-&0&0&0&-&1&- \\
&&&&&&&&&&&&&&&1&0&-&1&-&1&0&0 \\
&&&&&&&&&&&&&&&0&1&-&1&1&-&0&0
\end{array}\right]$$

Define $F(4t+4,5)$ to be the matrix formed from $E(4t+2,5)$ by adding two rows and two columns and replacing the $4 \times 4$ sub-matrix in the bottom right hand corner of $E(4t+2,5)$ with the following $6 \times 6$ sub-matrix:

$$\begin{array}{cccccc}
1 & - & 0 & 0 & 1 & 0 \\
- & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & - & - \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & - & 1 & - & 1 \\
0 & 1 & - & 1 & 1 & -
\end{array}$$

Let

$$D(16,5) = \begin{bmatrix}
1&1&1&1&1&&&&&&&&&&& \\
1&-&0&0&0&1&1&1&&&&&&&& \\
1&0&-&0&0&-&0&0&1&1&&&&&& \\
1&0&0&-&0&0&-&0&-&0&1&&&&& \\
1&0&0&0&-&0&0&-&0&-&-&&&&& \\
&1&-&0&0&1&0&0&0&0&0&1&1&&& \\
&1&0&-&0&0&1&0&0&0&0&-&0&1&& \\
&1&0&0&-&0&0&1&0&0&0&0&-&-&& \\
&&1&-&0&0&0&0&1&0&0&1&0&0&1& \\
&&1&0&-&0&0&0&0&1&0&0&1&0&-& \\
&&&1&-&0&0&0&0&0&1&0&0&1&1& \\
&&&&&1&-&0&1&0&0&-&0&0&0&1 \\
&&&&&1&0&-&0&1&0&0&-&0&0&- \\
&&&&&&1&-&0&0&1&0&0&-&0&1 \\
&&&&&&&&1&-&1&0&0&0&-&- \\
&&&&&&&&&&&1&-&1&-&1
\end{bmatrix}$$

where each unspecified position on $D(16,5)$ is 0.

Finally we exhibit the unique $W(n,5)$'s with $n=6$ and $n=8$.

$$W(6,5) = \begin{bmatrix}
0&1&1&1&1&1 \\
1&0&1&-&-&1 \\
1&1&0&1&-&- \\
1&-&1&0&1&- \\
1&-&-&1&0&1 \\
1&1&-&-&1&0
\end{bmatrix}$$

$$W(8,5) = \begin{bmatrix}
1&1&1&1&1&0&0&0 \\
1&1&-&-&0&1&0&0 \\
1&-&-&1&0&0&1&0 \\
1&-&1&-&0&0&0&1 \\
1&0&0&0&-&-&-&- \\
0&1&0&0&-&-&1&1 \\
0&0&1&0&-&1&1&- \\
0&0&0&1&-&1&-&1
\end{bmatrix}$$

THEOREM 5. *Any $W(2n,5)$ is equivalent to*

$$
\begin{array}{cccccccc}
\oplus & W(6,5) & \oplus & W(8,5) & \oplus & D(16,5) & \oplus & (\oplus\, E(4t_i+2,5)) \\
x & & y & & d & & i & t_i \\
\text{copies} & & \text{copies} & & \text{copies} & & \text{copies} & \text{copies} \\
 & & & & \oplus & & & (\oplus\, F(4t_j+4,5)) \\
 & & & & j & & & t_j \\
 & & & & \text{copies} & & & \text{copies}
\end{array}
$$

*where $t_i \geq 2$ and $t_j \geq 2$.*

PROOF: In constructing a $W(2n,5)$, we can assume that

(a) at least two other rows intersect with the first row in four places or
(b) no rows intersect any other row in four places or
(c) exactly one row intersects the first row in four places.

It is straightforward to check that in case (a) the only possible such weighing matrices are the unique $W(2n,5)$ with $n = 3$ and 4. Also, in case (b) the only possible such weighing matrix is $D(16,5)$ (see Example 4).

In case (c), we can assume that the top left hand corner of $W$ has the form

$$
\begin{array}{cccccc|cccc}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & - & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & - & 0 & 0 & 0 & 0 & - & - & 0 & 1 \\
1 & 0 & 0 & 0 & - & - & 0 & 0 & - & - \\
0 & 1 & 0 & 0 & - & - & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & 1 & - & 0 & 0 & x & y & & \\
0 & 0 & 1 & - & 0 & 0 & & & & \\
0 & 0 & 1 & 0 & - & 1 & & & & \\
0 & 0 & 0 & 1 & - & 1 & & & &
\end{array}
$$

To be orthogonal to row 3, either $x \neq 1$ or $y \neq 1$. If $x = 1$ and $y = 0$ then this matrix can only be completed to $E(10,5)$. If $x = 1$ and $y = -1$ then this matrix can only be completed to $F(12,5)$. Finally, if $x = 0$ and $y = 0$ then this matrix can only be completed to $E(4t+2,5)$ or $F(4t+4,5)$. In any case $W(2n,5)$ is of the required form.

From Example 2 we have shown that $W(7,4)$ and $B(8,4)$ are the only $W(n,4)$ with $x_{2i} = 0$ for all $i > 1$. For more general $k$ we can obtain a bound on $n$ in such a case.

LEMMA 6. *Suppose that there exists a* $W(n,k)$ *with* $x_{2i} = 0$ *for all* $i > 1$. *Then*

$$n \geq k^2 - 4k + 7.$$

PROOF: From the intersection pattern condition,

$$x_0 + x_2 = n - 1$$

and

$$x_2 = \frac{k(k-1)}{2}$$

so

$$x_0 = n - k(k-1)/2 - 1.$$

Schematically the non-zero elements can be illustrated as

$$\left[\begin{array}{c|c} 1\,1\,\ldots\,1 & 0\,0\,\ldots\,0 \\ \hline & X \\ A & k-2 \text{ ones/row} \\ \hline & Y \\ 0 & k \text{ ones/row} \end{array}\right]$$

The intersection between any pair of rows in $A$ is 0 or 1 (since $x_4 = 0$) and thus the intersection between any pair of rows in $X$ is 0 or 1. Thus to complete $X$ (with $k-2$ ones per row) we need at least $(k-3)^2+(k-3)+1$ columns (considering projective planes). Thus

$$(k-3)^2 + (k-3) + 1 + k \leq n.$$

EXAMPLE 7: We can obtain some results concerning $W(12,6)$ matrices using a classification involving the intersection patterns.

Clearly $x_6 \leq 1$ and so $x_0 \leq 2$. Also

$$x_0 + x_2 + x_4 + x_6 = 1$$

and

$$2x_2 + 4x_4 + 6x_6 = 30.$$

These observations yield the following possible intersection patterns.

| $x_0$ | $x_2$ | $x_4$ | $x_6$ | *Type* |
|---|---|---|---|---|
| 2 | 3 | 6 | 0 | *D* |
| 1 | 5 | 5 | 0 | *E* |
| 0 | 7 | 4 | 0 | *F* |
| 2 | 4 | 4 | 1 | *G* |
| 1 | 6 | 3 | 1 | *D* |
| 0 | 8 | 2 | 1 | *H* |

We note that intersection patterns $(2,3,6,0)$ and $(1,6,3,1)$ (type $D$) arise in the same matrices; this can only be seen by choosing one of the rows intersecting with the first row in zero places in a matrix with pattern $(2,3,6,0)$ as the distinguished row (instead of choosing the first row), thus obtaining the pattern $(1,6,3,1)$. As yet we have found no $W(12,6)$ of type $D$. However at least 4 inequivalent $W(12,6)$'s exist, as the following are of types $E$, $F$, $G$ and $H$.

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & - & - & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & - & - & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & - & 1 & 0 & 1 & - & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & - & - & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & - & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
- & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & - & 0 & 0 & 0 & 0 & 1 & 0 & 1 & - & - & 1 \\
0 & 0 & - & 0 & 0 & 0 & 1 & 1 & 0 & 1 & - & - \\
0 & 0 & 0 & - & 0 & 0 & 1 & - & 1 & 0 & 1 & - \\
0 & 0 & 0 & 0 & - & 0 & 1 & - & - & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & - & 1 & 1 & - & - & 1 & 0
\end{bmatrix}$$

A $W(12,6)$ with intersection pattern $(1,5,5,0)$ whose transpose has intersection pattern $(1,5,5,0)$ (Type $E$).

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
0 & 1 & 1 & 0 & 1 & - & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & - & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & - & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
0 & - & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & - \\
- & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & - & 0 \\
1 & 0 & - & 1 & 1 & 0 & 1 & 0 & 0 & - & 0 & 0 \\
\hline
0 & - & 0 & 0 & 0 & - & 0 & 1 & 1 & 0 & - & 1 \\
- & 0 & 0 & 0 & - & 0 & 1 & 0 & 1 & - & 1 & 0 \\
0 & 0 & - & - & 0 & 0 & 1 & 1 & 0 & 1 & 0 & - \\
\hline
0 & - & 0 & 0 & 0 & 1 & 0 & 1 & - & 0 & 1 & 1 \\
- & 0 & 0 & 0 & 1 & 0 & 1 & - & 0 & 1 & 0 & 1 \\
0 & 0 & - & 1 & 0 & 0 & - & 0 & 1 & 1 & 1 & 0
\end{array}\right]$$

**A $W(12,6)$ with intersection pattern $(0,7,4,0)$ whose transpose has intersection pattern $(0,7,4,0)$ (Type $F$).**

$$\begin{bmatrix}
0 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & - & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
- & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
 & & & & & & & & & & & \\
0 & - & - & 0 & 1 & - & 1 & 0 & 0 & - & 0 & 0 \\
- & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & - & 0 \\
- & - & 0 & - & 0 & 1 & 0 & 0 & 1 & 0 & 0 & - \\
 & & & & & & & & & & & \\
- & 0 & 0 & - & 0 & 0 & 0 & 1 & - & 0 & 1 & 1 \\
0 & - & 0 & 0 & - & 0 & 1 & - & 0 & 1 & 0 & 1 \\
0 & 0 & - & 0 & 0 & - & - & 0 & 1 & 1 & 1 & 0 \\
 & & & & & & & & & & & \\
- & 0 & 0 & 1 & 0 & 0 & 0 & - & - & 0 & 1 & - \\
0 & - & 0 & 0 & 1 & 0 & - & 0 & - & 1 & - & 0 \\
0 & 0 & - & 0 & 0 & 1 & - & - & 0 & - & 0 & 1
\end{bmatrix}$$

**A $W(12,6)$ with intersection pattern $(2,4,4,1)$ whose transpose has intersection pattern $(2,4,4,1)$ (Type $G$).**

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & - & 0 & 1 & - & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & - & 0 & - & 1 & 0 & 1 & - & 0 & 0 & 0 & 0 \\
1 & 0 & - & 0 & 0 & 0 & - & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & - & 0 & 0 & 0 & - & 0 & - & - & - & 0 \\
0 & 1 & - & 0 & 0 & 0 & 1 & 0 & 1 & - & 0 & 1 \\
0 & 1 & - & 0 & 0 & 0 & 1 & 0 & - & 1 & 0 & - \\
0 & 0 & 0 & 1 & 0 & - & 0 & - & 1 & 0 & - & - \\
0 & 0 & 0 & 1 & 0 & - & 0 & - & - & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & - & 0 & 1 & 0 & 1 & - & 1 \\
0 & 0 & 0 & 0 & 1 & - & 0 & 1 & 0 & - & 1 & -
\end{bmatrix}
$$

**A $W(12,6)$ with intersection pattern $(0,8,2,1)$ (Type H) whose transpose has intersection pattern $(1,5,5,0)$ (Type E).**

### *2.2. Smith Normal Form.*

The Smith Normal Form (SNF) of a weighing matrix $W(n,k)$ can be obtained by using elementary row and column operations to diagonalize $W(n,k)$ over the integers; equivalently we pre- and post-multiply $W(n,k)$ by integer matrices with determinant 1 until a diagonal matrix is achieved (see, for example [**12**]). It is known that the SNF of $W(n,k)$ is diag $(a_1, a_2, \ldots, a_n)$ where $a_i \mid a_{i+1}$ for $1 \le i \le n-1$ and $a_1 a_2 \ldots a_n$ det $W = k^{n/2}$; we say that $W$ is $Z$-equivalent to diag $(a_a, a_2, \ldots, a_n)$.

THEOREM 8. *Any $W(n,k)$ is $Z$-equivalent to* diag $(a_1, a_2, \ldots, a_n)$ *where* $a_{n+1-i} a_i = k$.

PROOF: By Theorem 10.5 of Wallis [**17**] any $W(n,k)$ is $Z$-equivalent to diag $(a_a, a_2, \ldots, a_n)$ where $a_i \mid a_{i+1}$ for $1 \le i \le n-1$. Let

$$PWQ = \text{diag}\,(a_1, a_2, \ldots, a_n).$$

Since $WW^T = kI_n$,

$$(PWQ)(Q^{-1}W^TP^{-1}) = kI_n$$

so

$$Q^{-1}W^TP^{-1} = \text{diag}\,(k/a_1, k/a_2, \ldots, k/a_n).$$

Thus $k/a_{i-1}$ divides $k/a_i$ for $1 \le i \le n$ and so as $W$ and $W^T$ have the same invariant factors,

$$k/a_i = a_{n+1-i} \quad \text{for} \quad 1 \le i \le n.$$

Clearly if two $W(n,k)$ have different SNF's then they are inequivalent, so we can use this in finding inequivalent weighing matrices. For convenience, we represent the SNF $\operatorname{diag}(\underbrace{t_1,\ldots,t_1}_{a_1 \text{ times}},\underbrace{t_2,\ldots,t_2}_{a_2 \text{ times}},\ldots)$ by $\operatorname{diag}(t_1^{a_1}, t_2^{a_2}, \ldots)$.

LEMMA 9. *Suppose $A$ is a $W(n,k)$ where $k = \prod_i p_i^{a_i}$ is the prime decomposition of $k$. Let $S_j = \{i \mid a_i = j\}$ and $t_j = \prod_{i \in S_j} p_i$. So $k = t_1 t_2^2 t_3^3 \ldots t_q^q$. Then the SNF of $A$ can be obtained by taking the product of the SNF's of a $W(n, t_j^j)$ for each $j = 1, \ldots, q$. For $i = 2j$ or $2j+1$ the SNF of a $W(n, t_j^j)$ will be*

$$\operatorname{diag}(1^{a_{i,0}}, t_i^{a_{i,1}}, (t_i^2)^{a_{i,2}}, \ldots, (t_i^i)^{a_{i,i}})$$

*where (i) if $i$ is even then*

$$a_{i,s} = a_{i,i-s} \quad \text{for} \quad 0 \le s \le j-1 \quad \text{and} \quad a_{i,j} = \frac{n}{2} - \sum_{s=0}^{j-1} a_{i,s}$$

*and (ii) if $i$ is odd then*

$$a_{i,s} = a_{i,i-s} \quad \text{for} \quad 0 \le s \le j.$$

*Lemma 9 strongly restricts the theoretically possible SNFs of weighing matrices as the following corollaries suggest.*

COROLLARY 10. *If $k$ is square free then the SNF of any $W(n,k)$ is*

$$\operatorname{diag}\left(1^{n/2}, k^{n/2}\right).$$

COROLLARY 11. *The SNF of a $W(n,k)$ with $k = t_2^2$ is*

$$\operatorname{diag}\left(1^{a_{2,0}}, t_2^{n-2a_{2,0}}, (t_2^2)^{a_{2,0}}\right)$$

LEMMA 12. *If $A$ is a $W(n,k)$ with SNF $\operatorname{diag}(a_1,\ldots,a_n)$ and $B$ is a $W(m,k)$ with SNF $\operatorname{diag}(b_1,\ldots,b_m)$ then $A \oplus B$ is a $W(n+m,k)$ with*

*SNF* diag $(a_1, \ldots, a_n, b_1, \ldots, b_m)$ *(with diagonal elements reordered to be in non-decreasing order).*

PROPOSITION 13. *A* $W(4n+2i,4)$, $i \in \{0,1\}$ *with SNF* diag $(1^a, 2^{4n+2i-2a}, 4^a)$ *exists iff*

(i) $n \le a \le 2n$ *if* $n$ *is even and* $i = 0$ *and*
(ii) $n + i \le a \le 2n - 1 + i$ *otherwise.*

PROOF: First we note that

$W(4,4)$ has SNF diag $(1, 2^2, 4)$
$W(6,4)$ has SNF diag $(1^2, 2^2, 4^2)$,
$C(2i,4)$ has SNF diag $(1^{i-1}, 2^2, 4^{i-1})$ and
$B(8,4)$ has SNF diag $(1^4, 4^4)$.

Therefore $\oplus W(4,4) + C(4n+2n-4b, 4)$ has SNF diag $(1^{2n+i-1-b}, 2^{2b+2}, 4^{2n+i-1-b})$ for $0 \le b \le n-1$, and if $n$ is even and $i = 0$ then $\overset{\oplus}{_{n/2}} B(8,4)$ has SNF diag $(1^{2n}, 4^{2n})$. The proposition now follows from these observations together with Theorem 3.

COROLLARY 14. *A* $W(4n+2i+7,4)$, $i \in \{0,1\}$ *with SNF* diag $(1^a, 2^{2n+7-2a}, 4^a)$ *exists iff*

(i) $n + 3 \le a \le 2n + 3$ *if* $n$ *is even and* $i = 0$ *and*
(ii) $n + 3 + i \le a \le 2n + 2 + i$ *otherwise.*

PROOF: This follows from Proposition 13 and the fact that $W(7,4)$ has SNf diag $(1^3, 2, 4^3)$.

COROLLARY 15. *For any* $n$*, there exist at least* $n$ *inequivalent* $W(2n+2i,4)$ *and* $W(4n+2i+7,4)$ *for* $i \in \{0,1\}$.

*2.3 Other techniques.*

1. *Profile*. As in [1], we consider the absolute value of the generalized inner product of four rows $i$, $j$, $k$ and $\ell$ of a weighing matrix $W = (w_{i,j})$ to be

$$P_{i,j,k,\ell} = \sum_{x=1}^{n} |w_{i,x} w_{j,x} w_{k,x} w_{\ell,x}| .$$

Define

$$\pi(m) = |\{i,j,k,\ell \mid P_{i,j,k,\ell} = m\}|$$

to be the *profile* of the weighing matrix. Matrices with different profiles for some $m$ are inequivalent.

2. *Largest Integer* [10]. Equivalence operations are used to

(i) ensure that $w_{1,1} = 1$

(ii) rearrange the rows and columns so that each row has as many 1's to the left as possible, then −1's are given priority and finally 0's.

The matrix is then treated as an integer base 3 where 1's are evaluated as 2, −1's as 1 and 0's as 0, starting at the top left corner. For example

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{bmatrix} \quad \text{becomes} \quad 2220210220110212_3 .$$

If $A$ and $B$ give rise to different integers then they are inequivalent. We have not used this test.

3. _Codes_. Weighing matrices can be used as the basis vectors of linear codes over $GF(3)$. Two matrices giving rise to codes of different minimum distance are inequivalent. Seberry and Wehrhahn [**18**] have found two inequivalent $W(13,9)$ using this method.

4. _Character_. For $i \neq j$ write $A(i,j)$ for the set of columns of a given $W(n,k)$, $n \geq 8$ on which rows $i$ and $j$ agree in their non-zero elements. The *character* of $W$ is the largest number of distinct pairs of rows of $W$ having the same agreement sets. The character was significant in establishing the inequivalence of $W(8n, 8n)$ though is of limited use in comparing weighing matrices when $k$ is small compared to $n$. We have not used this test.

5. _Groups_. Matrices with different automorphism groups generated by the monomial matrices are equivalent.

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## APPENDIX A

## Inequivalent $W(n,k)$ for $n \leq 20$

| (n) | (k) | (#) | SNF | PROFILE |
|---|---|---|---|---|
| 2 | 1 | unique | $1^2$ | |
| | 2 | unique | $1^1 2^1$ | |
| 3 | 1 | unique | $1^3$ | |
| 4 | 1 | unique | $1^4$ | |
| | 2 | unique | $1^2 2^2$ | |
| | 3 | unique | $1^2 3^2$ | |
| | 4 | unique | $1^1 2^2 4^1$ | $\pi(4)=1$ |
| 5 | 1 | unique | $1^5$ | |
| 6 | 1 | unique | $1^6$ | |
| | 2 | unique | $1^3 2^3$ | |
| | 4 | unique | $1^2 2^2 4^2$ | $\pi(2)=3, \pi(0)=12$ |
| | 5 | unique | $1^3 5^3$ | $\pi(2)=15$ |
| 7 | 1 | unique | $1^7$ | |
| | 4 | unique | $1^3 2^1 4^3$ | $\pi(1)=7, \pi(0)=28$ |
| 8 | 1 | unique | $1^8$ | |
| | 2 | unique | $1^4 2^4$ | |
| | 3 | unique | $1^4 3^4$ | |
| 8 | 4 | 3 | $1^2 2^4 4^2$ | $\pi(4)=2, \pi(0)=68$ |
| | | | $1^3 2^2 4^3$ | $\pi(2)=4, \pi(0)=66$ |
| | | | $1^4 4^4$ | $\pi(1)=8, \pi(0)=62$ |
| | 5 | unique | $1^4 5^4$ | $\pi(4)=2, \pi(1)=32, \pi(0)=36$ |
| | 6 | unique | $1^4 6^4$ | $\pi(4)=6, \pi(2)=24, \pi(0)=40$ |
| | 7 | unique | $1^4 7^4$ | $\pi(4)=28, \pi(0)=42$ |
| | 8 | unique | $1^1 2^3 4^3 8^1$ | $\pi(8)=14, \pi(0)=56$ |
| 9 | 1 | unique | $1^9$ | |
| 10 | 1 | unique | $1^{10}$ | |
| | 2 | unique | $1^5 2^5$ | |
| | 4 | 2 | $1^3 2^4 4^3$ | $\pi(4)=1, \pi(2)=3, \pi(0)=206$ |
| | | | $1^4 2^2 4^4$ | $\pi(2)=5, \pi(0)=205$ |
| | 5 | unique | $1^5 5^5$ | $\pi(2)=5, \pi(1)=40, \pi(0)=165$ |
| | 8 | unique | $1^4 2^1 4^1 8^4$ | $\pi(6)=5, \pi(4)=20, \pi(2)=125, \pi(0)=60$ |
| | 9 | unique | $1^4 3^2 9^4$ | $\pi(6)=30, \pi(2)=180$ |
| 11 | 1 | unique | $1^{11}$ | |
| | 4 | unique | $1^4 2^3 4^4$ | $\pi(4)=1, \pi(1)=7, \pi(0)=322$ |

| (n) | (k) | (*) | SNF | PROFILE |
|---|---|---|---|---|
| 12 | 1 | unique | $1^{12}$ | |
| | 2 | unique | $1^{6}2^{6}$ | |
| | 3 | unique | $1^{6}3^{6}$ | |
| | 4 | 5 | $1^{3}2^{6}4^{3}$ | $\pi(4)=3, \pi(0)=492$ |
| | | | $1^{4}2^{4}4^{4}$ | $\pi(2)=6, \pi(0)=489$ |
| | | | $1^{4}2^{4}4^{4}$ | $\pi(4)=1, \pi(2)=4, \pi(0)=490$ |
| | | | $1^{5}2^{2}4^{5}$ | $\pi(4)=1, \pi(1)=8, \pi(0)=486$ |
| | | | $1^{5}2^{2}4^{5}$ | $\pi(2)=6, \pi(0)=489$ |
| | 5 | 2 | $1^{6}5^{6}$ | $\pi(2)=30, \pi(0)=465$ |
| | | | $1^{6}5^{6}$ | $\pi(2)=6, \pi(1)=48, \pi(0)=441$ |
| | 6 | $\geq 2$ | $1^{6}6^{6}$ | $\pi(2)=30, \pi(1)=120, \pi(0)=345$ |
| | | | $1^{6}6^{6}$ | $\pi(4)=4, \pi(2)=18, \pi(1)=104, \pi(0)=369$ |
| | 7 | $\geq 1$ | $1^{6}7^{6}$ | $\pi(4)=6, \pi(2)=66, \pi(1)=192, \pi(0)=231$ |
| | 8 | $\geq 3$ | $1^{2}2^{4}4^{4}8^{2}$ | $\pi(4)=6, \pi(2)=105, \pi(0)=384$ |
| | | | $1^{4}2^{2}4^{2}8^{4}$ | $\pi(5)=8, \pi(4)=22, \pi(2)=112, \pi(1)=128, \pi(0)=225$ |
| | | | $1^{4}2^{2}4^{2}8^{4}$ | $\pi(6)=6, \pi(4)=9, \pi(2)=192, \pi(0)=288$ |
| | 9 | $\geq 3$ | $1^{4}3^{4}9^{4}$ | $\pi(5)=24, \pi(4)=18, \pi(3)=24, \pi(2)=216, \pi(1)=48, \pi(0)=165$ |
| | | | $1^{5}3^{2}9^{5}$ | $\pi(5)=18, \pi(4)=18, \pi(3)=42, \pi(2)=216, \pi(1)=36, \pi(0)=165$ |
| | | | $1^{6}9^{6}$ | $\pi(5)=24, \pi(4)=18, \pi(3)=24, \pi(2)=216, \pi(1)=48, \pi(0)=165$ |
| | 10 | $\geq 3$ | $1^{6}10^{6}$ | $\pi(8)=3, \pi(4)=120, \pi(2)=120, \pi(0)=252$ |
| | | | $1^{6}10^{6}$ | $\pi(8)=15, \pi(4)=120, \pi(2)=120, \pi(0)=240$ |
| | | | $1^{6}10^{6}$ | $\pi(8)=3, \pi(6)=24, \pi(4)=120, \pi(2)=96, \pi(0)=252$ |
| | 11 | unique | $1^{6}11^{6}$ | $\pi(4)=330, \pi(0)=165$ |
| | 12 | unique | $1^{1}2^{5}6^{5}12^{1}$ | $\pi(4)=495$ |
| 13 | 1 | unique | $1^{13}$ | |
| | 4 | unique | $1^{5}2^{3}4^{5}$ | $\pi(2)=3, \pi(1)=7, \pi(0)=705$ |
| | 9 | $\geq 2$ | $1^{3}3^{7}9^{3}$ | $\pi(3)=234, \pi(0)=481$ |
| | | | $1^{6}3^{1}9^{6}$ | $\pi(3)=78, \pi(2)=312, \pi(1)=156, \pi(0)=169$ |
| 14 | 1 | unique | $1^{14}$ | |
| | 2 | unique | $1^{7}2^{7}$ | |
| | 4 | 6 | $1^{4}2^{6}4^{4}$ | $\pi(4)=2, \pi(2)=3, \pi(0)=996$ |
| | | | $1^{5}2^{4}4^{5}$ | $\pi(4)=1, \pi(2)=5, \pi(0)=995$ |
| | | | $1^{5}2^{4}4^{5}$ | $\pi(2)=7, \pi(0)=994$ |

| (n) | (k) | (*) | SNF | PROFILE |
|---|---|---|---|---|
| 14 | 4 | (cont) | $1^6 2^2 4^6$ | #(1)=14, #(0)=987 |
| | | | $1^6 2^2 4^6$ | #(2)=3, #(1)=8, #(0)=990 |
| | | | $1^6 2^2 4^6$ | #(2)=7, #(0)=994 |
| | 5 | 3 | $1^7 5^7$ | #(4)=2, #(2)=15, #(1)=32, #(0)=952 |
| | | | $1^7 5^7$ | #(1)=70, #(0)=931 |
| | | | $1^7 5^7$ | #(2)=7, #(1)=56, #(0)=938 |
| | 8 | ≥ 3 | $1^3 2^4 4^4 8^3$ | #(4)=21, #(1)=224, #(0)=756 |
| | | | $1^6 2^1 4^1 8^6$ | #(6)=7, #(4)=7, #(2)=231, #(0)=756 |
| | | | $1^6 2^1 4^1 8^6$ | #(4)=21, #(1)=224, #(0)=756 |
| | 9 | ≥ 2 | $1^6 3^2 9^6$ | #(3)=84, #(2)=273, #(1)=378, #(0)=266 |
| | | | $1^6 3^2 9^6$ | #(6)=3, #(6)=14, #(3)=64, #(2)=156, #(1)=384, #(0)=380 |
| | 10 | ≥ 2 | $1^7 10^7$ | #(8)=7, #(6)=21, #(5)=28, #(4)=84, #(3)=98, #(2)=[illegible], |
| | | | | #(1)=322, #(0)=182 |
| | | | $1^7 10^7$ | #(6)=21, #(4)=77, #(2)=441, #(0)=462 |
| | 13 | unique | $1^7 13^7$ | #(6)=273, #(2)=728 |
| 15 | 1 | unique | $1^{15}$ | |
| 15 | 4 | 3 | $1^5 2^5 4^5$ | #(4)=2, #(1)=7, #(0)=1356 |
| | | | $1^6 2^3 4^6$ | #(2)=4, #(1)=7, #(0)=1354 |
| | | | $1^7 2^1 4^7$ | #(1)=15, #(0)=1350 |
| | 9 | ≥ 1 | $1^7 3^1 9^7$ | #(5)=6, #(4)=12, #(3)=54, #(2)=204, #(1)=702, #(0)=387 |
| 16 | 1 | unique | $1^{16}$ | |
| | 2 | unique | $1^8 2^8$ | |
| | 3 | unique | $1^8 3^8$ | |
| | 4 | 10 | $1^4 2^8 4^4$ | #(4)=4, #(0)=1816 |
| | | | $1^5 2^6 4^5$ | #(4)=2, #(2)=4, #(0)=1814 |
| | | | $1^5 2^6 4^5$ | #(4)=1, #(2)=6, #(0)=1813 |
| | | | $1^6 2^4 4^6$ | #(4)=1, #(2)=6, #(0)=1813 |
| | | | $1^6 2^4 4^6$ | #(4)=2, #(1)=8, #(0)=1810 |
| | | | $1^6 2^4 4^6$ | #(2)=8, #(0)=1812 |
| | | | $1^6 2^4 4^6$ | #(2)=8, #(0)=1812 |
| | | | $1^7 2^2 4^7$ | #(2)=4, #(1)=8, #(0)=1808 |
| | | | $1^7 2^2 4^7$ | #(2)=8, #(0)=1812 |

| (n) | (k) | (#) | SNF | PROFILE |
|---|---|---|---|---|
| 16 | 4 | (cont) | $1^8 4^8$ | $\pi(1)=16, \pi(0)=1804$ |
| | 5 | 4 | $1^8 5^8$ | $\pi(1)=80, \pi(0)=1740$ |
| | | | $1^8 5^8$ | $\pi(4)=4, \pi(1)=64, \pi(0)=1752$ |
| | | | $1^8 5^8$ | $\pi(2)=8, \pi(1)=64, \pi(0)=1748$ |
| | | | $1^8 5^8$ | $\pi(2)=20, \pi(1)=40, \pi(0)=1760$ |
| 16 | 6 | ≥ 3 | $1^8 6^8$ | $\pi(4)=12, \pi(2)=48, \pi(0)=1760$ |
| | | | $1^8 6^8$ | $\pi(4)=4, \pi(1)=224, \pi(0)=1592$ |
| | | | $1^8 6^8$ | $\pi(4)=2, \pi(1)=232, \pi(0)=1586$ |
| | 7 | ≥ 3 | $1^8 7^8$ | $\pi(4)=8, \pi(1)=528, \pi(0)=1284$ |
| | | | $1^8 7^8$ | $\pi(4)=56, \pi(0)=1764$ |
| | | | $1^8 7^8$ | $\pi(4)=12, \pi(2)=48, \pi(1)=320, \pi(0)=1440$ |
| | 8 | ≥ 5 | $1^2 2^6 4^6 8^2$ | $\pi(8)=28, \pi(0)=1792$ |
| | | | $1^4 2^4 4^4 8^4$ | $\pi(4)=24, \pi(2)=256, \pi(0)=1540$ |
| | | | $1^5 2^3 4^3 8^5$ | $\pi(6)=8, \pi(4)=8, \pi(2)=264, \pi(0)=1540$ |
| | | | $1^8 8^8$ | $\pi(4)=56, \pi(1)=560, \pi(0)=1204$ |
| | | | $1^8 8^8$ | $\pi(4)=20, \pi(2)=96, \pi(1)=656, \pi(0)=1048$ |
| | 9 | ≥ 4 | $1^4 3^8 9^4$ | $\pi(4)=36, \pi(2)=288, \pi(1)=720, \pi(0)=776$ |
| | | | $1^8 9^8$ | $\pi(8)=28, \pi(1)=896, \pi(0)=896$ |
| | | | $1^8 9^8$ | $\pi(4)=68, \pi(2)=208, \pi(1)=576, \pi(0)=968$ |
| | | | $1^8 9^8$ | $\pi(4)=20, \pi(2)=352, \pi(1)=720, \pi(0)=728$ |
| | 10 | ≥ 5 | $1^8 10^8$ | $\pi(8)=28, \pi(4)=16, \pi(2)=544, \pi(0)=1232$ |
| | | | $1^8 10^8$ | $\pi(4)=4, \pi(1)=224, \pi(0)=1592$ |
| | | | $1^8 10^8$ | $\pi(4)=120, \pi(2)=432, \pi(1)=480, \pi(0)=788$ |
| | | | $1^8 10^8$ | $\pi(4)=8, \pi(2)=24, \pi(0)=1788$ |
| | | | $1^8 10^8$ | $\pi(4)=84, \pi(2)=696, \pi(0)=1040$ |
| 16 | 11 | ≥ 2 | $1^8 11^8$ | $\pi(8)=28, \pi(4)=104, \pi(3)=192, \pi(2)=192, \pi(1)=576,$ |
| | | | | $\pi(0)=728$ |
| | | | $1^8 11^8$ | $\pi(5)=8, \pi(4)=184, \pi(3)=136, \pi(2)=384, \pi(1)=576,$ |
| | | | | $\pi(0)=532$ |
| | 12 | ≥ 2 | $1^4 2^4 6^4 12^4$ | $\pi(12)=4, \pi(8)=72, \pi(4)=288, \pi(0)=1456$ |
| | | | $1^8 12^8$ | $\pi(8)=28, \pi(5)=64, \pi(4)=256, \pi(3)=64, \pi(2)=320, \pi(1)=400,$ |
| | | | | $\pi(0)=688$ |

| (n) | (k) | (*) | SNF | PROFILE |
|---|---|---|---|---|
| 16 | 13 | ≥ 2 | $1^8 13^8$ | #(12)=4,#(8)=72,#(6)=96,#(4)=208,#(2)=480,#(1)=192, |
| | | | | #(0)=768 |
| | | | $1^8 13^8$ | #(12)=2,#(9)=8,#(8)=42,#(7)=8,#(6)=120,#(4)=[illegible], |
| | | | | #(2)=456,#(1)=176,#(0)=774 |
| | 14 | ≥ 1 | $1^8 14^8$ | #(12)=28,#(8)=224,#(2)=336,#(0)=1232 |
| | 15 | ≥ 1 | $1^8 15^8$ | #(12)=56,#(8)=168,#(4)=504,#(0)=1092 |
| | 16 | 5(M.H) | $1^1 2^4 4^6 8^4 16^1$ | #(0)=1680,#(16)=140 |
| | | | $1^1 2^5 4^4 8^5 16^1$ | #(0)=1488,#(8)=256,#(16)=76 |
| | | | $1^1 2^6 4^2 8^6 16^1$ | #(0)=1392,#(8)=384,#(16)=44 |
| | | | $1^1 2^7 8^7 16^1$ | #(0)=1344,#(8)=448,#(16)=28 |
| | | | $1^1 2^7 8^7 16^1$ | #(0)=1344,#(8)=448,#(16)=28 |
| 17 | 1 | unique | $1^{17}$ | |
| | 4 | 2 | $1^6 2^5 4^6$ | #(4)=1,#(2)=3,#(1)=7,#(0)=2369 |
| | | | $1^7 2^3 4^7$ | #(2)=5,#(1)=7,#(0)=2368 |
| | 9 | unknown | | |
| 18 | 1 | unique | $1^{18}$ | |
| | 2 | unique | $1^9 2^9$ | |
| | 4 | 11 | $1^5 2^8 4^5$ | #(4)=3,#(2)=3,#(0)=3054 |
| | | | $1^6 2^6 4^6$ | #(4)=2,#(2)=5,#(0)=3053 |
| | | | $1^6 2^6 4^6$ | #(4)=1,#(2)=7,#(0)=3052 |
| | | | $1^6 2^6 4^6$ | #(2)=9,#(0)=3051 |
| | | | $1^7 2^4 4^7$ | #(2)=9,#(0)=3051 |
| | | | $1^7 2^4 4^7$ | #(2)=1,#(1)=74,#(0)=2985 |
| | | | $1^7 2^4 4^7$ | #(4)=1,#(2)=7,#(0)=3052 |
| | | | $1^7 2^4 4^7$ | #(2)=9,#(0)=3051 |
| | | | $1^7 2^4 4^7$ | #(4)=1,#(2)=3,#(1)=8,#(0)=3048 |
| | | | $1^8 2^2 4^8$ | #(2)=5,#(1)=8,#(0)=3047 |
| | | | $1^8 2^2 4^8$ | #(2)=9,#(0)=3051 |
| | 5 | 4 | $1^9 5^9$ | #(2)=45,#(0)=3015 |
| | | | $1^9 5^9$ | #(4)=2,#(2)=5,#(1)=72,#(0)=2981 |
| | | | $1^9 5^9$ | #(2)=9,#(1)=72,#(0)=2979 |
| | | | $1^9 5^9$ | #(2)=21,#(1)=48,#(0)=2991 |

| (n) | (k) | (#) | SWF | PROFILE |
|---|---|---|---|---|
| 18 | 8 | ≥ 2 | $1^5 2^4 4^4 8^5$ | $\pi(8)=14, \pi(6)=5, \pi(4)=20, \pi(2)=125, \pi(0)=2896$ |
| | | | $1^8 2^1 4^1 8^8$ | $\pi(6)=9, \pi(4)=9, \pi(2)=297, \pi(0)=2745$ |
| | 9 | ≥ 1 | $1^8 3^2 9^8$ | $\pi(8)=2, \pi(6)=10, \pi(4)=40, \pi(2)=176, \pi(1)=960, \pi(0)=1872$ |
| | 10 | ≥ 2 | $1^9 10^9$ | $\pi(4)=45, \pi(3)=72, \pi(2)=486, \pi(1)=1224, \pi(0)=1233$ |
| | | | $1^9 10^9$ | $\pi(6)=9, \pi(4)=63, \pi(2)=801, \pi(0)=2187$ |
| | 13 | ≥ 1 | $1^9 13^9$ | $\pi(8)=9, \pi(6)=27, \pi(5)=54, \pi(4)=306, \pi(3)=306, \pi(2)=1060,$ |
| | | | | $\pi(1)=576, \pi(0)=702$ |
| | 16 | ≥ 1 | $1^9 16^9$ | $\pi(14)=9, \pi(10)=36, \pi(8)=36, \pi(6)=612, \pi(4)=144,$ |
| | | | | $\pi(2)=1627, \pi(0)=396$ |
| | 17 | unique | $1^9 17^9$ | $\pi(6)=1224, \pi(2)=1836$ |
| 19 | 1 | unique | $1^{19}$ | |
| | 4 | 5 | $1^6 2^7 4^6$ | $\pi(4)=3, \pi(1)=7, \pi(0)=3866$ |
| | | | $1^7 2^5 4^7$ | $\pi(2)=6, \pi(1)=7, \pi(0)=3863$ |
| | | | $1^7 2^5 4^7$ | $\pi(4)=1, \pi(2)=4, \pi(1)=7, \pi(0)=3864$ |
| | | | $1^8 2^3 4^8$ | $\pi(4)=1, \pi(1)=15, \pi(0)=3860$ |
| | | | $1^8 2^3 4^8$ | $\pi(2)=6, \pi(1)=7, \pi(0)=3863$ |
| | 9 | ≥ 1 | $1^9 3^1 9^9$ | $\pi(2)=171, \pi(1)=2052, \pi(0)=1653$ |
| 20 | 1 | unique | $1^{20}$ | |
| | 2 | unique | $1^{10} 2^{10}$ | |
| | 3 | unique | $1^{10} 3^{10}$ | |
| | 4 | 10 | $1^5 2^{10} 4^5$ | $\pi(4)=5, \pi(0)=4840$ |
| | | | $1^6 2^8 4^6$ | $\pi(4)=3, \pi(2)=4, \pi(0)=4838$ |
| | | | $1^6 2^8 4^6$ | $\pi(4)=2, \pi(2)=6, \pi(0)=4837$ |
| | | | $1^7 2^6 4^7$ | $\pi(4)=3, \pi(1)=8, \pi(0)=4834$ |
| | | | $1^7 2^6 4^7$ | $\pi(4)=1, \pi(22)=8, \pi(0)=4836$ |
| | | | $1^7 2^6 4^7$ | $\pi(4)=1, \pi(2)=8, \pi(0)=4836$ |
| | | | $1^7 2^6 4^7$ | $\pi(4)=2, \pi(2)=6, \pi(0)=4837$ |
| | | | $1^7 2^6 4^7$ | $\pi(2)=6, \pi(1)=8, \pi(0)=4831$ |
| | | | $1^8 2^4 4^8$ | $\pi(4)=1, \pi(2)=8, \pi(0)=4836$ |
| | | | $1^8 2^4 4^8$ | $\pi(4)=1, \pi(2)=4, \pi(1)=8, \pi(0)=4832$ |
| | | | $1^8 2^4 4^8$ | $\pi(2)=10, \pi(0)=4835$ |
| | | | $1^8 2^4 4^8$ | $\pi(2)=10, \pi(0)=4835$ |

| (n) | (k) | (*) | SNF | PROFILE |
|---|---|---|---|---|
| 20 | 4 (cont) | | $1^8 2^4 4^8$ | #(2)=10,#(0)=4835 |
| | | | $1^8 2^4 4^8$ | #(2)=10,#(0)=4835 |
| | | | $1^8 2^4 4^8$ | #(2)=3,#(1)=14,#(0)=4828 |
| | | | $1^9 2^2 4^9$ | #(4)=1,#(1)=16,#(0)=4828 |
| | | | $1^9 2^2 4^9$ | #(2)=6,#(1)=8,#(0)=4831 |
| | | | $1^9 2^2 4^9$ | #(2)=10,#(0)=4835 |
| | 5 | 6 | $1^{10} 5^{10}$ | #(2)=10,#(1)=80,#(0)=4755 |
| | | | $1^{10} 5^{10}$ | #(4)=2,#(2)=6,#(1)=80,#(0)=4757 |
| | | | $1^{10} 5^{10}$ | #(2)=15,#(1)=70,#(0)=4760 |
| | | | $1^{10} 5^{10}$ | #(4)=1,#(2)=6,#(1)=80,#(0)=4758 |
| | | | $1^{10} 5^{10}$ | #(2)=10,#(1)=80,#(0)=4755 |
| | | | $1^{10} 5^{10}$ | #(2)=22,#(1)=56,#(0)=4767 |
| | 6 | ≥ 1 | $1^{10} 6^{10}$ | #(4)=10,#(2)=42,#(1)=104,#(0)=4689 |
| | | | $1^{10} 6^{10}$ | #(4)=6,#(2)=54,#(1)=120,#(0)=4665 |
| | | | $1^{10} 6^{10}$ | #(2)=10,#(1)=280,#(0)=4555 |
| | | | $1^{10} 6^{10}$ | #(4)=4,#(2)=22,#(1)=216,#(0)=4603 |
| | 7 | ≥ 2 | $1^{10} 7^{10}$ | #(4)=6,#(2)=22,#(1)=608,#(0)=4209 |
| | | | $1^{10} 7^{10}$ | #(4)=34,#(3)=2,#(2)=72,#(1)=220,#(0)=4517 |
| | 8 | ≥ 5 | $1^3 2^7 4^7 8^3$ | #(8)=17,#(4)=60,#(0)=4768 |
| | | | $1^5 2^5 4^5 8^5$ | #(8)=14,#(5)=8,#(4)=22,#(2)=112,#(1)=128,#(0)=4561 |
| | | | $1^5 2^5 4^5 8^5$ | #(8)=14,#(6)=4,#(4)=10,#(2)=179,#(0)=4638 |
| | | | $1^8 2^2 4^2 8^8$ | #(6)=10,#(4)=40,#(2)=250,#(0)=4545 |
| | | | $1^8 2^2 4^2 8^8$ | #(5)=8,#(4)=22,#(2)=144,#(1)=624,#(0)=4047 |
| | 9 | ≥ 7 | $1^8 3^4 9^8$ | #(6)=60,#(2)=360,#(0)=4425 |
| | | | $1^8 3^4 9^8$ | #(2)=420,#(1)=1280,#(0)=3146 |
| | | | $1^8 3^4 9^8$ | #(2)=166,#(1)=2092,#(0)=2587 |
| | | | $1^{10} 9^{10}$ | #(2)=166,#(1)=2092,#(0)=2587 |
| | | | $1^{10} 9^{10}$ | #(6)=10,#(4)=40,#(2)=250,#(1)=1120,#(0)=3425 |
| | | | $1^{10} 9^{10}$ | #(5)=8,#(4)=30,#(3)=8,#(2)=172,#(1)=1528,#(0)=3099 |
| | | | $1^{10} 9^{10}$ | #(4)=28,#(3)=24,#(2)=232,#(1)=1376,#(0)=3185 |
| | 10 | ≥ 2 | $1^{10} 10^{10}$ | #(6)=60,#(2)=360,#(1)=1680,#(0)=2745 |
| | | | $1^{10} 10^{10}$ | #(5)=8,#(4)=52,#(3)=20,#(2)=576,#(1)=1396,#(0)=2773 |
| | 11 | ≥ 1 | $1^{10} 11^{10}$ | #(6)=60,#(4)=20,#(2)=1040,#(1)=1260,#(0)=2445 |

| (n) | (k) | (*) | SNF | PROFILE |
|---|---|---|---|---|
| 20 | 12 | ≥ 4 | $1^{8}2^{2}6^{2}12^{8}$ | $\pi(8)=1,\pi(6)=19,\pi(5)=28,\pi(4)=126,\pi(3)=164,$ |
| | | | | $\pi(2)=1343,\pi(1)=1336,\pi(0)=1828$ |
| | | | $1^{9}2^{1}6^{1}12^{9}$ | $\pi(6)=10,\pi(5)=28,\pi(4)=133,\pi(3)=216,\pi(2)=1314,$ |
| | | | | $\pi(1)=1244,\pi(0)=1900$ |
| | | | $1^{10}12^{10}$ | $\pi(6)=8,\pi(5)=28,\pi(4)=144,\pi(3)=220,\pi(2)=1280,$ |
| | | | | $\pi(1)=1240,\pi(0)=1925$ |
| | | | $1^{9}2^{1}6^{1}12^{9}$ | $\pi(8)=1,\pi(6)=18,\pi(5)=32,\pi(4)=199,\pi(3)=156,\pi(2)=1078,$ |
| | | | | $\pi(1)=1340,\pi(0)=2021$ |
| | 13 | ≥ 1 | $1^{10}13^{10}$ | $\pi(8)=1,\pi(6)=21,\pi(5)=41,\pi(4)=280,\pi(3)=399,\pi(2)=1317,$ |
| | | | | $\pi(1)=1464,\pi(0)=1313$ |
| | 14 | ≥ 3 | $1^{10}14^{10}$ | $\pi(8)=4,\pi(7)=3,\pi(6)=78,\pi(5)=62,\pi(4)=481,\pi(3)=450,$ |
| | | | | $\pi(2)=1488,\pi(1)=1109,\pi(0)=1170$ |
| | | | $1^{10}14^{10}$ | $\pi(8)=4,\pi(7)=6,\pi(6)=84,\pi(5)=52,\pi(4)=488,\pi(3)=464,$ |
| | | | | $\pi(2)=1402,\pi(1)=1102,\pi(0)=1243$ |
| | | | $1^{10}14^{10}$ | $\pi(8)=4,\pi(7)=6,\pi(6)=84,\pi(5)=72,\pi(4)=416,\pi(3)=456,$ |
| | | | | $\pi(2)=1466,\pi(1)=1090,\pi(0)=1221$ |
| | 15 | ≥ 3 | $1^{10}15^{10}$ | $\pi(9)=2,\pi(8)=26,\pi(7)=22,\pi(6)=102,\pi(5)=158,\pi(4)=694,$ |
| | | | | $\pi(3)=386,\pi(2)=1576,\pi(1)=712,\pi(0)=1167$ |
| | | | $1^{10}15^{10}$ | $\pi(9)=2,\pi(8)=18,\pi(7)=22,\pi(6)=106,\pi(5)=178,\pi(4)=678,$ |
| | | | | $\pi(3)=382,\pi(2)=1620,\pi(1)=696,\pi(0)=1143$ |
| | | | $1^{10}15^{10}$ | $\pi(12)=1,\pi(10)=3,\pi(19)=2,\pi(8)=24,\pi(7)=34,\pi(6)=113,$ |
| | | | | $\pi(5)=137,\pi(4)=659,\pi(3)=409,\pi(2)=1474,\pi(1)=698,$ |
| | | | | $\pi(0)=1291$ |
| | 16 | ≥ 1 | $1^{8}2^{1}4^{2}8^{1}16^{8}$ | $\pi(12)=8,\pi(9)=24,\pi(8)=42,\pi(7)=52,\pi(6)=240,\pi(5)=180,$ |
| | | | | $\pi(4)=828,\pi(3)=204,\pi(2)=1488,\pi(1)=324,\pi(0)=1455$ |
| 20 | 17 | ≥ 2 | $1^{10}17^{10}$ | $\pi(11)=4,\pi(10)=12,\pi(9)=2,\pi(8)=126,\pi(7)=12,\pi(6)=352,$ |
| | | | | $\pi(5)=94,\pi(4)=1522,\pi(3)=142,\pi(2)=1196,\pi(1)=66,$ |
| | | | | $\pi(0)=1317$ |
| | | | $1^{10}17^{10}$ | $\pi(12)=8,\pi(10)=8,\pi(9)=12,\pi(8)=106,\pi(7)=28,\pi(6)=416,$ |
| | | | | $\pi(5)=96,\pi(4)=1364,\pi(3)=64,\pi(2)=1256,\pi(1)=120,$ |
| | | | | $\pi(0)=1367$ |
| | 18 | ≥ 3 | $1^{10}18^{10}$ | $\pi(12)=16,\pi(10)=16,\pi(8)=388,\pi(6)=280,\pi(4)=1968,$ |

| (n) | (k) | (*) | SNF | PROFILE |
|---|---|---|---|---|
| 20 | 18 | (cont) | | π(2)=424,π(0)=1753 |
| | | | $1^{10}18^{10}$ | π(10)=40,π(8)=300,π(6)=160,π(4)=2560,π(2)=520, |
| | | | | π(0)=1265 |
| | | | $1^{10}18^{10}$ | π(12)=16,π(10)=24,π(8)=361,π(6)=200,π(4)=2188, |
| | | | | π(2)=496,π(0)=1560 |
| | 19 | ≥ 1 | $1^{10}19^{10}$ | π(12)=90,π(8)=390,π(4)=3330,π(0)=1035 |
| | 20 | 3(N.N) | $1^{1}2^{9}10^{9}20^{1}$ | π(4)=4560,π(12)=265 |

## APPENDIX B

## Inequivalent $W(n,4)$ and $W(n,5)$, $n \le 20$

($+_i W(n,k)$ refers to the direct sum of i copies of $W(n,k)$)

(i) k = 4

| Type | order | number | decomposition | SNF |
|---|---|---|---|---|
| B1 | 4 | 1 | | $1^1 2^2 4^1$ |
| B2 | 5 | 0 | | |
| B3 | 6 | 1 | | $1^2 2^2 4^2$ |
| B4 | 7 | 1 | | $1^3 2^1 4^3$ |
| B5 | 8 | 3 | $+_2W(4,4)$ | $1^2 2^4 4^2$ |
| B6 | | | $C(8,4)$ | $1^3 2^2 4^3$ |
| B7 | | | $B(8,4)$ | $1^4 4^4$ |
| B8 | 9 | 0 | | |
| B9 | 10 | 2 | $W(6,4) + W(4,4)$ | $1^3 2^4 4^3$ |
| B10 | | | $C(10,4)$ | $1^4 2^2 4^4$ |
| B11 | 11 | 1 | $W(7,4) + W(4,4)$ | $1^4 2^3 4^4$ |
| B12 | 12 | 5 | $+_3W(4,4)$ | $1^3 2^6 4^3$ |
| B13 | | | $C(8,4) + W(4,4)$ | $1^4 2^4 4^4$ |
| B14 | | | $+_2W(6,4)$ | $1^4 2^4 4^4$ |
| B15 | | | $C(12,4)$ | $1^5 2^2 4^5$ |
| B16 | | | $B(8,4) + W(4,4)$ | $1^5 2^2 4^5$ |
| B17 | 13 | 1 | $W(7,4) + W(6,4)$ | $1^5 2^3 4^5$ |
| B18 | 14 | 6 | $W(6,4) +_2 W(4,4)$ | $1^4 2^6 4^4$ |
| B20 | | | $B(8,4) + W(6,4)$ | $1^6 2^2 4^6$ |
| B21 | | | $C(8,4) + W(6,4)$ | $1^5 2^4 4^5$ |
| | | | $C(10,4) + W(4,4)$ | $1^5 2^4 4^5$ |
| B22 | | | $+_2W(7,4)$ | $1^6 2^2 4^6$ |
| B23 | | | $C(14,4)$ | $1^6 2^2 4^6$ |
| B24 | 15 | 3 | $W(7,4) +_2 W(4,4)$ | $1^5 2^5 4^5$ |
| B25 | | | $C(8,4) + W(7,4)$ | $1^6 2^3 4^6$ |
| B26 | | | $B(8,4) + W(7,4)$ | $1^7 2^1 4^7$ |
| B27 | 16 | 10 | $+_4W(4,4)$ | $1^4 2^8 4^4$ |
| B28 | | | $C(8,4) +_2 W(4,4)$ | $1^5 2^6 4^5$ |
| B29 | | | $+_2W(6,4) + W(4,4)$ | $1^5 2^6 4^5$ |
| B30 | | | $B(8,4) +_2 W(4,4)$ | $1^6 2^4 4^6$ |
| B31 | | | $+_2C(8,4)$ | $1^6 2^4 4^6$ |
| B32 | | | $C(10,4) + W(6,4)$ | $1^6 2^4 4^6$ |

| Type | Order | Number | Decomposition | SNF |
|---|---|---|---|---|
| B33 | | | C(12,4) + W(4,4) | $1^6 2^4 4^6$ |
| B34 | | | C(8,4) + B(8,4) | $1^7 2^2 4^7$ |
| B35 | | | C(16,4) | $1^7 2^2 4^7$ |
| B36 | | | $+_2$B(8,4) | $1^8 4^8$ |
| B37 | 17 | 2 | W(7,4) + W(6,4) + W(4,4) | $1^6 2^5 4^6$ |
| B38 | | | C(10,4) + W(7,4) | $1^7 2^3 4^7$ |
| B39 | 18 | 11 | W(6,4) $+_3$ W(4,4) | $1^5 2^8 4^5$ |
| B40 | | | C(10,4) $+_2$ W(4,4) | $1^6 2^6 4^6$ |
| B41 | | | C(8,4) + W(6,4) + W(4,4) | $1^6 2^6 4^6$ |
| B42 | | | $+_3$W(6,4) | $1^6 2^6 4^6$ |
| B43 | | | C(14,4) + W(4,4) | $1^7 2^4 4^7$ |
| B44 | | | C(12,4) + W(6,4) | $1^7 2^4 4^7$ |
| B45 | | | C(10,4) + C(8,4) | $1^7 2^4 4^7$ |
| B46 | | | B(8,4) + W(6,4) + W(4,4) | $1^7 2^4 4^7$ |
| B47 | | | $+_2$W(7,4) + W(4,4) | $1^7 2^4 4^7$ |
| B48 | | | C(18,4) | $1^8 2^2 4^8$ |
| B49 | | | C(10,4) + B(8,4) | $1^8 2^2 4^8$ |
| B50 | 19 | 5 | W(7,4) $+_3$ W(4,4) | $1^6 2^7 4^6$ |
| B51 | | | W(7,4) + C(8,4) + W(4,4) | $1^7 2^5 4^7$ |
| B52 | | | W(7,4) $+_2$ W(6,4) | $1^7 2^5 4^7$ |
| B53 | | | W(7,4) + C(12,4) | $1^8 2^3 4^8$ |
| B54 | | | W(7,4) + B(8,4) + W(4,4) | $1^8 2^3 4^8$ |
| B55 | 20 | 18 | $+_5$W(4,4) | $1^5 2^{10} 4^5$ |
| B56 | | | $+_2$W(6,4) $+_2$W(4,4) | $1^6 2^8 4^6$ |
| B57 | | | C(8,4) $+_3$ W(4,4) | $1^6 2^8 4^6$ |
| B58 | | | B(8,4) $+_3$ W(4,4) | $1^7 2^6 4^7$ |
| B59 | | | C(8,4) $+_2$ W(6,4) | $1^7 2^6 4^7$ |
| B60 | | | $+_2$C(8,4) + W(4,4) | $1^7 2^6 4^7$ |
| B61 | | | C(10,4) + W(6,4) + W(4,4) | $1^7 2^6 4^7$ |
| B62 | | | C(12,4) $+_2$ W(4,4) | $1^7 2^6 4^7$ |
| B63 | | | $+_2$W(7,6) + W(6,4) | $1^8 2^4 4^8$ |
| B64 | | | B(8,4) $+_2$ W(6,4) | $1^8 2^4 4^8$ |
| B65 | | | C(8,4) + B(8,4) + W(4,4) | $1^8 2^4 4^8$ |
| B66 | | | $+_2$ C(10,4) | $1^8 2^4 4^8$ |
| B67 | | | C(12,4) + C(8,4) | $1^8 2^4 4^8$ |
| B68 | | | C(14,4) + W(6,4) | $1^8 2^4 4^8$ |
| B69 | | | C(16,4) + W(4,4) | $1^8 2^4 4^8$ |

| Type | Order | Number | Decomposition | SNF |
|---|---|---|---|---|
| B70 | | | $+_2B(8,4) + W(4,4)$ | $1^9 2^2 4^9$ |
| B71 | | | $C(12,4) + B(8,4)$ | $1^9 2^2 4^9$ |
| B72 | | | $C(20,4)$ | $1^9 2^2 4^9$ |
| | 21 | 6 | | |
| | 22 | 22 | | |
| | 23 | 9 | | |
| | 24 | 33 | | |

(ii) k = 5 (SNF is always $1^{\frac{1}{2}n}5^{\frac{1}{2}n}$)

| Type | Order | Number | Decomposition |
|---|---|---|---|
| B73 | 6 | 1 | W(6,5) |
| B74 | 8 | 1 | W(8,5) |
| B75 | 10 | 1 | E(10,5) |
| B76 | 12 | 2 | F(12,5) |
| B77 | | | $+_2W(6,5)$ |
| B78 | 14 | 3 | D(14,5) |
| B79 | | | E(14,5) |
| B80 | | | W(8,5) + W(6,5) |
| B81 | 16 | 4 | D(16,5) |
| B82 | | | F(16,5) |
| B83 | | | E(10,5) + W(6,5) |
| B84 | | | $+_2W(8,5)$ |
| B85 | 18 | 4 | E(18,5) |
| B86 | | | F(12,5) + W(6,5) |
| B87 | | | E(10,5) + W(8,5) |
| B88 | | | $+_3W(6,5)$ |
| B89 | 20 | 6 | F(20,5) |
| B90 | | | E(14,5) + W(6,5) |
| B91 | | | F(12,5) + W(8,5) |
| B92 | | | $+_2E(10,5)$ |
| B93 | | | $W(8,5 +_2 W(6,5)$ |
| B94 | | | D(14,5) + W(6,5) |

## APPENDIX C

### Substructures used to form weighing matrices

| Type | Form | Comment |
|---|---|---|
| C1 | $A$ | may be circulant. |
| C2 | $\begin{matrix} A & B \\ -B & A \end{matrix}$ | A and B must have same order,<br>A and B may be equivalent or circulant. |
| C3 | $\begin{matrix} A & B \\ B^T & -A^T \end{matrix}$ | A and B both must be circulant,<br>A and B may be equivalent,<br>$X^T$ : the transpose of X. |
| C4 | $\begin{matrix} A & I \\ I & -A^T \end{matrix}$ | A may be circulant,<br>I : identity matrix. |
| C5 | $\begin{matrix} A & 0 \\ 0 & B \end{matrix}$ | A and B may have different order,<br>A and B may be equivalent or circulant. |
| C6 | $\begin{matrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{matrix}$ | A, B, C and D all must have same order,<br>A, B, C and D may be circulant,<br>A, B, C and D may be equivalent. |
| C7 | $\begin{matrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{matrix}$ | A, B, C and D all must be circulant,<br>A, B, C and D may be equivalent,<br>XR : the back circulant of X,<br>$X^T$ : the transpose of X. |
| C8 | $\begin{matrix} A & A & I & B \\ A & -A & -B & I \\ I & B & A & A \\ -B & I & A & -A \end{matrix}$ | A : skew symmetric,<br>i.e., $A^T = -A$<br>B is either a 0 matrix or an identix<br>matrix. |

## APPENDIX D

### Weighing matrices of order 16

To make substantial progress in constructing designs of order 16 we first consider those with considerable structure, such as those used in constructing orthogonal designs (see Geramita and Seberry (1979) Sections 5.8 and 5.9).

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ - & 0 & - & 1 \\ - & 1 & 0 & - \\ - & - & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - \\ - & - & 0 & 0 \\ - & 1 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & - & 0 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - \end{bmatrix} \quad d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\beta = \begin{bmatrix} 0 & 1 & 1 & 1 \\ - & 0 & - & 1 \\ - & 1 & 0 & - \\ - & - & 1 & 0 \end{bmatrix} \quad \delta = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - \\ - & - & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & - & 0 \end{bmatrix}$$

We are now in a position to construct weighing matrices of order 16, using the Williamson array

$$\begin{bmatrix} X & Y & Z & W \\ -Y & X & W & -Z \\ -Z & -W & X & Y \\ -W & Z & -Y & X \end{bmatrix}$$

because $AB^T = BA^T$; $AC^T = CA^T$; $AD^T = DA^T$; $BC^T = CB^T$; $BD^T = DB^T$; but $CD^T = -DC^T$

Thus we can obtain weighing matrices of order 16 and weight $k$ by making the following choices for $X, Y, Z, W$.

Some results on order 16 and weight $k$ were tested with various combinations of the matrices $A$, $B$, $C$, $D$, $O$, $a$, $c$, $d$, $\beta$, $\delta$, $\mu$: those that gave examples of equivalence classes are listed.

| k # | XYZW Intersection | SHF | Profile | |
|---|---|---|---|---|
| 4 ≥2 | ADOO (12,0,3) | $1^4 2^8 4^4$ | $\pi(4)=4, \pi(0)=1816$ | |
| | BDOO (9,6,0) | $1^8 4^8$ | $\pi(1)=16, \pi(0)=1804$ | |
| | CCDO (12,0,3) | $1^4 2^8 4^4$ | $\pi(4)=4, \pi(0)=1816$ | |
| | CDDO (9,6,0) | $1^8 4^8$ | $\pi(1)=16, \pi(0)=1804$ | |
| | DDDO (12,0,3) | $1^4 2^8 4^4$ | $\pi(4)=4, \pi(0)=1816$ | |
| 5 ≥2 | ADOO (8,4,3) | $1^8 5^8$ | $\pi(4)=4, \pi(1)=64, \pi(0)=1752$ | |
| | BCDO (8,4,3) | $1^8 5^8$ | $\pi(4)=4, \pi(1)=64, \pi(0)=1752$ | |
| | BODO (5,10,0) | $1^8 5^8$ | $\pi(1)=80, \pi(0)=1740$ | |
| | CCDO (8,4,3) | $1^8 5^8$ | $\pi(4)=4, \pi(1)=64, \pi(0)=1752$ | |
| | CDDO (5,10,0) | $1^8 5^8$ | $\pi(1)=80, \pi(0)=1740$ | |
| 6 ≥3 | ACDO (8,0,6,1) | $1^8 6^8$ | $\pi(4)=12, \pi(2)=48, \pi(0)=1760$ | |
| | ADDO (3,9,3,0) | $1^8 6^8$ | $\pi(4)=4, \pi(1)=224, \pi(0)=1592$ | |
| | BBDO (8,0,6,1) | $1^8 6^8$ | $\pi(4)=12, \pi(2)=48, \pi(0)=1760$ | |
| | BOOμ (0,15,0,0) | $1^8 6^8$ | $\pi(4)=2, \pi(1)=232, \pi(0)=1586$ | * |
| | CCCO (8,0,6,1) | $1^8 6^8$ | $\pi(4)=12, \pi(2)=48, \pi(0)=1760$ | |
| | CCDO (8,0,6,1) | $1^8 6^8$ | $\pi(4)=12, \pi(2)=48, \pi(0)=1760$ | |
| | βδμO (3,9,3,0) | $1^8 6^8$ | $\pi(4)=4, \pi(1)=224, \pi(0)=1592$ | |
| 7 ≥2 | ABOO (8,0,0,7) | $1^8 7^8$ | $\pi(4)=56, \pi(0)=1764$ | |
| | ACDO (2,6,6,1) | $1^8 7^8$ | $\pi(4)=12, \pi(2)=48, \pi(1)=320, \pi(0)=1440$ | |

*This is not an SBIBD (16, 6, 2), as only one particular intersection pattern is considered.

| k * | XYZW Intersection | SNF | Profile |
|---|---|---|---|
| | BBDD (2,6,6,1) | $1^8 7^8$ | π(4)=12,π(2)=48,π(1)=320,π(0)=1440 |
| | BCCD (2,6,6,1) | $1^8 7^8$ | π(4)=12,π(2)=48,π(1)=320,π(0)=1440 |
| | B∂μμ (2,6,6,1) | $1^8 7^8$ | π(4)=12,π(2)=48,π(1)=320,π(0)=1440 |
| 8 ≥4 | AADD (8,0,0,0,7) | $1^2 2^6 4^6 8^2$ | π(8)=28,π(0)=1792 |
| | ABDD (1,7,0,7,0) | $1^8 8^8$ | π(4)=56,π(1)=560,π(0)=1204 |
| | ACCD (2,0,12,0,1) | $1^4 2^4 4^4 8^4$ | π(4)=24,π(2)=256,π(0)=1540 |
| | ACDd (0,4,9,2,0) | $1^8 8^8$ | π(4)=20,π(2)=96,π(1)=656,π(0)=1048 |
| | BBCD (0,4,9,2,0) | $1^8 8^8$ | π(4)=20,π(2)=96,π(1)=656,π(0)=1048 |
| | BBDD (2,0,12,0,1) | $1^4 2^4 4^4 8^4$ | π(4)=24,π(2)=256,π(0)=1540 |
| | CCCC (8,0,0,0,7) | $1^2 2^6 4^6 8^2$ | π(8)=28,π(0)=1792 |
| | B∂∂μ (0,4,9,2,0) | $1^8 8^8$ | π(4)=20,π(2)=96,π(1)=656,π(0)=1048 |
| 9 ≥4 | AADD (0,8,0,0,7) | $1^8 9^8$ | π(8)=28,π(1)=896,π(0)=896 |
| | ABCD (0,2,6,6,1) | $1^8 9^8$ | π(4)=68,π(2)=208,π(1)=576,π(0)=968 |
| | ABDD (0,2,6,6,1) | $1^8 9^8$ | π(4)=68,π(2)=208,π(1)=576,π(0)=968 |
| | ACCd (0,2,6,6,1) | $1^8 9^8$ | π(4)=68,π(2)=208,π(1)=576,π(0)=968 |
| | BBBD (0,0,9,6,0) | $1^4 3^8 9^4$ | π(4)=36,π(2)=288,π(1)=720,π(0)=776 |
| | BCCC (0,8,0,0,7) | $1^8 9^8$ | π(8)=28,π(1)=896,π(0)=896 |
| | BB∂μ (0,2,6,6,1) | $1^8 9^8$ | π(4)=20,π(2)=352,π(1)=720,π(0)=728 |
| 10 ≥2 | AACD (0,0,8,0,6,1) | $1^8 10^8$ | π(8)=28,π(4)=16,π(2)=544,π(0)=1232 |
| | AADD (0,0,8,0,6,1) | $1^8 10^8$ | π(8)=28,π(4)=16,π(2)=544,π(0)=1232 |
| | ABBD (0,0,3,9,3,0) | $1^8 10^8$ | π(4)=120,π(2)=432,π(1)=480,π(0)=768 |
| | ACCC (0,0,8,0,6,1) | $1^8 10^8$ | π(8)=28,π(4)=16,π(2)=544,π(0)=1232 |
| | BBBD (0,0,3,9,3,0) | $1^8 10^8$ | π(4)=120,π(2)=432,π(1)=480,π(0)=768 |
| | BBCC (0,0,8,0,6,1) | $1^8 10^8$ | π(8)=28,π(4)=16,π(2)=544,π(0)=1232 |
| 11 ≥2 | AABD (0,0,0,8,4,3) | $1^8 11^8$ | π(8)=28,π(4)=104,π(3)=192,π(2)=192, |
| | | | π(1)=576,π(0)=728 |
| | AACD (0,0,0,8,4,3) | $1^8 11^8$ | π(8)=28,π(4)=104,π(3)=192,π(2)=192, |
| | | | π(1)=576,π(0)=728 |
| | AADD (0,0,0,5,10,0) | $1^8 11^8$ | π(5)=8,π(4)=184,π(3)=136,π(2)=384, |
| | | | π(1)=576,π(0)=532 |
| | ABCC (0,0,0,8,4,3) | $1^8 11^8$ | π(8)=28,π(4)=104,π(3)=192,π(2)=192, |
| | | | π(1)=576,π(0)=728 |
| | BBBC (0,0,0,8,4,3) | $1^8 11^8$ | π(8)=28,π(4)=104,π(3)=192,π(2)=192, |
| | | | π(1)=576,π(0)=728 |
| 12 ≥2 | AAAD (0,0,0,0,12,0,3) | $1^4 2^4 6^4 12^4$ | π(12)=4,π(8)=72,π(4)=288,π(0)=1456 |

| k | # | XYZW intersection | SNF | Profile |
|---|---|---|---|---|
| | | AABD (0,0,0,0,9,6) | $1^8 12^8$ | #(8)=28,#(5)=64,#(4)=256,#(3)=64, |
| | | | | #(2)=320,#(1)=400,#(0)=688 |
| | | AACC (0,0,0,0,12,0,3) | $1^4 2^4 6^4 12^4$ | #(12)=4,#(8)=72,#(4)=288,#(0)=1456 |
| | | ABBC (0,0,0,0,9,6) | $1^8 12^8$ | #(8)=28,#(5)=64,#(4)=256,#(3)=64, |
| | | | | #(2)=320,#(1)=400,#(0)=688 |
| | | BBBB (0,0,0,0,12,0,3) | $1^4 2^4 6^4 12^4$ | #(12)=4,#(8)=72,#(4)=288,#(0)=[illegible] |
| 13 | ≥2 | AAAD (0,0,0,0,0,12,3,0) | $1^8 13^8$ | #(12)=4,#(8)=72,#(6)=96,#(4)=208, |
| | | | | #(2)=480,#(1)=192,#(0)=768 |
| | | AABC (0,0,0,0,0,12,3,0) | $1^8 13^8$ | #(12)=2,#(9)=8,#(8)=42,#(7)=8, |
| | | | | #(6)=120,#(4)=234,#(2)=456,#(1)=176, |
| | | | | #(0)=774 |
| | | ABBB (0,0,0,0,0,12,3,0) | $1^8 13^8$ | #(12)=2,#(9)=8,#(8)=42,#(7)=8, |
| | | | | #(6)=120,#(4)=234,#(2)=456,#(1)=176, |
| | | | | #(0)=774 |
| 14 | ≥1 | AAAC (0,0,0,0,0,0,14,1) | $1^8 13^8$ | #(12)=28,#(8)=224,#(2)=336,#(0)=1232 |
| | | AABB (0,0,0,0,0,0,14,1) | $1^8 13^8$ | #(12)=28,#(8)=224,#(2)=336,#(0)=1232 |
| 15 | ≥1 | AAAB | $1^8 15^8$ | #(12)=56,#(8)=168,#(4)=504,#(0)=1092 |
| 16 | ≥1 | AAAA | $1^1 2^4 4^6 8^4 16^1$ | #(16)=140,#(0)=1680 |

## APPENDIX E

## $W(n,9)$ for $n \leq 20$

We give some known $W(n,9)$ designs of small order. Where the design is constructed from circulant matrices and not otherwise indicated the weighing matrix has structure

$$\begin{bmatrix} A & B \\ B^T & -A^T \end{bmatrix}$$

$$W(10,9) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & - & 1 & - & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & 1 & 0 & 1 & - & - & - & 1 & - \\ 1 & - & - & 1 & 0 & 1 & 1 & - & 1 & - \\ 1 & 1 & - & - & 1 & 0 & 1 & - & - & 1 \\ 1 & - & 1 & - & 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & - & - & - & 1 & 0 & 1 & 1 \\ 1 & - & - & 1 & 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & - & 1 & - & 1 & 1 & 0 \end{bmatrix}$$

$W(11,9)$ does not exist.

$$W(12,9) = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&0&0&0 \\ 1&1&1&1&-&-&-&-&0&1&0&0 \\ 1&1&-&-&1&-&1&-&0&0&1&0 \\ 1&-&-&1&-&-&1&1&0&0&0&1 \\ 1&-&1&-&0&0&1&0&-&1&-&- \\ 1&1&0&-&0&0&-&1&-&-&-&1 \\ 1&-&0&1&1&0&-&0&-&-&1&- \\ 1&-&0&-&0&1&-&0&1&1&1&1 \\ 1&0&-&0&-&1&0&-&1&-&-&- \\ 0&1&-&0&-&1&0&1&-&1&1&- \\ 0&0&1&-&-&-&0&1&1&-&1&- \\ 0&0&1&0&-&1&1&-&-&-&1&1 \end{bmatrix} \qquad W(12,9) = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&0&0&0 \\ 1&1&1&1&-&-&-&-&0&1&0&0 \\ 1&1&-&-&1&-&1&-&0&0&1&0 \\ 1&-&-&1&-&-&1&1&0&0&0&1 \\ 1&-&1&-&-&0&0&0&1&-&1&- \\ 1&1&0&-&-&1&0&0&-&-&-&1 \\ 1&-&0&1&1&0&0&-&-&-&-&- \\ 1&-&0&-&1&0&-&0&1&1&-&1 \\ 1&0&-&0&0&1&-&1&-&1&1&- \\ 0&1&-&0&0&-&-&1&1&-&-&- \\ 0&0&1&-&0&-&1&1&-&1&-&- \\ 0&0&1&0&1&-&-&1&-&-&1&1 \end{bmatrix}$$

$W(13,9)$: there are two known circulant matrices. They have first rows:

$$\begin{matrix} 0&1&0&1&1&0&0&-&-&1&1&-&1 \end{matrix}$$

and

$$\begin{matrix} 0&-&0&-&1&0&0&1&1&-&1&1&1 \end{matrix}$$

Hain (1977) proved these were inequivalent by showing there was no group operation to send one first row to the other while preserving the positions of the zeros. Seberry and Wehrhahn (1978) showed that the linear codes over GF(3) generated by these two weighing matrices have different minimum distances and so the weighing matrices are different.

$W(14,9)$: uses the circulant matrices with first rows:

$$\begin{matrix} 0&1&1&-&1&-&- \end{matrix}$$

and

$$\begin{matrix} 0&1&1&0&1&0&0 \end{matrix}$$

in the structure indicated above.

The following symmetric $W(14,9)$ is from Geramita and Seberry p.331.

$$W(14,9) = \begin{bmatrix} 0&-&-&1&0&0&0&0&1&-&1&1&1&- \\ -&0&-&1&0&0&0&0&-&1&1&1&-&1 \\ -&-&0&1&0&0&0&0&1&-&-&-&-&1 \\ 1&1&1&0&0&0&0&0&1&-&1&1&-&1 \\ 0&0&0&0&0&1&1&-&1&1&1&-&-&- \\ 0&0&0&0&1&0&1&-&-&-&-&1&-&- \\ 0&0&0&0&1&1&0&-&-&-&1&-&1&1 \\ 0&0&0&0&-&-&-&0&-&-&1&-&-&- \\ 1&-&1&1&1&-&-&-&0&1&0&0&0&0 \\ -&1&-&-&1&-&-&-&1&0&0&0&0&0 \\ 1&1&-&1&1&-&1&1&0&0&0&-&0&0 \\ 1&1&-&1&-&1&-&-&0&0&-&0&0&0 \\ 1&-&-&-&-&-&1&-&0&0&0&0&0&1 \\ -&1&1&1&-&-&1&-&0&0&0&0&1&0 \end{bmatrix}$$

The intersection patterns of these two matrices are (0,0,3,10,0) and (0,0,6,4,3) showing they are inequivalent.

$$W(15,9) = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & - & - & - \\
0 & 0 & 0 & 1 & 1 & 1 & - & - & - & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & - & 1 & 0 & 0 & 0 & 0 & 1 & - & 0 & - & 1 \\
1 & 1 & 1 & - & 1 & 0 & 0 & 0 & 0 & 1 & - & 0 & - & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & - & 0 & 0 & 0 & - & 0 & 1 & 1 & 0 & - \\
0 & - & 1 & 0 & 0 & 0 & 1 & 0 & 0 & - & 1 & 1 & - & 1 & 1 \\
0 & - & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & - & 1 & 1 & - \\
0 & - & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & - & 1 & 1 & - & 1 \\
1 & 0 & - & 0 & 1 & - & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & - & 1 & - & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & - & - & 0 & 1 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
- & 1 & 0 & 0 & - & 1 & - & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
- & 1 & 0 & - & 1 & 0 & 1 & 1 & - & 0 & 1 & 0 & 1 & 0 & 0 \\
- & 1 & 0 & 1 & 0 & - & 1 & - & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

W(16,9); these are treated in Appendix D.

W(17,9): this is not known.

W(18,9): from Geramita and Seberry P.333

$$W(18,9) = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & - & - & 1 & 1 & - & - & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\
1 & 1 & - & - & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - \\
1 & 1 & - & - & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & 1 \\
1 & - & 1 & - & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - \\
1 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & - & 0 & 0 & - & 1 \\
1 & - & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 1 & - & 1 & - & 0 & 0 \\
1 & - & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & 1 & - & - & 0 & 1 & - & 1 & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & - & 1 & 0 & - & - & 1 & 1 & - \\
0 & 0 & 0 & 0 & - & 1 & - & 1 & 0 & 0 & 1 & - & 1 & - & - & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & - & 1 & - & 1 & 0 & 0 & - & 1 & - & 1 & 0 & - & 0 & 0 \\
0 & 0 & 0 & 0 & - & 1 & 1 & - & 1 & - & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 \\
0 & 0 & 0 & 0 & - & 1 & 1 & - & - & 1 & 0 & 0 & - & - & 0 & 0 & 0 & -
\end{bmatrix}$$

W(19,9): A. Mathon has signed a (19,9,4) design

$$W(19,9) = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & - & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & - \\
1 & 1 & 0 & 1 & 0 & - & 0 & 0 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & - & 0 & 0 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & - & 0 \\
1 & - & 0 & 0 & 0 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & - & 1 \\
1 & 0 & - & 0 & 1 & 0 & 1 & 0 & - & 0 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - \\
1 & 0 & 0 & - & 1 & 1 & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & - & 1 & 0 \\
1 & - & 0 & 0 & - & 0 & 0 & 0 & 1 & 1 & 0 & - & 1 & 0 & 1 & - & 0 & 0 & 0 \\
1 & 0 & - & 0 & 0 & - & 0 & 1 & 0 & 1 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & - & 0 & 0 & - & 1 & 1 & 0 & - & 1 & 0 & 1 & - & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & - & 1 & 0 & 0 & 0 & - & - & 0 & - & - \\
0 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 1 & 0 & - & 0 & - & - & 0 & - \\
0 & 0 & 0 & 0 & - & 1 & 0 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & - & 0 \\
0 & 0 & 1 & - & 0 & 0 & 0 & 0 & - & 1 & 0 & - & - & 1 & 0 & 0 & 0 & - & - \\
0 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & - & 0 & 1 & 0 & - & 0 & - \\
0 & 1 & - & 0 & 0 & 0 & 0 & - & 1 & 0 & - & - & 0 & 0 & 0 & 1 & - & - & 0 \\
0 & 0 & - & 1 & 0 & 1 & - & 0 & 0 & 0 & 0 & - & - & 0 & - & - & 1 & 0 & 0 \\
0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & - & 0 & - & - & 0 & - & 0 & 1 & 0 \\
0 & - & 1 & 0 & 1 & - & 0 & 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 & 0 & 1
\end{bmatrix}$$

W(20,9): from Geramita and Seberry P.209

W(20,9) can be constructed from the negacyclic with the first row:

1 1 0 0 - 1 - - 0 - 0 0 0 - 0 0 0 0 1 0

The following W(20,9) are constructed using four circulants:

1) 0 1 0 0 1, 1 1 0 0 -, 0 1 0 0 1, 0 1 0 0 -

2) 0 1 0 0 1, 1 1 0 0 -, 0 0 1 - 0, 0 0 1 1 0

3) 0 1 1 1 -, 0 1 1 - 1, 1 0 0 0 0, 0 0 0 0 0

4) 1 0 1 - 0, 0 0 1 1 0, 1 1 0 0 0, 1 - 0 0 0

5) 0 1 - - 1, - 1 1 1 1, 0 0 0 0 0, 0 0 0 0 0

6) 0 1 - - 1, 0 1 0 0 1, 0 0 1 1 0, 1 0 0 0 0