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A New Construction for Williamson-type Matrices

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Abstract. It is shown that if q is a prime power then there are Williamson-type matrices of order
(i) $\frac{1}{2}q^2(q+1)$ when $q \equiv 1 \pmod 4$,
(ii) $\frac{1}{4}q^2(q+1)$ when $q \equiv 3 \pmod 4$ and there are Williamson-type matrices of order $\frac{1}{4}(q+1)$.
This gives Williamson-type matrices for the new orders 363, 1183, 1805, 2601, 3174, 5103. The construction can be combined with known results on orthogonal designs to give an Hadamard matrix of the new order $33396 = 4 \cdot 8349$.

1. Introduction

We use a result of Wallis [16] used to construct SBIBD configurations with parameters $(q^2(q+2), q(q+1), q)$ for q a prime power to construct Williamson-type matrices. For all definitions the reader is referred to Geramita and Seberry [4].

This paper has been motivated by results of Mathon [7] and Seberry and Whiteman [11].

Previous relevant results on Williamson matrices can be found in Mukhopadhyay [9], Seberry [10], Seberry Wallis [13, 14, 15], and Yamada [17, 18]. This paper uses extensively q $(0,1)$ matrices $R_0, R_1, \ldots, R_{q-1}$ of order q^2 (q a prime or prime power) which satisfy

$$R_i R_j^T = J \times J \qquad i \neq j \tag{i}$$

$$\sum_{i=0}^{q-1} R_i R_i^T = q^2 I \times I + q(J-I) \times J, \tag{ii}$$

where I is the identity matrix and J is the matrix of all ones. These R_i's have been used before to construct Hjelmslev planes (see Dembowski [1] for the definition of Hjelmslev planes). In a construction called the "Craig construction" the R_i, $i = 0, \ldots q-1$ and $R_q = I \times J$ are substituted for the ones of an incidence matrix A of a projective plane of order q, such that each row and column contains exactly one R_i, and if the $q^2 \times q^2$ matrix of all zeros is substituted for each zero of A, then the

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resultant matrix is the incidence matrix of a uniform Hjelmslev plane of order q. The R_i's have been called auxilliary matrices (see Drake and Lenz [3]).

In J. Wallis [16] it is shown how such matrices can be used to construct symmetric balanced incomplete block designs with parameters $(q^2(q+2), q^2+q, q)$. The existence of this family of block designs was also established by Takeuchi (1963) and Ahrens and Szekeres (1969) but it is the matrices of Wallis' construction that interest us.

In the interests of completeness we give the following theorem and the proof by Glynn [5]:

Theorem (Proof by Glynn). *Suppose the exists a projective plane of order q then there exists a set of q $(0,1)$-matrices $R_0, R_1, \ldots, R_q$ satisfying*

$$R_iR_j^T = J \times J \qquad i \neq j \tag{i}$$

$$\sum_{i=0}^{q-1} R_iR_i^T = q^2 I \times I + q(J-I) \times J. \tag{ii}$$

In particular the matrices $R_0, R_1, \ldots, R_q$ exist whenever q is a prime power.

Proof. Let Π be a projective plane of order q, let l be a line of Π and label the $q+1$ points of l, $P_0, \ldots, P_q$. Also label the q lines through P_q of the affine plane Π^1 $a_1, \ldots, a_q$ and label the q lines through P_0 of Π^1, $b_1, \ldots, b_q$. Thus each point of $a_i \cap b_j$ of Π^1 can be labelled P_{ij}, where $i, j \in \{1, \ldots, q\}$. For each $k = 0, \ldots, q$ define a $q^2 \times q^2$ $(0,1)$-matrix R_k as follows. The q^2 rows of each R_k are labelled in order $(1,1)$, $(1,2)$, $\ldots, (1,q), (2,1), \ldots, (2,q), \ldots, (q,q)$ and so are the columns. Define the matrix R_k to have a one in the position with row (i_1, j_1) and column (i_2, j_2) if and only if $(i_1, j_1) = (i_2, j_2)$ or the line joining the distinct points $a_{i_1} \cap b_{j_1}$ and $a_{i_2} \cap b_{j_2}$ passes through P_k. An element of R_k is zero otherwise. It is straight-forward to check, using the definition of projective plane, that with $R_q = I \times J$ and $R_0 = J \times I$ the required equations are satisfied. □

Seberry and Whiteman [11] have shown that for q a prime power (i) and (ii) can be satisfied.

2. The case $q \equiv 3 \pmod 4$.

Take a skew-Hadamard matrix of order $q+1$ (see Wallis [12] for definitions and results) and normalize so the first row consists of $+1$, the first column (except for the first element consists of -1) and the remainder of the matrix is written $B = I + S$ where S is skew-symmetric (ie B is of order q and comprises the (i,j) elements of the normalized skew-Hadamard matrix $i, j = 2, \ldots, q+1$) then

$$BJ = J, \quad BB^T = (q+1)I - J.$$

Define $S_i = (I \times B)R_i$ and $S_q = B \times J$ so that

$$S_iS_j^T = (I \times B)R_iR_j^T(I \times B^T) = J \times J, \quad i \neq j, \quad i, j \neq q \tag{iii}$$

$$S_iS_q^T = (I \times B)R_i(B^T \times J) = (BP^{ab})_{ab}(b_{cd}J)_{cd} \tag{iv}$$

where BP^{ab} means the ab-th element of $(I \times B)R_i$ is B times some permutation matrix P^{ab} and b_{cd} is the cd-th element of B. Since $BP^{ab}J = J$ the product $S_iS_q^T$ depends on $\sum_c b_{cd} = +1$ and hence

$$S_iS_q^T = J \times J \qquad i \neq q.$$

$$\begin{aligned}\sum_{i=0}^{q} S_iS_i^T &= \sum_{i=0}^{q-1} (I \times B)R_iR_i^T(I \times B^T) + qBB^T \times J \\ &= (I \times BB^T)(q^2I \times I + q(J - I) \times J) + qBB^T \times J \\ &= q^2I \times BB^T + q(J - I) \times J + qBB^T \times J \\ &= q^2(q+1)I \times I.\end{aligned} \tag{v}$$

Furthermore we note $S_0, \ldots, S_q$ are $(1, -1)$ matrices.

Now suppose W_1, W_2, W_3, W_4 are Williamson type matrices or suitable matrices of order t (see Geramita and Seberry [4] for definitions and results) then W_i are ± 1 matrices such that

$$W_iW_j^T = W_jW_i^T \qquad i, j = 1, 2, 3, 4$$

$$\sum_{i=1}^{4} W_iW_i^T = 4tI_t.$$

Let $W_i = (w_{ab}^i)$ and define $Y_j, j = 1, \ldots, 4$ to be the circulant matrices with first rows

$$[S_{(j-1)t+1}, \ldots, S_{jt}] \qquad j = 1, 2, 3, 4, \qquad t = \tfrac{1}{4}(q+1)$$

and $X_i = W_i * Y_i$ where $*$ is the element by element product or Hadamard product. Now

$$X_iX_j^T = W_iW_j^T * (J \times J) = X_jX_i^T \qquad i \neq j$$

and

$$\sum_{i=1}^{4} X_iX_i^T = I \times \sum_{k=1}^{q+1} S_kS_k^T = q^2(q+1)I.$$

Hence we have shown

Theorem 1. *Let $q \equiv 3 \pmod 4$ be a prime power. Then if there are Williamson-type matrices of order $\frac{1}{4}(q+1)$ there are Williamson-type matrices of order $\frac{1}{4}q^2(q+1)$.*

Remark. Putting $q = 11$, 19, 23 and 27 in the theorem gives Williamson-type matrices for the new orders, 363, 1805, 3174 and 5103.

Corollary 2. *If there exists an orthogonal design of type (a, a, a, a) in order $4a$ (a Baumert-Hall array of order $4a$) and Williamson-type matrices of order $\frac{1}{4}(q+1)$ where $q \equiv 3 \pmod 4$ is a prime power there is an Hadamard matrix of order $q^2(q+1)a$.*

Remark. Since there is an orthogonal design of type (23, 23, 23, 23) in order 92 and Williamson-type matrices of order 363 there is an Hadamard matrix of the new order $12 \cdot 11^2 \cdot 23 = 33396$.

Lemma 3. *There is an Hadamard matrix of order $q^2(q+1)$, $q \equiv 3 \pmod 4$ a prime power.*

Proof. Let $H = (h_{ij})$ be an Hadamard matrix of order $q + 1$. Form a latin square LS from the $q + 1$ matrices $S_0, \ldots, S_q$ for example LS might be

$$\begin{bmatrix} S_0 & S_1 & S_2 & \cdots & S_{q-1} & S_q \\ S_1 & S_2 & S_3 & \cdots & S_q & S_0 \\ \vdots & \vdots & & & \vdots & \vdots \\ S_q & S_0 & S_1 & \cdots & S_{q-2} & S_{q-1} \end{bmatrix}.$$

We then take the Hadamard product of the latin square with the Hadamard matrix H to form a matrix of order $q^2(q + 1)$ with entries ± 1, for example we might now have

$$\begin{bmatrix} h_{11}S_0 & h_{12}S_1 \; h_{13}S_2 & \cdots & h_{1q}S_{q-1} & h_{1,q+1}S_q \\ h_{21}S_1 & h_{22}S_2 \; h_{23}S_3 & \cdots & h_{2q}S_q & h_{2,q+1}S_0 \\ \vdots & \vdots & & \vdots & \vdots \\ h_{q1}S_{q-1} & & & h_{qq}S_{q-3} & h_{q,q+1}S_{q-2} \\ h_{q+1,1}S_q & & \cdots & h_{q+1,q}S_{q-2} & h_{q+1,q+1}S_{q-1} \end{bmatrix}.$$

This last matrix is the required Hadamard matrix of order $q^2(q + 1)$ because the product across the rows is

$$\sum_{j=1}^{q+1} h_{ji}S_{a_{ji}}h_{ki}S_{b_{ki}}^T = \sum_{j=1}^{q+1} h_{ji}h_{ki}J = 0, \quad \text{for } j \neq k, \; S_{a_{ji}} \neq S_{b_{ki}},$$

while the product of a row of the signed last matrix with itself gives

$$\sum_{j=1}^{q+1} h_{ij}^2 S_{a_{ij}} S_{a_{ij}}^T = q^2(q + 1)I. \qquad \square$$

Example 1. Let $W_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $W_2 = W_3 = W_4 = \begin{bmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{bmatrix}$, so $t = 3$.

$$Y_1 = \begin{bmatrix} S_1 & S_2 & S_3 \\ S_3 & S_1 & S_2 \\ S_2 & S_3 & S_1 \end{bmatrix}, \qquad Y_2 = \begin{bmatrix} S_4 & S_5 & S_6 \\ S_6 & S_4 & S_5 \\ S_5 & S_6 & S_4 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} S_7 & S_8 & S_9 \\ S_9 & S_7 & S_8 \\ S_8 & S_9 & S_7 \end{bmatrix}, \qquad Y_4 = \begin{bmatrix} S_{10} & S_{11} & S_0 \\ S_0 & S_{10} & S_{11} \\ S_{11} & S_0 & S_{10} \end{bmatrix}$$

and

$$X_1 = Y_1, \quad X_2 = \begin{bmatrix} S_4 & -S_5 & -S_6 \\ -S_6 & S_4 & -S_5 \\ -S_5 & -S_6 & S_4 \end{bmatrix}, \quad X_3 = \begin{bmatrix} S_7 & -S_8 & -S_9 \\ -S_9 & S_7 & -S_8 \\ -S_8 & -S_9 & S_7 \end{bmatrix},$$

$$X_4 = \begin{bmatrix} S_{10} & -S_{11} & -S_0 \\ -S_0 & S_{10} & -S_{11} \\ -S_{11} & -S_0 & S_{10} \end{bmatrix}.$$

Now if the S_i are 12 $(1, -1)$ matrices of order 11^2 as constructed above (with subscripts increased by 1)

$$S_i S_j^T = J \qquad i \neq j$$

$$\sum_{i=0}^{11} S_i S_i^T = 11^2 \cdot 12 I \times I.$$

Thus $X_i X_j^T = -J \times J, j = 2, 3, 4$

$$X_i X_j^T = \begin{bmatrix} 3J & -J & -J \\ -J & 3J & -J \\ -J & -J & 3J \end{bmatrix} \qquad i, j = 2, 3, 4$$

and $\sum_{i=1}^{4} X_i X_i^T = \sum_{j=0}^{11} S_i S_i^T \times I = 11^2 \cdot 12 I \times I$.

Hence X_1, X_2, X_3, X_4 are Williamson-type matrices of order 363.

3. The case $q \equiv 1 \pmod 4$

Normalize a symmetric conference matrix of order $q + 1$ (see Wallis [12] for definitions and results) so its first row and column are all $+1$. Write $B + I$ where B is symmetric for the matrix of order q comprising the (i,j) elements of the normalized symmetric conference matrix $i, j = 2, \ldots, q + 1$. Let

$$B_+ = I + B \quad \text{and} \quad B_- = I - B$$

be the $(1, -1)$ matrices satisfying

$$B_+ J = B_- J = J, \quad B_+ B_+^T + B_- B_-^T = 2(q + 1)I - 2J.$$

Define $Q_i = (I \times B_+)R_i, Q_q = B_+ \times J, i = 0, \ldots, q - 1$

$$Q_{q+1+i} = (I \times B_-)R_i, \quad Q_{2q+1} = B_- \times J$$

and that

$$Q_i Q_j^T = J \times J \qquad i, j \in \{0, \ldots, 2q\}, i \neq j. \qquad (i,j) \neq \{(i, q), (i, 2q + 1), (q, 2q + 1), (i, q + 1 + i)\} \tag{vi}$$

$$Q_i Q_q^T = Q_i Q_{2q+1}^T = J \times J \quad \text{(as in the proof of (iv) above)}, i \neq j, i \neq 2q + 1. \tag{vii}$$

$$Q_q Q_{2q+1}^T = Q_{2q+1} Q_q^T, \quad Q_i Q_{q+1+i}^T = Q_{q+1+i} Q_i^T \tag{viii}$$

$$\sum_{i=0}^{2q+1} Q_i Q_i^T = 2q^2(q + 1)I. \tag{ix}$$

We now proceed as before with Williamson-type matrices of order $\frac{1}{2}(q + 1)$. Since Q_i and Q_{q+1+i} will not be in the same matrix X_i we need only to consider $X_1 X_3^T$ and $X_2 X_4^T$ to establish the existence of Williamson-type matrices of order $\frac{1}{2}q^2(q + 1)$.

Now

$$X_1X_3^T = \begin{bmatrix} w_{11}^1Q_0 & w_{12}^1Q_1 & \cdots & w_{1t}^1Q_{t-1} \\ w_{21}^1Q_{t-1} & & \cdots & w_{2t}^1Q_{t-2} \\ \vdots & & & \vdots \\ w_{t1}^1Q_1 & & \cdots & w_{tt}^1Q_0 \end{bmatrix} \begin{bmatrix} w_{11}^3Q_{2t}^T & \cdots & w_{t1}^3Q_{2t+1}^T \\ w_{12}^3Q_{2t+1}^T & \cdots & w_{t2}^3Q_{2t+2}^T \\ \vdots & & \vdots \\ w_{1t}^3Q_{3t-1}^T & & w_{tt}^3Q_{2t}^T \end{bmatrix}$$

$$= X_3X_1^T \qquad t = \tfrac{1}{2}(q+1)$$

because of the properties of the Williamson-type matrices and (viii).

Thus we have

Theorem 4. *Let $q \equiv 1 \pmod 4$ be a prime power. Then there exist Williamson-type matrices of order $\frac{1}{2}q^2(q+1)$.*

Proof. When $q \equiv 1 \pmod 4$ is a prime power Williamson matrices of order $\frac{1}{2}(q+1)$ always exist (see for example Geramita and Seberry [4] p. 94, Corollary 4.43). Furthermore the conference matrix of order $q+1$ always exists. Thus the construction above always gives the required result.

Remark. Putting $q = 13$ and 17 gives new Williamson-type matrices of orders 1183 and 2601.

Corollary 5. *Let $q \equiv 1 \pmod 4$ be a prime power. Then there is an Hadamard matrix of order $2q^2(q+1)$.*

Proof. Put the Williamson-type matrices just constructed into an orthogonal design of type (1, 1, 1, 1) in order 4.

Example 2. Let $q = 5$ and $Q_0, \ldots, Q_{11}$ be as just described. Let W_1, W_2, W_3, W_4 be as in the previous example. Then proceeding exactly as before we

$$X_1 = \begin{bmatrix} Q_0 & Q_1 & Q_2 \\ Q_2 & Q_0 & Q_1 \\ Q_1 & Q_2 & Q_0 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} Q_3 & -Q_4 & -Q_5 \\ -Q_5 & Q_3 & -Q_4 \\ -Q_4 & -Q_5 & Q_3 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} Q_6 & -Q_7 & -Q_8 \\ -Q_8 & Q_6 & -Q_7 \\ -Q_7 & -Q_8 & Q_6 \end{bmatrix}, \qquad X_4 = \begin{bmatrix} Q_9 & -Q_{10} & -Q_{11} \\ -Q_{11} & Q_9 & -Q_{10} \\ -Q_{10} & -Q_{11} & Q_9 \end{bmatrix}$$

thus

$$X_1X_2^T = -J \times J,$$
$$X_1X_3^T = I_3 \times (Q_0Q_6^T - Q_1Q_7^T - Q_2Q_8^T) + (J - I) \times -J,$$
$$X_1X_4^T = -J \times J$$
$$X_2X_3^T = I_3 \times 3J + (J - I) \times -J = X_3X_4^T,$$
$$X_2X_4^T = I_3 \times (Q_3Q_9^T + Q_4Q_{10}^T + Q_5Q_{11}^T) \times (J - I) \times -J$$

which are all symmetric.

Let $U_j = \{3j-3, 3j-2, 3j-1\}, j = 2, 3, 4$. Then

$$\sum_{i=1}^{4} X_i X_i^T = X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = \left(\sum_{i=0}^{2} Q_i Q_i^T \times I_4 + 3J_{25} \times (J - I)_4\right)$$

$$+ 3\left(\sum_{i \in U_j} Q_i Q_i^T \times I_4 - J_{25} \times (J - I)_4\right) \qquad j = 2, 3, 4$$

$$= \sum_{i=0}^{11} Q_i Q_i^T \times I_4 = 12 \cdot 25 I = 300 I.$$

Thus $X_1, \ldots, X_4$ are Williamson-type matrices of order 75.

Corollary 6. *Let $q \equiv 1 \pmod 4$ be a prime power. Suppose there exists an orthogonal design of type (a, a, a, a) in order $4a$. Then there exists an Hadamard matrix of order $2aq^2(q+1)$.*

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