Journal of Statistical Planning and Inference 11 (1985) 373-379
North-Holland

GENERALIZED BHASKAR RAO DESIGNS OF BLOCK SIZE THREE

Jennifer SEBERRY

Department of Computer Science, University of Sydney, N.S.W. 2006, Australia

Received 4 May 1982; revised manuscript received 2 April 1984
Recommended by N.M. Singhi

Abstract: We show that the necessary conditions

$$\lambda \equiv 0 \pmod{|G|},$$

$$\lambda(v-1) \equiv 0 \pmod{2},$$

$$\lambda v(v-1) \equiv \begin{cases} 0 \pmod{6} & \text{for } |G| \text{ odd,} \\ 0 \pmod{24} & \text{for } |G| \text{ even,} \end{cases}$$

are sufficient for the existence of a generalized Bhaskar Rao design GBRD$(v, b, r, 3, \lambda; G)$ for the elementary abelian group G, of each order $|G|$.

AMS Subject Classifications: Primary 05B99; Secondary, 05B05, 05B30, 62K10.

Key words: Bhaskar Rao designs.

1. Introduction

For the definitions and notations used in this paper we refer the reader to Lam and Seberry (1984). Since the underlying structure for a GBRD$(v, 3, \lambda; G)$ is a BIBD$(v, 3, \lambda)$ we have the following necessary conditions for existence:

$$\lambda \equiv 0 \pmod{|G|}, \tag{1.1}$$

$$\lambda(v-1) \equiv 0 \pmod{2}, \tag{1.2}$$

$$\lambda v(v-1) \equiv 0 \pmod{6}. \tag{1.3}$$

In Lam and Seberry (1984) the extra condition

$$|G| \equiv 0 \pmod{2} \quad \Rightarrow \quad b \equiv 0 \pmod{4} \tag{1.4}$$

was established.

We also use the notation $\mathrm{EA}(H) = \mathrm{EA}(\prod_{i=1}^{n} p_i^{r_i})$ for the elementary abelian group $Z_{p_1} \times Z_{p_1} \times \cdots \times Z_{p_1} \times \cdots \times Z_{p_n} \times Z_{p_n} \times \cdots \times Z_{p_n}$ where Z_{p_i} occurs r_i times with $\prod_{i=1}^{n} p_i^{r_i}$ the prime decomposition of the order of the group H.

In this paper we establish that the necessary conditions are sufficient for the existence of a GBRD$(v, 3, \lambda; \mathrm{EA}(H))$.

Lam and Seberry (1984), building on the results of Seberry (1982, 1984), proved:

Theorem 1.1. *The necessary conditions* (1.1)–(1.4) *are sufficient for the existence of a* GBRD$(v, 3, \lambda; G)$ *when*

(i) $|G|$ *is odd,*

(ii) $G = Z_2^r$,

(iii) $G = Z_2^r \times H$ *where* $3 \nmid |H|$ *and* $r \geq 1$.

In this paper we establish existence for the remaining group orders.

2. Existence of GBRD on $Z_2 \times Z_3$

Theorem 2.1. *The necessary conditions*

$$\lambda \equiv 0 \pmod 6,$$

$$\lambda v(v-1) \equiv 0 \pmod{24}$$

are sufficient for the existence of a GBRD$(v, 3, \lambda; Z_2 \times Z_3)$.

Proof. From Section 1 these are clearly necessary conditions: in the case $\lambda \equiv 6 \pmod{12}$ they become $v(v-1) \equiv 0 \pmod 4$ and for $\lambda \equiv 0 \pmod{12}$ there is no condition on v. Hence we consider these cases separately.

Case 1: $\lambda = 6$. The necessary condition becomes $v(v-1) \equiv 0 \pmod 4$. We first establish the existence of GBRD$(v, 3, 6; Z_6 = Z_2 \times Z_3)$ *for* $v \in K_4^1 = \{4, 5, 8, 9, 12\}$. Using $0, 1, 2, 3, 4, 5$ for the elements, additively, of Z_6 we have that

$$D_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 2 & 3 & 4 & 5 & . & . & . & 0 & 0 & 0 \\ 0 & 2 & 1 & . & . & . & 3 & 4 & 5 & 3 & 2 & 4 \\ . & . & . & 3 & 2 & 1 & 5 & 4 & 0 & 1 & 5 & 3 \end{pmatrix}$$

is a GBRD$(4, 3, 6; Z_6)$. The initial blocks developed as

$$D_5 = \{(0_0, 1_0, 2_1), (0_0, 1_2, 2_5), (0_0, 1_4, 3_4), (0_0, 1_5, 3_2)\} \pmod{5,\ Z_6},$$

$$D_8 = \{(0_0, 1_0, 2_1), (0_0, 1_2, 2_5), (0_0, 1_4, 3_4), (0_0, 1_5, 3_2), (0_0, 2_2, 4_0), (\infty_0, 1_1, 4_2),$$
$$(\infty_0, 1_3, 4_0), (\infty_0, 1_4, 5_5)\} \pmod{7,\ Z_6},$$

$$D_{12} = \{(0_0, 1_1, 10_3), (0_0, 2_1, 9_3), (0_0, 3_1, 8_3), (0_0, 4_1, 7_3), (0_0, 1_0, 5_5), (0_0, 4_0, 5_4),$$
$$(0_0, 3_4, 5_0), (0_0, 3_0, 4_2), (0_0, 1_5, 3_5), (\infty_0, 0_5, 2_5), (\infty_0, 0_1, 5_4),$$
$$(\infty_0, 0_2, 5_3)\} \pmod{11,\ Z_6},$$

give the required designs for $u = 5, 8, 12$. A GBRD$(9, 3, 6; Z_6)$ with a subdesign on 4 points is

$$D_9 = \left(\begin{array}{ccccc|ccccc|ccccc|ccccc|c}
e & e & e & e & e & & & & & & & & & & & & & & & & \\
 & & & & & e & e & e & e & e & & & & & & & & & & & D_4 \\
 & & & & & & & & & & e & e & e & e & e & & & & & & \\
 & & & & & & & & & & & & & & & e & e & e & e & e & \\
\hline
x & s & & & & x & s & & & & x & s & & & & x & s & & & & \\
y & & s & & & & y & s & & & & & x & s & & & & x & s & & \\
 & y & & s & & & & y & s & & y & & & & s & & & & t & x & 0 \\
 & & t & & x & & & & t & x & & t & & y & & & y & & & t & \\
 & & & y & t & & & & & y & & & y & & t & y & & t & & &
\end{array}\right)$$

where

$$e = (0, 0, 0),$$

$$x = (0, 1, 2), \qquad s = (3, 4, 5), \qquad y = (0, 2, 1), \qquad t = (3, 5, 4),$$

so that

$$x \cdot y^{-1} = (0, 5, 1), \qquad x \cdot t^{-1} = (3, 2, 4),$$

$$s \cdot t^{-1} = (0, 5, 1), \qquad s \cdot y^{-1} = (3, 2, 4).$$

Now using Theorem 2.2 of Lam and Seberry (1984) with Hanani's theorem (Hall (1967), Lemma 15.5.1) we only need to establish the existence of GBRD$(v, 3, 6; Z_6)$ for $v \in K_4^1 = \{4, 5, 8, 9, 12\}$ to establish the existence of all GBRD$(u, 3, 6; Z_6)$ for $u \equiv 0$ or 1 (mod 4), $u \geq 4$. Thus we have all these designs and by taking t copies (t odd) we have GBRD$(v, 3, 6t; Z_6)$ for all $\lambda \equiv 6$ (mod 12).

Case 2: $\lambda = 12$. There are now no necessary conditions on v. We observe that if $v = 2p + 1$, p a positive integer, then GBRD$(2p + 1, 3, 12; Z_6)$ are obtained by developing the blocks (modulo $2p + 1, Z_6$)

$$(0_0, i_1, 2p + 1 - i_4), \quad i = 1, \ldots, 2p,$$

$$(0_0, j_1, 2p + 1 - j_1), \ (0_0, j_2, 2p + 1 - j_2), \quad j = 1, \ldots, p.$$

For $v = 2p + 2$, $p \geq 2$ integer, a GBRD$(2p + 2, 3, 12; Z_6)$ can be obtained by developing the following blocks (modulo $2p + 1, Z_6$):

$$(0_0, i_1, 2p + 1 - i_4), \quad i = 3, \ldots, 2p,$$

$$(0_0, j_1, 2p + 1 - j_1), \ (0_0, j_2, 2p + 1 - j_2), \quad j = 1, \ldots, p,$$

$$(\infty_0, 0_0, 2p - 1_3), \ (\infty_0, 0_1, 2p - 3_4), \ (\infty_0, 0_2, 2p_0),$$

$$(\infty_0, 0_1, 2p - 1_5), \ (\infty_0, 0_2, 1_3), \ (\infty_0, 0_4, 2_5).$$

The case for $v = 4$ is obtained by taking two copies of the GBRD$(4, 3, 6; Z_6)$ given in case 1 above.

Hence we have constructed a GBRD$(v, 3, 12; Z_6)$ for every v. Taking t copies gives us GBRD$(v, 3, 12t; Z_6)$ for every $\lambda = 12t \equiv 0 \pmod{12}$. Hence we have the theorem. □

3. Existence of GBRD on $Z_2 \times Z_2 \times Z_3$

Theorem 3.1. *The necessary condition* $\lambda \equiv 0 \pmod{12}$ *is sufficient for the existence of a* GBRD$(v, 2, 12t;$EA$(12))$.

Proof. For the group $G = $ EA(12) the only necessary condition for the existence of a GBRD$_G(v, 3, \lambda)$ is that $\lambda \equiv 0 \pmod{12}$.

To establish existence we use Theorem 2.2 of Lam and Seberry (1984) with Hanani's theorem (Hall (1967), Lemma 15.5.2). Thus we must establish the existence of GBRD$(v, 3, 12;$EA$(12))$ for $v \in K_4^2 = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 22, 23\}$ to establish existence for all $v \geq 4$.

From Lam and Seberry (1984) we see that GBRD$(v, 3, 4, ; Z_2 \times Z_2)$, say A, exists for $v \equiv 0, 1 \pmod 3$. Writing $W = (w_{ij})$ for

$$W = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{3.1}$$

the generalized Hadamard matrix GH$(3, Z_3)$, we have, replacing every A_{ij} of A by the vector of ordered pairs $[(A_{ij}, w_{i1}), (A_{ij}, w_{i2}), (A_{ij}, w_{i3})]$, the required design. Hence we have the existence of GBRD$(v, 3, 12; Z_2 \times Z_2 \times Z_3)$ for $v = 4, 6, 7, 9, 10, 12, 15, 18, 19, 22$.

Similarly, from Seberry (1982), we have a GBRD$(v, 3, 3; Z_3)$ for all odd v. Thus, combining these designs with three rows of

$$H = \begin{pmatrix} e & e & e & e \\ e & a & b & ab \\ e & b & ab & a \\ e & ab & a & b \end{pmatrix}, \quad a^2 = b^2 = e, \ ab = ba, \tag{3.2}$$

the generalized Hadamard matrix GH$(4, Z_2 \times Z_2)$, we have the existence of GBRD$(v, 3, 12,$ EA$(12))$ for all odd v and in particular for $v = 5$, 11 and 23.

A GBRD$(8, 3, 12;$EA$(12))$ is constructed by the use of Theorem 2.2 with the GBRD$(8, 4, 3; Z_3)$ obtained by developing the initial blocks $(\infty_1, 1_1, 2_{\omega^2}, 4_\omega)$ and $(0_1, 1_\omega, 2_\omega, 4_\omega)$ (mod 7, Z_3), and the GBRD$(4, 3, 4; Z_2 \times Z_2)$

$$\begin{pmatrix} e & e & e & e & e & e & \cdot & \cdot \\ e & a & b & ab & \cdot & \cdot & e & e \\ e & b & \cdot & \cdot & a & ab & a & b \\ \cdot & \cdot & e & ab & b & a & b & ab \end{pmatrix}, \quad a^2 = b^2 = e, \ ab = ba.$$

A GBRD(14, 3, 12;EA(12)) is obtained by developing the following initial blocks (modulo 13, EA(12)):

$$(0_1, i_a, 13-i_{ab\omega^2}),\ (0_1, i_{a\omega}, 13-i_{ab\omega}),\ (0_1, i_\omega, 13-i_\omega),\quad i=1,\ldots,6,$$

$$(0_a, i_b, 13-i_\omega),\quad i=3,4,5,6,$$

$$(\infty_1, 1_1, 12_{ab}),\ (\infty_1, 1_{b\omega}, 12_{ab\omega^2}),\ (\infty_1, 2_{ab\omega}, 11_{a\omega^2}),$$

$$(\infty_1, 0_a, 1_b),\ (\infty_1, 0_{\omega^2}, 2_{a\omega}),\ (\infty_1, 0_{b\omega^2}, 1_\omega),$$

where $a^2=b^2=\omega^3=1$ and all elements commute.

Thus a GBRD(v, 3, 12;EA(12)) exists for all $v \in K_4^2$ and hence, for all v. GBRD for $\lambda=12t$ are obtained by taking t copies of the design for $\lambda=12$, giving the result. □

4. Existence of GBRD on $Z_2\times Z_2\times Z_2\times Z_3$

Theorem 4.1. *The necessary condition* $\lambda\equiv 0$ (mod 24), *is sufficient for the existence of* GBRD(v, 3, 24t;EA(24)).

Proof. For the group EA(24) the only necessary condition is $\lambda\equiv 0$ (mod 24). Hence, as in the previous section, it is only necessary to establish the existence of GBRD(v, 3, 24;EA(24)) for $v\in K_4^2$.

From Seberry (1982) a GBRD(v, 3, 3;Z_3) exists for all odd v and so we use these designs with three rows of the generalized Hadamard matrix GH(8, Z_2^3) = GBRD(3, 3, 8;$Z_2\times Z_2\times Z_2$), say

$$\begin{bmatrix} e & e & e & e & e & e & e & e \\ e & a & b & ab & c & ac & bc & abc \\ e & b & bc & c & ac & abc & ab & a \end{bmatrix},\quad a^2=b^2=c^2=e, \tag{4.1}$$

where the elements commute, in Theorem 2.2 of Lam and Seberry (1984) to obtain GBRD(u, 3, 24;EA(24)) for all odd u.

From Theorem 2.1 a GBRD(u, 3, 6;Z_6) exists for $u\equiv 0, 1, 4, 5, 8, 9$ (mod 12) and so may be used with three rows of (4.2), the GH(4, Z_2^2) = GBRD(3, 3, 4;$Z_2\times Z_2$) in Theorem 2.2 of Lam and Seberry (1984) to obtain GBRD(u, 3, 24;EA(24)) for these u and in particular for $u\in\{4, 8, 12\}$. From Lam and Seberry (1984) we see a GBRD(u, 3, 8;$Z_2\times Z_2\times Z_2$) exists for $u(u-1)\equiv 0$ (mod 3): these designs may be used with (3.1), the GH(3, Z_3) = GBRD(3, 3, 3;Z_3) in Theorem 2.2 of Lam and Seberry (1984) to obtain GBRD(u, 3, 24) for these u including $u\in\{6, 10, 18, 22\}$.

There exists a GBRD(13, 3, 24;EA(24)). It is obtained by developing the following blocks modulo 13:

$$(0_1, i_\omega, 13-i_\omega),\ (0_1, i_{a\omega}, 13-i_{b\omega}),\ (0_1, i_{b\omega}, 13-i_{c\omega}),\ (0_1, i_{ac\omega}, 13-i_{a\omega}),$$

$$(0_1, i_{ac\omega^2}, 13-i_{abc\omega^2}),\ (0_1, i_{bc\omega}, 13-i_{abc\omega}),\ (0_1, i_{ab\omega}, 13-i_{bc\omega}),$$

$$(0_1, i_{c\omega}, 13-i_{ab\omega}),\quad i=1,2,3,4,5,6.$$

A GBRD(14, 3, 24;EA(24)) can be obtained by developing the same blocks modulo 13 except that the last initial block, $(0_1, i_{c\omega i}, 13 - i_{ab\omega})$, $i = 1, 2, \ldots, 6$, should be replaced by

$$(0_1, 5_{c\omega}, 8_{ab\omega}),\ (0_1, b_{c\omega}, 7_{ab\omega}),$$

$$(\infty_1, 0_{ac\omega^2}, 1_a),\ (\infty_1, 0_b, 2_{bc\omega}),\ (\infty_1, 0_{ac}, 3_{a\omega}),\ (\infty_1, 0_{b\omega^2}, 4_{bc}),$$

$$(\infty_1, 0_1, 1_{ab\omega^2}),\ (\infty_1, 0_{abc}, 2_{c\omega^2}),\ (\infty_1, 0_{a\omega^2}, 3_{b\omega}),\ (\infty_1, 0_{bc\omega^2}, 4_{ac\omega}),$$

$$(\infty_1, 0_\omega, 2_{abc\omega}),\ (\infty_1, 0_{\omega^2}, 4_{abc\omega^2}),\ (\infty_1, 0_{ab\omega}, 6_{c\omega}),\ (\infty_1, 0_{ab}, 8_c),$$

which should also be developed modulo 13.

Hence a GBRD(v, 3, 24;EA(24)) exists for all $v \in K_4^2$ and hence, as in the proof of Theorem 3.1 for all v. GBRD for $\lambda = 24t$ are obtained by taking t copies of the design for $\lambda = 12$ giving the result. □

5. Existence of Bhaskar Rao designs with block size three

Theorem 5.1. *The necessary conditions* (1.1)–(1.4) *are sufficient for the existence of a generalized Bhaskar Rao design* GBRD(v, 3, λ;G) *on the elementary abelian group G for every order* $|G|$.

Proof. From Theorem 1.1 it is merely necessary to consider the cases $G = Z_2^r \times H$, $r \geq 1$ where $3 \mid |H|$. We have three basic cases:

(i) $|G| = 6p$, p odd, with necessary condition $pv(v-1) \equiv 0 \pmod 4$,

(ii) $|G| = 12p$, p odd, with no condition on v,

(iii) $|G| = 24p$, p odd, with no condition on v.

We use Theorems 2.1, 3.1 and 4.1 respectively to obtain the design for $|G| = 6$, 12 and 24 and makes repeated use of theorem 2.2 of Lam and Seberry (1984) to obtain the result. For orders $2^s p$, $s \geq 4$, use $|G| = 12$ or 24 and repeated use of GBRD(3, 3, 4;$Z_2 \times Z_2$) according as s is even or odd. □

6. Application

Proceeding as in Lam and Seberry (1984) and Seberry (1982, 1984) we now have:

Theorem 6.1. *Whenever* $\lambda \equiv 0 \pmod g$, $\lambda(v-1) \equiv 0 \pmod 2$, $\lambda v(v-1) \equiv 0 \pmod 6$ *for g odd or* $\lambda v(v-1) \equiv 0 \pmod{24}$ *for g even there exists a regular group divisible design with parameters* $(vg, bg, r, 3, \lambda_1 = 0, \lambda_2 = \lambda/g, m = v, n = g)$.

References

M. Hall, Jr. (1967). *Combinatorial Mathematics*. Blaisdell, Waltham, MA.

C. Lam and J. Seberry (1984). Generalized Bhaskar Rao designs. *J. Statist. Plann. Inference* 10, 83–95.

J. Seberry (1982). Some families of partially balanced incomplete block designs. In: *Combinatorial Mathematics IX*, Lecture Notes in Mathematics No. 952. Springer, Berlin–Heidelberg–New York, 378-386.

J. Seberry (1984). Regular group divisible designs and Bhaskar Rao designs with block size three. *J. Statist. Plann. Inference* 10, 69–82.