

ON BHASKAR RAO DESIGNS OF BLOCK SIZE FOUR

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ABSTRACT. We show that Bhaskar Rao designs of type BRD($v, b, r, 4, 6$) exist for $v \equiv 0, 1 \pmod{5}$ and of type BRD($v, b, r, 4, 12$) exist for all $v \geq 4$.

Let A, B and $A+B$ be $v \times b$ matrices with entries 0, 1. Then $X = A - B$ is said to be a *Bhaskar Rao design* with parameters BRD(v, b, r, k, λ) when the following matrix equations are satisfied :

$$XX^T = rI \quad \dots (1)$$

$$(A+B)(A+B)^T = (r-\lambda)I + \lambda J \quad \dots (2)$$

$$J(A+B) = kJ. \quad \dots (3)$$

X is a $v \times b$ matrix with entries 0, +1, -1 with row inner product zero and which, when the -1 elements are replaced by +1, becomes the incidence matrix of a BIBD(v, b, r, k, λ). For example the following matrix is a BRD(6, 15, 10, 4, 6) :

Example 1 : There exists a BRD(6, 15, 10, 4, 6). Write—for -1

1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
1	1	1	-	-	-	0	0	0	0	1	1	1	1	0
1	1	0	1	0	0	-	-	-	0	1	-	-	0	1
1	0	-	0	-	0	1	1	0	-	-	-	0	1	1
0	0	1	-	0	-	1	0	-	1	-	0	-	-	1
0	1	0	0	1	-	0	1	-	-	0	-	1	-	-

These designs were first studied by Bhaskar Rao [1, 2] and may be used to obtain group divisible PBIBD with parameters

$$\begin{aligned} v^* &= 2v, & b^* &= 2b, & r^* &= r, & k^* &= k, \\ \lambda_1 &= 0, & \lambda_2 &= \lambda/2, & m &= 2, & n &= v. \end{aligned}$$

The necessary conditions for the existence of a $BRD(v, b, r, k, \lambda)$ are for $k = 4$,

$$\begin{aligned}\lambda(v-1) &= r(k-1) \\ bk &= vr \quad \dots (4)\end{aligned}$$

and other restrictions on the parameters have been found (see [3, 4]) when $k \neq 4$. They have also been studied by Vyas [5] and Singh [6]. We use the notation $BRD(v, k, \lambda)$ for $BRD(v, b, r, k, \lambda)$ as b and r are dependent on v, k, λ .

In this paper we use the following known results (see [7]) restricted to the group Z_2 :

Theorem 1: *Suppose there exists a $BRD(k, j, \lambda_B)$ and*

- (i) *a $BRD(v, k, \lambda_A)$ then there exists a $BRD(v, j, \lambda_A \lambda_B)$;*
- (ii) *a $BIBD(v, k, \lambda)$ then there exists a $BRD(v, j, \lambda \lambda_B)$.*

Or, as is obtained in a similar fashion:

Corollary 2: *Suppose there exists a pairwise balanced design $B(K, \lambda, v)$ where $K = \{k_1, \dots, k_b\}$ and a $BRD(k_i, j, \mu)$ for each $k_i \in K$ then there exists a $BRD(v, j, \lambda \mu)$.*

The next result is a slight improvement on the result of Lam and Seberry [7] where the existence of $k-1$ mutually orthogonal latin squares was required. The result may be proved by adjusting the matrix in the proof of the original theorem.

Theorem 3: *Suppose there exists a $BRD(u, k, \lambda)$ with a subdesign on w points (the values $w = 0$ and 1 are allowed), a $BRD(v, k, \lambda)$ and $k-2$ mutually orthogonal Latin squares then there exists a $BRD(v(u-w)+w, k, \lambda)$ with subdesigns on u, w and v points.*

Remark 4: In this paper we are interested in the case $k = 4$, so we only need a pair of orthogonal latin squares and hence $u-w$ may take on any value other than 2 or 6.

Hanani's theorem stated on p. 250 of Hall [8] states

Theorem 5 (Hanani): *Let $u \equiv 0, 1 \pmod{5}$ then $u \in B(K_5^1, 1)$ where $K_5^1 = \{5, 6, 10, 11, 15, 16, 20, 35, 36, 40, 70, 71, 75, 76\}$.*

Remark: We now see that if $u \equiv 0, 1 \pmod{5}$ and there exists a $BRD(k_i, 4, 6)$ for every $k_i \in K_5^1$ then we have the existence of a $BRD(u, 4, 6)$ using either Theorem 1 or Corollary 2 with Theorem 5.

The main theorem: First we establish:

Theorem 6: *Let $p \geq 5$, odd, be a prime or prime power. Then there is a $BRD(p, \frac{1}{2}p(p-1), 2(p-1), 4, 6)$.*

Proof: Let g be a generator of the multiplicative group, G , of $GF(p)$. Consider the initial sets, writing \bar{g}^a for g^a with the non-identity element of Z_2 attached,

$$D_i = (\bar{0}, g^i, g^{i+1}, g^{i+2}) \text{ where } i = 0, 1, \dots, \frac{1}{2}(p-3).$$

The differences from D_i are

$$E_i = \{\bar{g}^i, \bar{g}^{i+1}, \bar{g}^{i+2}, \bar{g}^{i(p-1)+i}, \bar{g}^{i(p-1)+i+1}, \bar{g}^{i(p-1)+i+2}, g^i(g-1), g^i(g^2-1), g^{i+1}(g-1), g^{i+1}(g^2-1), g^{i(p-1)+i}(g-1), g^{i(p-1)+i}(g^2-1), g^{i(p-1)+i+1}(g-1)\}.$$

As i runs through $0, 1, \dots, \frac{1}{2}(p-3)$ the totality of elements from all E_i including repetitions is 3 copies of G and 3 copies of G with the non-identity element attached, that is,

$$\sum_{i=0}^{\frac{1}{2}(p-3)} E_i = 3G + 3\bar{G},$$

giving the result.

Theorem 7: A BRD($v, 4, 6$) exists for $v \equiv 0, 1 \pmod{5}$.

Proof: By the remark after Theorem 5 it is merely necessary to show the existence of a BRD($u, 4, 6$) for $u \in K_5^1$. These are obtained as given in Table 1 by developing the indicated initial blocks. First we exhibit a BRD(8, 4, 6):

Example 2: There is a BRD(8, 28, 14, 4, 6).

1	1	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	1	0	0																										
1	1	0	0	<i>I</i>	<i>-I</i>	<i>-I</i>	<i>-I</i>	<i>J-I</i>	<i>B</i>	<i>B</i>	<i>B</i>																		
1	1	0	0																										
0	0	1	1																										
0	0	1	1	<i>B</i>	<i>I</i>	<i>-I</i>	<i>-B</i>	<i>B</i>	<i>J-I</i>	<i>I</i>	<i>I</i>																		
0	0	1	1																										
0	0	1	1	0	0	0	- - -	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$$B = \begin{bmatrix} 0 & 1 & - \\ - & 0 & 1 \\ 1 & - & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

TABLE 1.

treatments	construction
5	Prime, Theorem 6.
6	Example 1.
7	Prime, Theorem 6.
8	Example 2.
9	Theorem 6.
10	$(\infty, 0, 2, 4), (\overline{\infty}, 0, 1, 3), (0, \bar{1}, \bar{2}, 5), (0, \bar{2}, 3, 6), (0, 1, \bar{2}, \bar{5}) \pmod 9$.
11	Prime, Theorem 6.
15	Form a BRD $(7, 4, 2)$ by developing $(\bar{0}, 1, 2, 4) \pmod 7$. Use this BRD with a BIBD $(15, 7, 3)$ in Theorem 1 (ii).
16	$(\infty, 0, 2, 6), (\overline{\infty}, 0, 3, 8), (\bar{0}, 4, \bar{5}, 7), (\bar{0}, 4, 5, \bar{7}), (\bar{0}, 6, 7, \bar{11}), (\bar{0}, 1, 5, 7), (\bar{0}, 2, 3, 9), (\bar{0}, 1, 3, 6) \pmod{15}$;
20	$(\infty, 0, 2, 6), (\overline{\infty}, 0, 13, 16), (\bar{0}, 3, \bar{5}, 11), (\bar{0}, 3, \bar{10}, 16), (\bar{0}, 4, 8, \bar{12}), (\bar{0}, 5, 12, 13), (\bar{0}, 2^{5+a}, 2^{9+a}, 2^{7+a}), a \in \{1, 2, 3, 4\}, \pmod{19}$;
35	$35 = 5 \times 7$, Theorem 3.
36	$36 = 5(8-1)+1$, Theorem 3.
40	$40 = 8 \times 5$, Theorem 3.
70	$70 = 7 \times 10$, Theorem 3.
71	Prime, Theorem 6.
75	$75 = 5 \times 15$, Theorem 3.
76	$76 = 15(6-1)+1$, Theorem 3.

Before we proceed to the case $\lambda = 12$ we establish the existence of a few more $BRD(v, 4, 6)$:

Theorem 8 : *There exist $BRD(v, 4, 6)$ for $v \in \{12, 14, 18, 22, 24, 32, 33, 38\}$.*

Proof: Use Table 2 developing the initial blocks indicated.

TABLE 2.

number of treatments	construction
12	$(\infty, 0, \bar{3}, 5), (\infty, 0, \bar{4}, \bar{7}), (3, \bar{4}, 5, 9), (1, 3, 4, 5), (\bar{1}, 3, 4, \bar{8}), (1, \bar{3}, 4, \bar{9}) \pmod{11}$;
14	$(\infty, 0, 1, 3), (\infty, 0, 2, 5), (\bar{0}, 2, 4, 8), (\bar{0}, 2, 3, 7), (\bar{0}, 1, 2, \bar{6}), (\bar{0}, 1, 5, \bar{8}), (\bar{0}, 1, 4, \bar{7}) \pmod{13}$;
18	$(\infty, 0, 3, 4), (\infty, 0, 5, 7), (\bar{0}, 1, 5, \bar{8}), (0, \bar{6}, 8, 13), (\bar{0}, 5, \bar{6}, 8), (\bar{0}, 5^a, 5^{a+1}, 5^{a+2}), a \in \{2, 3, 4, 6\}, \pmod{17}$;
22	$(\infty, 0, 2, 6), (\infty, 0, 9, 10), (0, \bar{6}, \bar{7}, 16), (0, \bar{4}, \bar{6}, 14), (\bar{0}, 1, \bar{3}, 10), (\bar{0}, 1, 4, 9), (\bar{0}, 2, 5, 12), (\bar{0}, 4, 8, 15), (\bar{0}, 2, 3, 8), (\bar{0}, 1, 3, 16), (\bar{0}, 3, 7, 16), \pmod{21}$;
24	$(\infty, 0, 4, 12), (\infty, 0, 4, 10), (\bar{0}, \bar{1}, 2, 5), (\bar{0}, 2, 5, 10), (\bar{0}, 2, 4, 10), (\bar{0}, 3, \bar{7}, 10), (\bar{0}, 5^a, 5^{a+1}, 5^{a+2}), a \in \{5, 6, 7, 8, 10\}, \pmod{23}$;
32	$(\infty, 0, 2, 6), (\infty, 0, 1, 8), (\bar{0}, 3, \bar{4}, 15), (\bar{0}, \bar{1}, 6, 14), (\bar{0}, 3, \bar{7}, 9), (\bar{0}, 9, 12, 27), (0, 3^a, 3^{a+1}, 3^{a+2}), a \in \{2, 4, 5, 7, 8, 10, \dots, 14\}, \pmod{31}$;
33	$33 = 8(5-1)+1$, Theorem 3.
38	$(\infty, 0, 2, 12), (\infty, 0, 1, 17), (\bar{0}, 4, 17, 33), (\bar{0}, 10, \bar{12}, 13), (\bar{0}, 3, \bar{8}, 16), (\bar{0}, 4, 11, 12), (\bar{0}, 2^a, 2^{a+1}, 2^{a+2}), a \in \{1, 4, 5, 7, 8, 9, 11, \dots, 17\}, \pmod{37}$.

We use Hanani's theorem quoted from Hall [8, p. 248] to establish the result for BRD($v, 4, 12$).

Theorem 9 (Hanani) : Let $u \geq 4$ then $u \in B(K_4^2, 1)$ where

$$K_4^2 = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 22, 23\}.$$

Now we can show

Theorem 10 : A BRD($v, 4, 12$) exists for all $v \geq 4$.

Proof: We note any four distinct rows of an Hadamard matrix of order 12 gives the result for $v = 4$. Thus, just as in the remark after Theorem 5, it is merely necessary to show the existence of a BRD($u, 4, 12$) for $u \in K_4^2$.

Taking two copies of the BRD($u, 4, 6$) given above gives the result immediately.

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