GENERALISED BHASKAR RAO DESIGNS OF BLOCK SIZE FOUR

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Sydney, 2006, Australia

ABSTRACT. In this paper, it is shown that when $v \geq 40$, $\lambda = t|G|$ and $t > 1$ the necessary conditions

$$\lambda \equiv 0 \pmod{|G|}$$
$$\lambda(v-1) \equiv 0 \pmod{3}$$
$$\lambda v(v-1) \equiv 0 \pmod{12}$$

are sufficient for the existence of a generalised Bhaskar Rao design GBRD($v,b,r,4,\lambda;G$) for the elementary abelian group, G, of each order $|G|$. Sufficiency is established for most other cases with $t > 1$, subject to the extra condition

$$|G| \equiv 2 \pmod{4},\ v = 4 \Rightarrow t \text{ even}.$$

Substantial partial results are obtained in the case $t = 1$.

Introduction

Bhaskar Rao designs with elements 0, ±1 have been studied by a number of authors including Bhaskar Rao [1,2], Seberry [34], Singh [37], Sinha [38], Street [40], Street and Rodger [41] and Vyas [42]. Bhaskar Rao [1] used these designs to construct partially balanced designs and his technique was improved by Street and Rodger [41]. Another technique for studying partially balanced designs has involved looking at generalized orthogonal matrices which have elements from elementary abelian groups together with the element 0. Matrices with group elements as entries have been studied by Berman [3,4], Butson [5,6], de Launey [8], de Launey and Seberry [10], Delsarte and Goethals [11], Drake [13], Lam and Seberry [20], Rajkundlia [32,33,35], Shrikhande [36], and Street [39].

Recently Mackenzie and Seberry [21] have shown that such designs give maximal or the best known codes over ternary and q-ary alphabets.

* Research supported by a grant from the Australian Computer Research Board.

CONGRESSUS NUMERANTIUM, VOL. 41 (1984), pp. 229-294

Suppose we have a matrix W with elements from an elementary abelian group $G = \{h_1, h_2, \ldots, h_g\}$, where $W = h_1A_1 + h_2A_2 + \ldots + h_gA_g$; here $A_1, \ldots, A_g$ are vxb (0,1) matrices, and the Hadamard product $A_i * A_j$ $(i \neq j)$ is zero. Suppose $(a_{i1}, \ldots, a_{ib})$ and $(b_{j1}, \ldots, b_{jb})$ are the ith and jth rows of w; then we define ww^+ by

$$(WW^+)_{ij} = (a_{i1}, \ldots, a_{ib}) \cdot (b_{j1}^{-1}, \ldots, b_{jb}^{-1})$$

with $\cdot$ designating the scalar product. Then W is a *generalized Bhaskar Rao design or GBRD if*

(i) $$WW^+ = rI + \sum_{i=1}^{m} (c_iG)B_i$$

(ii) $$N = A_1 + \ldots + A_g \text{ satisfies } NN^T = rI + \sum_{i=1}^{m} \lambda_i B_i,$$

that is, N is the incidence matrix of a PBIBD(m), and (c_iG) gives the number of times a complete copy of the group G occurs.

Such a matrix will be denoted by $GBRD_G(v,b,r,k;\lambda_1, \ldots, \lambda_m; c_1, \ldots, c_m)$. In this paper we shall only be concerned with $m = 1$, $c = \lambda/g$, and $B_1 = J - I$. In this case N is the incidence matrix of a PBIBD(1), that is, a BIBD. Hence, the equations become:

(i) $$WW^+ = rI + \frac{\lambda G}{g}(J-I)$$

(ii) $$NN^T = (r-\lambda)I + \lambda J.$$

Thus W is a $GBRD_G(v,b,r,k,\lambda)$. Since $\lambda(v-1) = r(k-1)$ and $bk = vr$, we sometimes use the notation $GBRD(v,k,\lambda;G)$.

These matrices are generalizations of generalized weighing matrices and may be used in the construction of PBIBDs.

We use the following notation for initial blocks of a GBRD. We say $(a_\alpha, b_\beta, \ldots, c_\gamma)$ is an initial block, when the Latin letters are developed mod n and the Greek subscripts are the elements of the group, which will be placed in the incidence matrix in the positions indicated by the Latin letters. Thus, we place α in the $(i,a-1+i)$th position of the incidence matrix, β in the $(i,b-1+i)$th position, and so on.

We form the difference table of an initial block $(a_\alpha, b_\beta, \ldots, c_\gamma)$ by placing in the position headed by x_δ and by row y_η the element $(x-y)_{\delta\eta^{-1}}$ where $(x-y)$ is mod n and $\delta\eta^{-1}$ is in the abelian group.

A set of initial blocks will be said to form a *GBR difference set* (if there is one initial block) or *GBR supplementary difference sets* (if more than one) if in the totality of elements

$$(x-y)_{\delta\eta^{-1}} \pmod{n, G}$$

each non-zero element a_g, $a \pmod n$, $g \in G$, occurs $\lambda/|G|$ times. Examples of the use of these GBRSDs are given in Seberry [33].

This paper makes continual use of the following theorem.

THEOREM 1.1.1. *(Lam and Seberry) Suppose there exists a* $\text{GBRD}(k,j,\lambda_B;G_B)$ *and*

i) *a* $\text{GBRD}(v,k,\lambda_A;G_A)$, *then there exists a* $\text{BGRD}(v,j,\lambda_A\lambda_B;G_A \times G_B)$;

ii) *a* $\text{BIBD}(v,k,\lambda)$, *then there exists a* $\text{GBRD}(v,j,\lambda\lambda_B;G_B)$;

iii) j *rows of a generalized Hadamard matrix* $\text{GH}(h,H)$, *then there exists a* $\text{GBRD}(k,j,\lambda_B h,G_B \times H)$;

iv) $h \geq j$ *is a prime power, then there exists a* $\text{GBRD}(k,j,\lambda_B h;G_B \times H)$ *where* H *is the elementary abelian group of order* h.

We note that *generalized Hadamard matrices* $\text{GH}(h|G|, G)$ can be regarded as $\text{GBRD}(h|G|, h|G|, h|G|, G)$, and hence used in the above theorem since they exist for $h|G|$ a prime power and other orders (see Street [39], Seberry [31,32], and Dawson [7]). De Launey [8] has obtained some results on the non-existence of generalized Hadamard matrices.

Using results of Hanani and Wilson with those of Lam and Seberry we have:

COROLLARY 1.1.2 *Suppose there exists a pairwise balanced design*

$B[K,\lambda,v]$ *where* $K = \{k_1,\ldots,k_b\}$ *and a* $GBRD(k_j,j,\mu;G_B)$ *for each* $k_i \in K$, *then there exists a* $GBRD(v,j,\lambda\mu;G)$. *Hence*

i) *if* $u \equiv 0$ *or* $1 \pmod 4$, $u \geq 4$ *and there exists a* $GBRD(k,j,\lambda;G)$ *for* $k \in K_4^1 = \{4,5,8,9,12\}$, then there is a $GBRD(u,j,\lambda;G)$;

ii) *if* $u \geq 4$ *and there exists a* $GBRD(k,j,\lambda;G)$ *for all* $k \in K_4^2 = \{4,5,6,7,8,9,10,11,12,14,15,18,19,22,23\}$, *then there is a* $GBRD(u,j,\lambda;G)$;

iii) *if* $u \equiv 1 \pmod 4$ *and there exists a* $GBRD(k,j,\lambda;G)$ *for all* $k \in H_4^4 = \{5,9,13,17,29,33,49,57,89,93,129,137\}$, *then there exists a* $GBRD(u,j,\lambda;G)$;

iv) *if* $u \equiv 1 \pmod 3$ *and there exists a* $GBRD(k,j,\lambda;G)$ *for all* $k \in H_3^3 = \{4,7,10,19\}$, *then there exists a* $GBRD(u,j,\lambda;G)$.

The next result is a slight improvement on the result of Lam and Seberry (1983) where the existence of k-1 mutually orthogonal latin squares was required. The result may be proved by adjusting the matrix in the proof

THEOREM 1.1.3. *Suppose there exists a* $GBRD(u,k,\lambda;G)$ *with a subdesign on* w *points (the values* $w=0$ *and* 1 *are allowed), a* $GBRD(v,k,\lambda;G)$, *and* $k-2$ *mutually orthogonal Latin squares, then there exists a* $GBRD(v(u-w)+w,k,\lambda;G)$ *with subdesigns on* u,w, *and* v *points.*

Remark 1.1.4. In this paper we are interested in the case $k=4$; so we only need a pair of orthogonal latin squares; hence $u-w$ may have any value except 2 or 6.

1.2 Small Generating Sets.

In this section we prove those results on generating sets used in this paper. First, we make some definitions, using the notation in Wilson's paper [45].

<u>Definition 1.2.1.</u> *Let the sets* J,K, *and* L *be sets of integers (not necessarily finite).*

i) A *pairwise balanced design* (PBD[K,v]) is a pair (X,*A*) where X, $|X| = v$, is a set (of *points*) and *A* a class of subsets A, $|A| \in K$, of X (called *blocks*) such that any pair of distinct points of X is contained in exactly one of the blocks of A. If $K = \{k\}$, we write PBD[k,v] instead of PBD[{k},v].

ii) We let $\mathbb{B}(K)$ denote the set of integers v for which there exists a PBD[K,v]. If $L = \mathbb{B}(J)$, then J is said to *generate* L. K is said to be *closed* if $K = \mathbb{B}(K)$. An element ℓ of L is said to be *essential* if $\ell \notin \mathbb{B}(\{m \in L | m < \ell\})$. □

If L is closed and $\ell \in L$ is not essential, then ℓ may be removed from any generating set of L (Proposition 5.1, Wilson [45]). One aim is to find small generating sets for the following $L = \{5,8,9,12,13,...\}$, $\{4,10,13,16,19,...\}$.

<u>Definition 1.2.2.</u> *A group divisible design* (GDD) *on* v *points is a triple* (X,*S*,*A*) *where*

i) X is a set (of *points*),

ii) *S* is a class of non-empty subsets of X (called *groups*) which partition X,

iii) *A* is a class of subsets of X (called *blocks*), each containing at least two points,

iv) no block meets a group in more than one point,

v) each pair {x,y} of points not contained in a group is contained in precisely one block. □

There is a fundamental composition construction for GDD's.

<u>Construction 1.2.3. (Wilson [45]).</u>

Let (X,*S*,*A*) be a GDD and let a positive integral weight s_x be assigned to each point $x \in X$. Let $(S_x : x \in X)$ be pairwise disjoint sets with $|S_x| = s_x$. With the notation $S_Y = \bigcup_{x \in Y} S_x$ for $Y \subseteq X$, put

$$X^* = S_X, \qquad S^* = \{S_G : G \in \mathcal{S}\}.$$

For $A \in \mathcal{A}$, we have a natural partition $\Pi_A = (S_A, \{S_x | x \in A\})$; we suppose that for each block $A \in \mathcal{A}$, a GDD

$$(S_A, \{S_x | x \in A\}, \mathcal{B}_A)$$

is given, and put $\mathcal{A}^* = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A$. Then $(X^*, \mathcal{S}^*, \mathcal{A}^*)$ is a GDD.

There is a special class of GDDs.

<u>Definition 1.2.4</u>. *A transversal design* TD(n,t) is a GDD with n groups, each of size t, and block size n. □

THEOREM 1.2.5. *(MacNeish)*

If $n = q_1 \cdot q_2 \cdot \ldots \cdot q_r$ *is the prime decomposition of* $n > 1$, *then there exists a* TD(k,n) *whenever*

$$k \leq 1 + \min_{1 \leq i \leq r} q_r.$$

□

In constructing generating sets, we make considerable use of transversal designs.

LEMMA 1.2.6. *Let* T *be the set of integers* t *for which there exists a* $TD(v_0+1,t)$. *Then there exists an integer* $\sigma(v_0)$ *such that for all* $t \in T$ *there exists a* $t' \in T$ *with*

$$0 < t' - t \leq \sigma(v_0)$$

Proof. Let $p_1, p_2, \ldots, p_r$ be the primes less than v_0. By 1.2.5, if $t \equiv 1 \pmod{p_1 p_2 \cdots p_r}$, then a $TD(v_0+1,t)$ exists. So we may take $\sigma(v_0) = p_1 p_2 \cdots p_r$. □

Remark 1.2.7. In general, we can do better than $p_1p_2\ldots p_r$. For any integer $v_0 > 1$, define $\tau(v_0)$ to be the length of the longest sequence of successive numbers divisible by at least one prime less than v_0. Then $\tau(v_0) \le p_1p_2\ldots p_r$ and we may take $\sigma(v_0) = \tau(v_0)+1$. We note that $\tau(4) = \tau(5) = 3$ (check by considering the residues mod 6). □

We now deal with the set $L = \{5,8,9,12,13,16,17,\ldots\}$ by slightly altering a construction appearing in Wilson's paper [Lemma 5.1, 45].

LEMMA 1.2.8. *Suppose there exists a* GDD *on v points, with block sizes from* $\{5,6\}$. *Suppose further that the* GDD *has at least two groups and that all groups have at least two points. Then* $4v$ *is not essential in* $B(\{5\} \cup \{8,12,16,20,\ldots\})$.

Proof. We can produce two GDDs whose groups have size 4 and whose blocks have size 5 by deleting a point from each of the designs PBD[5,25] and PBD[5,21]. The first GDD has 6 groups and the second has 5 groups. We now use these two GDD s together with the one on v points in Construction 1.2.3 to produce a GDD on 4v points. This new GDD will have block size 5 and all its group sizes will be divisible by 4. We now produce a $\mathrm{PBD}[\{5\} \cup \{8,12,16,\ldots,4(v-1)\},4v]$. Let $(X,\mathcal{S},\mathcal{A})$ be the GDD on 4v points. We define the required PBD, $(Y,\mathcal{C})$ as follows:

$$Y = X \quad \text{and} \quad \mathcal{C} = \mathcal{A} \cup \{G \mid G \in \mathcal{S}\}.$$

Note that it was necessary that our GDD on v points had all group sizes greater than one;otherwise, some of new blocks, $G \in \mathcal{S}$, would contain only 4 points. □

THEOREM 1.2.9.

$$\{4v \mid v\ge 1\} \subset \mathbb{B}(\{5,9,13,17,29\} \cup \{4v \mid v=2,3,4,5,6,7,8,11,12,13,17,19,21,22,23,31,33\}).$$

Proof. Given a TD(6,t), we may construct a GDD satisfying Lemma 1.2.8 provided $t > 1$ and

$$5t \le v \le 6t \quad \text{with} \quad v \ne 5t + 1.$$

Now $\tau(5) = 3$; so, if a TD(6,t) exists, then a TD(6,t') exists for some t' satisfying $0 < t' - t \le 4$. If $t \ge 22$ then

$$6t \ge 5(t+4) + 2.$$

So GDDs satisfying Lemma 1.2.8 may be constructed for all $v \ge 5.23 + 2 = 117$. Hence 4v is not essential in $\mathbb{B}(\{5\} \cup \{4v \mid v \ge 2\})$ for $v \ge 117$. We can rule out the following cases as well

$t = 5$	$25 \le v \le 30$	$v \ne 26$
$t = 7$	$35 \le v \le 42$	$v \ne 36$
$t = 8$	$40 \le v \le 48$	
$t = 9$	$45 \le v \le 54$	
$t = 11$	$55 \le v \le 66$	$v \ne 56$
$t = 13$	$65 \le v \le 78$	
$t = 16$	$80 \le v \le 96$	$v \ne 81$
$t = 17$	$85 \le v \le 102$	
$t = 19$	$95 \le v \le 114$	
$t = 23$	$115 \le v \le 138$	$v \ne 116$

Thus

$$\mathbb{B}(\{5\} \cup \{4v \mid v = 2,3,\ldots\}) = \mathbb{B}(\{4v \mid v=2,3,\ldots,24,26,31,32,33,34,36,56,79,81,116\} \cup \{5\}).$$

We may eliminate many of these by allowing the block sizes of our PBDs on 4v points to come from {5,9,13,17,29} as well as from smaller multiples of 4. Table 1 (Appendix) shows how we eliminate certain values of v from the small set above to obtain the result. □

COROLLARY 1.2.10 $\{4v, 4v+1 \mid 4v, 4v+1 \ge 5\} = \mathbb{B}(\{5,9,13,17,29,33,49\} \cup \{4v \mid v=2,3,\ldots,8,11,12,13,17,21,22,23,31,33\})$.

Proof. By a theorem of Wilson [Theorem 5.1(iii), 45]

$$\{4v+1 \mid v \ge 1\} = \mathbb{B}(\{5,9,13,17,29,33,49,57,89,93,129,137\}).$$

57,89,93,129, and 137 may be removed if we allow block sizes 8,12, and 16. These constructions are given as follows:

57	Use the SBIBD(57,8,1).
89	Use TD(8,11). Add one row to form a PBD[{12,8},89].
93	See Lemma 1.3.4.
129	Add a row to a TD(16,8).
137	Take 7 rows from a group of a TD(9,16) and complete to form a PBD[{8,9,16},137].

□

We now find a small generating set for $\mathbb{B}\{4,10,13,16,19,\dots\}$.

THEOREM 1.2.11. $\mathbb{B}\{4,10,13,16,19,\dots\} = \mathbb{B}\{4,10,19,22,34,43,55,79,199\}$

Proof. Proceeding as before, see [Lemma 5.1,45], we find that $3v+1$ is not essential if there exists a TD(5,t) and

$$4t \le v \le 5t \quad \text{with} \quad v \ne 4t+2$$

Note that we rule out $v=4t+2$ because the PBD[K,3v+1] constructed in [Lemma 5.1, 45] would have a column of size 7.

Since $\tau(4) = 3$, we find that, provided $v > 78$, $3v+1$ is not essential. We may rule out the following v:

$t = 4$	$16 \le v \le 20$	$v \ne 18$
$t = 5$	$20 \le v \le 25$	$v \ne 22$
$t = 7$	$28 \le v \le 35$	$v \ne 30$
$t = 8$	$32 \le v \le 40$	
$t = 9$	$36 \le v \le 45$	
$t = 11$	$44 \le v \le 55$	$v \ne 46$
$t = 13$	$52 \le v \le 65$	
$t = 16$	$64 \le v \le 80$	$v \ne 66$
$t = 17$	$68 \le v \le 85$	

So our small set is

$$\{4,10,13,16,19,22,\dots,46\} \cup \{55,67,79,82,91,139,199\}.$$

We rule out many of these values in Table 2 (Appendix) to obtain the result.

□

We now obtain a theorem which will prove most useful when either designs do not exist, or cannot be constructed, for some, possibly essential, element v of a closed set. For example, GBRD($v,4,6;Z_2$) is known for all v where $5 \le v \le 40$ except for $v = 28,34$, and 39.

Theorem 1.2.14 (proved below) will be used in Section 4.3(Lemma 4.3.3) to prove

$$\{v|v \geq 5,\ v \neq 28,34,39\}$$
$$= \mathbb{B}(\{v|5 \leq v \leq 25\} \cup \{v|v \equiv \pm 1 (\text{mod } 6) \text{ and } 25 \leq v < 130\}).$$

Hence, the designs in question exist for all $v \geq 5$ except possibly for $v = 28,34,39$.

It is known that 28 is essential (see Section 1.3), while it remains undecided whether 34 and 39 are essential.

Before proving Theorem 1.2.14, we rephrase some standard results due to Hanani and Wilson.

Definition and Notation 1.2.12. Let S and K be sets of positive integers. Define

$$[v_o]S \oplus K = \{v|v = v_o s + k \text{ where } s \geq k\}.$$

Let s and t be integers, and let ${}_sS^t$ denote

$$\{v \mid s \leq v \leq t\} \cap S\ ;$$

let ${}_sS$ and S^t respectively denote the sets

$$\{v|v \geq s\} \cap S \quad \text{and} \quad \{v|0 \leq v \leq t\} \cap S.$$

THEOREM 1.2.13. *(Wilson, Hanani) Let* v_o *be a positive integer. Let* S *be a set of positive numbers such that for all* $t \in S$ *there exists a* $TD(v_o+1,t)$. *Let* K *be a set of positive integers containing the integers* v_o *and* v_o+1. *Then*

$$\mathbb{B}(S \cup K) \supseteq \{[v_o]\ S \oplus K\}.$$

Further, if $s+1 \in S \cup K$, *then* $\mathbb{B}(S \cup K) \supseteq \{v_o s,\ v_o s+1\}$.

Proof. Suppose $v \in [q]\ S \oplus K$; then $v = qs + k$ where $s \in S$, $k \in K$ and $s \geq k$. Form a GDD,$(X,\underline{S},\underline{A})$, on v points with block sizes from $\{q+1,q\}$ by removing all but k points from a group of a $TD(q+1,s)$. A PBD$(\{s,k,q+1,q\},v)$, $(X,\underline{B})$, may be formed by letting $\underline{B} = \underline{A} \cup \underline{S}$. □

THEOREM 1.2.14. *Let* $v_o \geq 2$ *be an integer. Let* S *be an increasing infinite sequence such that for all* $t \in S$ *there exists a* $TD(v_o+1,t)$.

Let K *be a set of positive integers containing* v_o *and* v_o+1. *Let* $k_o = \min_{k\varepsilon K} \{k\}$ *and suppose there exists a* $TD(v_o+1, t_o)$ *for some* t_o *not necessarily in* S. *Let* T *be the set of elements* $t \varepsilon S$, $t > v_o t_o + k_o$, *for which there does not exist a pair* $t' \varepsilon S$ *and* $k \varepsilon K$ *such that* $t' \geq t_o$ *and* $t = v_o t' + k$. *Then*

$$\mathbb{B}\left({}_{t_o}S^{v_o t_o + k_o - 1} \cup T \cup K \cup \{t_o\}\right) \supseteq [v_o] S \oplus K,$$

Finally let U *be the set of all* $v > v_o t_o + k_o$ *such that there does not exist a pair* $t' \varepsilon S$, $k \varepsilon K$ *satisfying* $t' > t_o$ *and* $v = v_o t' + k$; *then*

$$\mathbb{B}\left({}_{t_o}S^{v_o t_o + k_o - 1} \cup U \cup K \cup \{t_o\}\right) \supseteq \{v \geq v_o t_o + k_o\} .$$

Proof. Let ${}_{t_o}S = \{s_n\}_{n=0}^{\infty}$ and let N be the least integer n such that $s_{n+1} \geq v_o t_o + k_o$. Put $P_n = \{s_0, s_1, \ldots, s_{N+n}\}$. Note that ${}_{t_o}S^{v_o t + k_o - 1} = P_o$.

Now suppose

$$[v_o]P_n \oplus K \subseteq \mathbb{B}(P_o \cup T \cup K \cup \{t_o\}).$$

It is shown that

$$[v_o]P_{n+1} \oplus K \subseteq \mathbb{B}(P_o \cup T \cup K \cup \{t_o\}).$$

By Theorem 1.2.13, it is sufficient to show that

$$s_{N+n+1} \varepsilon \mathbb{B}(P_o \cup T \cup K \cup \{t_o\}).$$

If $s_{N+n+1} \varepsilon T$, then the result is immediate; if $s_{N+n+1} \notin T$, then there exist $t \varepsilon S$ and $k \varepsilon K$ such that $t > t_o$ and $s_{N+n+1} = v_o t + k$. Since $v_o \geq 2$, $t < s_{N+n+1}$; hence

$$s_{N+n+1} \subseteq [v_o]P_n \oplus K \subseteq \mathbb{B}(P_o \cup T \cup K \cup \{t_o\}).$$

To complete the induction and the proof of the first assertion of the theorem, one notes that by Theorem 1.2.13

$$[v_o]P_o \oplus K \subseteq \mathbb{B}(P_o \cup T \cup K \cup \{t_o\}).$$

Finally, since $T \subseteq U$ and

$$\{v \geq v_o t_o + k_o\} = ([v_o]S \oplus K) \cup U,$$

$$\{v \geq v_o t_o + k_o\} \subseteq \mathbb{B}({}_{t_o}S^{v_o t_o + k_o} \cup U \cup K \cup \{t_o\}).$$ □

Remark 1.2.15. The theorem is still true if S is finite, but then $T \cup K$ is infinite and generally unmanageable. Besides, nothing is gained over Theorem 1.2.13. Theorem 1.2.14 is particularly useful when there exists an integer $\tau(S)$ for an infinite sequence S with the property that, if $t \in S$, then there exists a $t' \in S$ such that $t' > t$ and $t'-t \leq \tau(S)$. If there exists such an integer, then for all $t > v_o t_o + k_o$ there exists a $t' \in S$, where $v_o t + k_o < t < v_o t + v_o \tau(S) + k_o$. It then becomes easy to determine T and $[v_o]S \oplus K$. By Remark 1.2.7, such sequences, S, satisfying the conditions of the theorem are readily available. □

The following result, needed in 7.3, is an application of Theorem 1.2.14.

LEMMA 1.2.16. *Let* $L = \{5,6,7,9,10,11,12,13,14,15,16,18,19,20\}$ *and* $S = \{v \mid v \equiv \pm 1 \pmod 6\}$. *Then*

$$\mathbb{B}(L \cup {}_{25}S^{130} \cup \{28\}) \supseteq \{v \geq 130\}.$$

Proof. PBD(K,v) will be needed for v = 21,25,47,52,53,54, and 63. There exists a BIBD(21,5,1) and a BIBD(25,5,1), and the TD(7,9), TD(6,9), and TD(7,7) may be adjusted to obtain the designs for v = 47,52,54, and 63. The design on 53 points is obtained from a TD(7,9) by removing 8 points from the first group and 2 from the next.

We now apply Theorem 1.2.14 with $v_o = 5$, $t_o = 25$, $S = \{v \mid v \equiv \pm 1 \pmod 6\}$, and $K = L \cup \{21,25,28\}$. To do this it is necessary to determine U. By Remark 1.2.15, for all $v \geq 130$ there exists a $t \in {}_{25}S$ such that $v = 5t+r$ where $0 \leq r \leq 25$, and if $r \in K$ then $v \notin U$. So if $v \in U$ then $v = 5t+f$ where $f = \{8,17,22,23,24\}$ and $t \equiv \pm 1 \pmod 6$. If $t \equiv -1 \pmod 6$, then $(t-4)$ and $(t+2) \in S$; if $t \equiv 1 \pmod 6$, then $(t-2)$ and $(t-6) \in S$. So the values below are not in U.

$t \equiv -1 \pmod 6$:

$$5t + 8 = 5(t-4) + 28 \qquad t \geq 35$$

$$5t + (17,22,23,24) = 5(t+2)+(7,12,13,14) \qquad t \geq 25$$

$t \equiv 1 \pmod 6$:

$$5t + 8 = 5(t-2) + 18 \qquad t \geq 31;$$

$$5t + 17 = 5(t-6) + 47 \qquad t \geq 55;$$

$$5t + (22,23,24) = 5(t-6) + (52,53,54) \qquad t \geq 61.$$

So one may draw up a table of the undisposed of values of v.

f \ v	25	29	31	37	43	49	55	61
8	133	153	-	-	-	-	-	-
17	142	-	172	202	232	262	292	-
22	147	-	177	207	237	267	297	327
23	148	-	178	208	238	268	298	328
24	149	-	179	209	239	269	299	329

By removing points from the first group of each of TD(11,13), TD(12,13), TD(10,19), TD(11,19), TD(10,25), TD(11,25), TD(16,19), TD(6,63), one can construct PBD($\mathbb{B}$(L $\cup$ ${}_{25}S^{130}$ $\cup$ {28}), v) for all v except v $\in$ {133,147,153,179,207,267}. Now there exists an SBIBD(133,12,1) and the required PBDs can be obtained for v = 153 and 207 by removing rows from TD(10,16) and TD(13,16) respectively. The three remaining values are dealt with below.

v	Construction
147	Remove 1 point from 2 distinct groups and 7 points from a third group of TD(12,13).
179	Remove 10 points from one group and 9 points from another of TD(12,16)
267	Remove 15 points from one group and 18 points from another of TD(12,25).

Since all elements of U are contained in $\mathbb{B}(L \cup \{28\} \cup {}_{25}S^{130})$, $\mathbb{B}(L \cup \{28\} \cup {}_{25}S^{130}) \supseteq \mathbb{B}(L \cup U \cup \{28\} \cup {}_{25}S^{130}) \supseteq \{v \mid v \geq 130\}$.

1.3 Some Pairwise Balanced Designs

We now prove the existence of some PBD designs we use later. First we prove that 28 is essential in the set $\{v \geq 5\}$.

LEMMA 1.3.1 *Suppose there exists a* PBD[K,v] *and that for some* $k \in K$, $k < v$, *there is at least one block of size* k. *Then, letting* $M = \min_{m \in k}\{m\}$ *we have,*

$$k(M-1) \leq v-1.$$

Proof. Since $k < v$, there is at least one point which is not in the block of k elements. Removing this point produces a GDD with groups of size $\geq M-1$. Since only one point of any block may be contained in any one group, $k(M-1) \leq v-1$. □

We include a similar lemma which is not needed for our result on 28.

LEMMA 1.3.2. *Suppose there exists a* PBD[K,v] *with two (or more) blocks of size* $k < v$; *then*

$$(k-1)(M-1) \leq v-k.$$

Proof. Let B_1 and B_2 be two blocks of size k. Remove B_1 from the design. This produces a GDD on $(v-k)$ points with at least k-1 groups of size $\geq M-1$. Thus $(k-1)(M-1) \leq v-k$. □

This lemma ensures that any PBD[K,34] has at most one block of size 8, while Lemma 1.3.1 excludes all block sizes $9 \leq k < 34$.

THEOREM 1.3.3. $v = 28$ *is an essential element of the set* $v \geq 5$.

Proof. If 28 is not essential, then there exists a PBD[K,28] with $k < 28$ for all $k \in K$. Now $M \geq 5$; so $4k \leq 27$. Hence, by lemma 1.3.1, $K = \{5,6\}$. Thus we need only disprove the non-existence of a PBD[{5,6},28].

Suppose there exists such a design. Let s be the number of 1s in the first row. Let a_5 of these s columns have precisely five 1s and a_6 have precisely six 1s.

$$s = a_6 + a_5 .$$

Now the first row has inner product 1 with each other row. So there are precisely $27 = v-1$ 1s under the s 1s in the first row. Counting column by column, there are $4a_5+5a_6$ 1s under the 1s in the first row.

$$27 = 4a_5 + 5a_6 .$$

The only possible solution to this equation is $a_5 = 3$ and $a_6 = 3$. So $s=6$. We can do this for any row.

We can now determine the number of columns. Because each row has precisely six 1s, there are precisely $6 \cdot 28 = 168$ 1s in the entire design. Now the first six rows are without loss of generality

```
1 1 1 1 1 1 1
1           1 1 1 1 1
1                     1 1 1 1 1
1                               1 1 1 1 1
1                                         1 1 1 1 1
1                                                   1 1 1 1 1
```

So there are at least 18 columns containing five 1s and 13 columns containing six 1s. The number of 1s in these columns alone is 168; so there can be no more columns.

Because this design has index 1, there are exactly $\frac{1}{2}(28.27)=378$ occurrences of a pair of 1s in a column. But we have determined that there are 13 columns with six 1s and 18 with 5. Counting the pairs of 1s column by column, one obtains only $13.\ 6.\ 5/2 + 18.5.4/2 = 375$. It follows that no PBD($\{5,6\}$,28) exists. □

LEMMA 1.3.4. $93 \in \mathbb{B}\{5,8,9,12\}$.

Proof. Take 9 rows of a $GH(11,Z_{11})$ and obtain the usual GDD by replacing the elements by their right regular matrix representations. Delete the first 7 rows. This gives a GDD on $92=8\times 11+4$ points with block sizes $k=8,9$. The group sizes are 8 of size 11 and 1 of size 4. Put in columns of 1s beside the groups to give a PBD($\{8,9,4,11\}$,92,1).

Place a row of 9 ones above these columns. Now we have a PBD({8,9,5,12},93,1) as required. □

LEMMA 1.3.5. *There exist:* PBD({11,12},44,3), PBD({10,11,12},43,3), PBD({9,11,12},42,3), PBD({8,9,11,12},35,3), PBD({8,9,10,12},34,3), PBD({7,8,9,12},33,3), PBD({7,8,9,11,12},32,3), PBD({7,8,9,10,12},31,3), PBD({7,9,12},30,3), PBD({6,7,9,11},29,3), PBD({6,7,9,10},28,3), PBD({6,9},27,3). *Also* 36,...,41 $\in \mathbb{B}(\{9,10,11,12\},3)$.

Proof. We use the following (corrected) BIBD(45,12,3) given by J. Wallis [43] where

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad L=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\qquad M=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

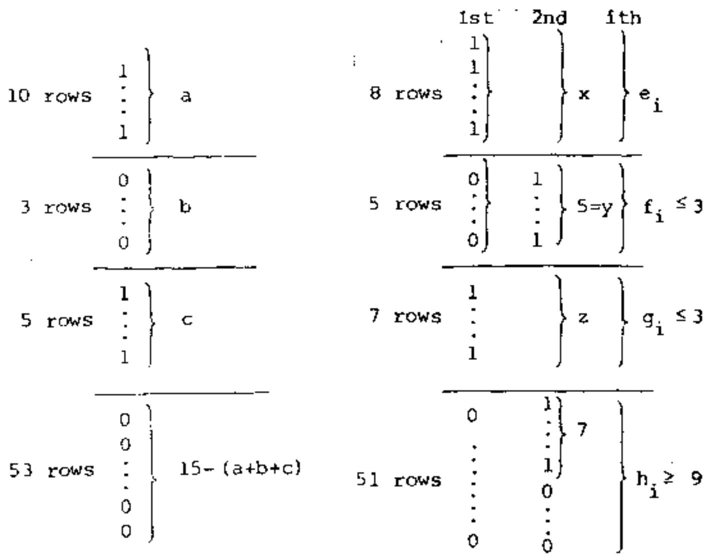
$$J = I + L + M ,$$

$$Y=\left[\begin{array}{ccc|ccc|ccc|ccc|ccc}
J&&&&&&I&I&I&I&I&I&I&I&I\\
&J&&&&&L&L&L&M&M&M&I&I&I\\
&&J&&&&M&M&M&L&L&L&I&I&I\\ \hline
&&&J&&&I&L&M&I&M&L&I&L&M\\
&&&&J&&L&M&I&M&L&I&I&L&M\\
&&&&&J&M&I&L&L&I&M&I&L&M\\ \hline
I&L&M&I&M&L&J&&&&&&I&M&L\\
L&M&I&M&L&I&&J&&&&&I&M&L\\
M&I&L&L&I&M&&&J&&&&I&M&L\\ \hline
I&M&L&I&L&M&I&M&L&J&&&&&\\
M&L&I&L&M&I&I&M&L&&J&&&&\\
L&I&M&M&I&L&I&M&L&&&J&&&\\ \hline
I&I&I&I&I&I&&&&I&L&M&J&&\\
M&M&M&L&L&L&&&&I&L&M&&J&\\
L&L&L&M&M&M&&&&I&L&M&&&J
\end{array}\right]$$

In each case the result for i, i = 27,...,44 is obtained by taking rows 46-i,...,45 of Y. □

LEMMA 1.3.6. *There exist:* PBD({12,13,14,15},68,3), PBD({5,9,10,11,12,13,14,15},58,3) *and* PBD({7,9,10,...,15},58,3).

Proof. We first construct the PBD({5,9,10,...,15},58,3). Haemers [16] has constructed an SBIBD(71,15,3). Construct its incidence matrix and rearrange the first column until the matrix appears as in Figure (I).



(I) (II)

Now the inner product of every column with the first is 3. So if the number of 1s in column j of the first figure and the first 10 rows is a, in rows 11 to 13 is b, in rows 14 to 18 is c, and in the last 53 rows is 15-(a+b+c), we have

$$a+c = 3, \quad 0 \le a,\ c \le 3, \quad 0 \le b \le 3.$$

So the number of ones in the last 58 rows, 5 and 15-a-b gives the PBD({5,9,...,15},58,3).

Now we construct a PBD({7,9,10,...,15},58,3). Arrange the incidence matrix of Haemers' design as in Figure (II). Again the inner product of every column with the first is 3. Let x,y,z be the number of ones in the second column of the second figure and with $y = 5$, $x+z = 3$. Let e_i, f_i, g_i, h_i be the number of ones in the ith column in the respective positions. Then $e_i+g_i = 3$ and $f_i \le 3$; so $h_i = 15-e_i-f_i-g_i \ge 9$. Hence the last 58 rows have 7,12+z or $g_i+h_i \ge 9$ ones.

The design on 68 points is obtained by removing three rows. □

LEMMA 1.3.7. *17* ε $B(\{5,6,8,9\},3)$; *22, 23* ε $\mathbb{B}(\{7,8,9\},3)$; *and* 24 ε $B(\{8,9\},3)$.

Proof. For $i = 22,23,24$, consider rows $26-i,\ldots,25$ of the following arrangement of the SBIBD(25,9,3) from J. Wallis [43], where J,I,L,M are as in Lemma 1.3.5, $e = (1,1,1)$, $\varepsilon = e^T$.

$$X = \begin{bmatrix}
 & & & e & e & c & & & \\
 & e & & e & & & & & e \\
 & e & & & e & & e & & \\
 & e & & & & e & & e & \\
 & & & I & I & I & J-I & J-I & J-I \\
\epsilon & \epsilon\quad\quad & \epsilon\quad\quad & I & I & I & I & I & I \\
\epsilon & \quad\epsilon\quad & \quad\epsilon\quad & I & M & L & I & M & L \\
\epsilon & \quad\quad\epsilon & \quad\quad\epsilon & I & L & M & I & L & M \\
 & I & J-I & I & & J-I & J-I & & I \\
 & I & J-I & J-I & I & & I & J-I & \\
 & I & J-I & & J-I & I & & I & J-I
\end{bmatrix},$$

For 17, consider rows 2,3,11,...,25 of this same SBIBD. □

LEMMA 1.3.8. *Let* $r = \lambda+t$. *Suppose there exists a* BIBD(v,b,r,k,λ) *and a* GBRD$(v,v,gt;G)$ *where* $g = |G|$. *Then there exists a* PBD$(\{v,gk\},vg,\lambda+t)$ *and a* PBD$(\{v,v-1,kg,kg-s\},vg-s,\lambda+t)$ *for* $s = 1,2,\ldots,g$.

Proof. Let X be the GBRD with entries replaced by their permutation matrix representations. Then X is a GDD$(vg,v,\lambda_1 = 0,\ \lambda_2 = t,\ m=g)$. Let Y be the matrix obtained from the BIBD by replacing its zeros and ones by the $g\times 1$ matrices of zeros and ones respectively. Then Y is a GDD$(vg,gk,\lambda_1 = r,\lambda_2 = \lambda,m=g)$. Thus $Z = [X,Y]$ is a PBD$(\{v,gk\},vg,\lambda+t)$. Removing the first s rows from Z gives the result. □

Now for all $q = 2^t$, there exists an SBIBD$(q^2+q+1,q+1,1)$. Also, for all prime powers p^r there exists a GH$(2p^r;EA(p^r))$; so if $2p^r \geq q^2+q+1$, there exists a GBRD$(q^2+q+1, q^2+q+1, qp^r;EA(p^r))$, because q is even Thus, we have

COROLLARY 1.3.9. *Let* $q = 2^s$ *and* p^r *be a prime power such that* $2p^r \geq q^2+q+1$. *Then there exists a* PBD$(\{q^2+q+1,(q+1)p^r\}, (q^2+q+1)p^r, q+1)$ *and a* PBD$(\{q^2+q+1,q^2+q,(q+1)p^r,(q+1)p^r-s\}, (q^2+p+1)p^r-s,q+1)$ for $s = 1,2,\ldots, p^r$.

Example 1.3.10 Let $q = 2$. Then there exists a BIBD(7,3,1) and a GH$(2p^r,EA(p^r))$ for all prime powers p^r. Thus we have PBD$(\{7,3p^r\},7p^r,3)$ and PBD$(\{7,6,3p^r,3p^r-s\}, 7p^r-s,3)$.

Thus $p^r = 4$ gives PBD({7,12},28,3) and PBD({7,6,12,11},27,3).

$p^r = 5$ gives PBD({7,15},35,3) and PBD({7,6,15,15-s},35-s,3) $s = 1,2,3$.

$p^r = 7$ gives PBD({7,6,21,16},44,3).

$p^r = 9$ gives PBD({7,6,27,22},58,3).

COROLLARY 1.3.11. *Let* p^r *be a prime power not equal to* 3 *or* 4. *Then there exists a*

$$\text{PBD}(\{21,..,21-s,5p^r,\ldots,5p^r-s\},21p^r-s,5),\ s=0,1,\ldots p^r.$$

Proof. J.E. Dawson has shown that GH$(4p^r;EA(p^r))$ exists for all prime powers $p \geq 5$. There exists an SBIBD(21,5,1). Hence there exists a PBD$(\{21,\ldots,21-s,p^r,\ldots,p^r-s\},21p^r-s,5)$ for all p^r except possibly 3 and 4 by removing the appropriate number of rows. □

Example 1.3.12 A GH$(20,Z_5)$ can be used to give a GDD$(105,21,\lambda_1=0,\lambda_2=4, m=5)$ by putting the rows of all ones of length 20 on top. Proceeding as before, using the BIBD(21,5,1), replacing elements by 1×5 columns of zeros or ones, respectively, we get a PBD({21,25},105,5).

2. The Existence of GBRD(4,4,t,G;G).

A GBRD$(4,4,t|G|;G)$ is equivalent to four orthogonal rows of index t for the group G, four rows of a GH$(4|G|;G)$, or a

$(|G|,4;t,G)$ difference matrix. Difference matrices have been studied by Jungnickel and Drake, and we refer the reader to their papers for a definition. Such matrices have wide applications giving mutually orthogonal latin squares, mutually orthogonal F-squares, orthogonal arrays, group divisible designs, transversal designs, and λ-geometries.

Generalized Hadamard matrices are known to exist for the following orders, where p is prime and $(Z_p)^i$ is the elementary abelian groups of order p^i.

i) $GH(p^{i+j},(Z_p)^i)$ for all $i \geq 1$, $j \geq 0$.

ii) $GH(2^m p^{\alpha k},(Z_p)^{\alpha})$ for all $0 \leq m \leq k$, $k \geq 1$, $\alpha \geq 1$.

iii) $GH(4p^{\alpha},(Z_p)^{\alpha})$ for all p^{α}.

If $p^t - 1 = r^s$ for some prime r, then there exist:

iv) $GH(P^{tk+\ell} r^{sj},(Z_p)^i)$ for all $1 \leq i \leq t$, $1 \leq j \leq k$, $\ell \geq i$ or $\ell = 0$;

v) $GH(2^m p^{\alpha k+ti+\ell} r^{sj},(Z_p)^{\alpha})$ for all $0 \leq m \leq k$, $k \geq 1$, $1 \leq \alpha \leq t$, $i \leq j \leq i$, $\ell \geq \alpha$ or $\ell = 0$.

The case for four orthogonal rows is as yet incomplete: the case to be decided is $|G| \equiv 3$ or $6 \pmod 9$ with $|G| \not\equiv 2 \pmod 4$ and $t = 1$. Part (i) of our theorem below indicates that the range of values of t for which a $GBRD(4,4,t|G|;G)$ exists depends on whether or not $|G| \equiv 2 \pmod 4$. This is because of a non-existence theorem proved by Drake [13].

THEOREM 2.1. [Drake, Theorem 1.10, 13] *Let* G *be a finite group with a cyclic non-trivial Sylow* 2-*subgroup* T. *Then there is no* $GBRD(3,3,t|G|;G)$ *for odd* t.

Noting that the existence of a $GBRD(4,4,t|G|;G)$ necessitates the existence of a $GBRD(3,3,t|G|;G)$, we are now able to prove

THEOREM 2.2. *Let* G *be a product of elementary abelian groups. Then* *the following statements are true.*

i) *There exists a* GBRD$(4,4,t|G|;G)$ *for some odd* t *if and only if the condition* $|G| \equiv 0,1,$ *or* $3 \pmod 4$ *is satisfied.*

ii) *If the condition in part (i) is not satisfied (i.e.,* $|G| \equiv 2 \pmod 4$*), then a* GBRD$(4,4,t|G|;G)$ *exists if and only if* $t > 0$ *is even.*

iii) *Suppose* $|G| \equiv 0,1,$ or $3 \pmod 4$, *then we have*

(a) $|G| \equiv 4 \pmod 8$ or $|G| \not\equiv 3,6 \pmod 9$ *implies the existence of* GBRD$(4,4,t|G|;G)$ *for all* $t \geq 1$;

(b) $|G| \equiv 3$ or $6 \pmod 9$ *is sufficient for the existence of a* GBRD$(4,4,t|G|;G)$ *for all* $t \geq 2$.

Proof.

i) The Sylow 2-subgroup of G is non-trivial and cyclic if and only if $|G| \not\equiv 0,1,3 \pmod 4$. So, by Drake's Theorem, if $|G| \equiv 2 \pmod 4$, then no GBRD$(4,4,t|G|;G)$ exists for t odd. To prove the converse, we need only prove the three remaining parts of the theorem.

ii) If $|G| \equiv 2 \pmod 4$, then consider $H = Z_2 \times G$. $|H| \equiv 4 \pmod 8$; so we may proceed to the proof of (iii).

iii) (a) If $|G| \not\equiv 3,6 \pmod 9$ and $|G| \not\equiv 2 \pmod 4$, then all the factors in the prime decomposition of $|G|$ are greater than 3. So we may use the first four rows of suitable generalised Hadamard matrices. Now suppose $|G| \equiv 4 \pmod 8$. If $|G| \not\equiv 3$ or $6 \pmod 9$, we are finished; so we may assume $G = Z_2 \times Z_2 \times Z_3 \times H$ where $2,3 \nmid |H|$ and H is elementary abelian. So it suffices to exhibit a GBRD$(4,4,12,Z_2 \times Z_2 \times Z_3)$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & w & w^2 & aw^2 & aw & b & ab & bw & bw^2 & abw^2 & abw \\ 1 & w & ab & bw & w^2 & bw^2 & a & aw & b & abw & aw^2 & abw^2 \\ 1 & w^2 & bw & b & abw & aw^2 & ab & abw^2 & aw & a & w & bw^2 \end{pmatrix}$$

(b) From the $GH(9;Z_3)$ and the $GH(6,Z_3)$, we may obtain a $GBRD(4,4,3t;Z_3)$ for all $t \geq 2$. Now suppose $|G| \equiv 3$ or 6 (mod 9). Let 2^t be the highest power of 2 dividing $|G|$. Since $|G| \not\equiv 2$ (mod 4), all the factors in the prime decomposition of $(|G|/3)$ are greater than 3. Using suitable generalised Hadamard matrices, we may obtain a $GBRD(4,4,t|G|;|G|)$ from a suitable $GBRD(4,4,3t;Z_3)$ for all $t \geq 2$. □

Remark 2.3. We note that, as a consequence of this theorem, the existence of $GBRD(4,4,tg;EA(g))$ is completely decided for $t \geq 2$.

3. Groups of odd order, $3 \nmid |G$

There is a generalized Hadamard matrix $GH(|G|,G)$ for every order $|G|$ which is a prime power. In particular there are four rows of a $GH(|G|,G)$ for every odd order $|G|$ where $3 \nmid |G|$. Taking the Kronecker product of these four rows we obtain a $GBRD(4,4,h;H)$ for every $h = p_1^{\alpha_1} p_2^{\alpha_2} \ldots$, where h is odd, $p_i \neq 3$ for any i, and $H = G_1 \times G_2 \ldots$, where G_i is the elementary abelian group of order $p_i^{\alpha i}$.

By Hanani's theorem

$$\lambda t(v-1) \equiv 0 \pmod 3 \quad \text{and} \quad \lambda tv(v-1) \equiv 0 \pmod{12}$$

are necessary and sufficient conditions for the existence of a $BIBD(v,4,\lambda t)$. Hence using Theorem 1.1.1 (ii) we have

THEOREM 3.1. *Suppose* h *is odd and that* $3 \nmid h$. *Then*

$$\lambda(v-1) \equiv 0 \pmod 3 \quad \text{and} \quad \lambda v(v-1) \equiv 0 \pmod{12}$$

are necessary and sufficient conditions for the existence of a generalized Bhaskar Rao design GBRD(v,4,λh,EA(h)).

Thus the necessary conditions as stated in the abstract are sufficient for the existence of a GBRD(v,b,r,4,λ;EA(h)) when 2 and 3 do not divide h. The remainder of the paper gives a partial solution of the existence question for the group EA(h) where 2 or 3 does not divide h.

4. Group of order 2.

4.1 The group Z_2 with $\lambda = 2$.

THEOREM 4.1.1. *Let* $v \equiv 1 \pmod 6$ *be a prime power. Then there exists a* GBRD$(v,4,2;Z_2)$.

Proof. Let x be a generator of the cyclic group of $GF(v)/\{0\}$ and $C_i = \{x^i, x^{2f+i}, x^{4f+i}\}$ where $v = 6f+1$; then

$$\{\bar{0},C_0\},\{\bar{0},C_1\},\ldots,\{\bar{0},C_{f-1}\}$$

are the required initial blocks which are developed to give the design.

These initial blocks give the differences

$$\pm C_0, \pm C_1, \ldots, \pm C_{f-1}$$

or, since $-1 \in C_f$,

$$C_0, \ldots, C_{f-1}, C_f, \ldots, C_{2f-1}$$

with the negative sign attached, that is, one copy of the cyclic group with the negative sign attached.

The differences with the positive sign attached are

$$\pm x^i(x^{2f}-1), \pm x^{2f+i}(x^{2f}-1), \pm x^i(x^{4f}-1) \;=\; \pm x^i(x^{4f}-x^{6f})$$

that is

$$\pm C_i(x^{2f}-1), \qquad i = 0,1,\ldots,f-1$$

which is one copy of the group. □

Example 4.1.2. The following are $GBRD(55,4,2;Z_2)$.

i) $\{\bar{0},1,52,53\}$, $\{\bar{0},4,6,50\}$, $\{\bar{0},7,21,47\}$, $\{\bar{0},9,33,39\}$, $\{\bar{0},10,26,44\}$, $\{\bar{0},12,32,40\}$, $\{\bar{0},13,30,35\}$, $\{\bar{0},14,27,37\}$, $\{\bar{0},17,24,36\}$.

ii) $\{\bar{0},1,2,4\}$, $\{\bar{0},3,7,12\}$, $\{\bar{0},5,11,27\}$, $\{\bar{0},6,25,37\}$, $\{\bar{0},15,22,35\}$, $\{\bar{0},8,26,36\}$, $\{\bar{0},14,31,39\}$, $\{\bar{0},9,23,38\}$, $\{\bar{0},10,21,42\}$.

They were found on the VAX by Vladimir Vasylenko and T. Mark Ellison respectively.

Remark 4.1.3. This means that these designs are know for $v \in \{7,13,19,25,31,37,43,49,55,61,67,73,79,85,91,97,103,109,115,121,127,133,139,151,157,163\}$ and not known for the following orders < 500 $\{145,205,265,319,355,415,493\}$ using Theorem 1.1.3.

Remark 4.1.4. The necessary condition is $v \equiv 1 \pmod 3$ but no designs are known for $v \equiv 4 \pmod 6$. It is easy to show that no $GBRD(4,4,2,Z_2)$ exists; while de Launey and Sarvate [9] have shown that there is no $GBRD(10,4,2;Z_2)$.

4.2 The Group Z_2 with $\lambda = 4$.

THEOREM 4.2.1. *The necessary condition* $v \equiv 1 \pmod 3$ *is sufficient for the existence of a* $GBRD(v,4,4;Z_2)$.

Proof. We have $\{1 \pmod 3\} = \mathbb{B}\{4,7,10,19\}$. For $v = 7,10$, and 19, let C be the matrix representation of a cyclic permutation. Replace the "signed group" elements $a^i, \bar{a}^i$, by C^i, $-C^i$, respectively, in the following matrices

$$\begin{pmatrix} 1\ldots1 & 1\ldots1 & 1\ldots1 & 1\ldots1 & & & \\ e & e & \bar{e} & \bar{e} & \bar{e}+a & e+a & \\ e & \bar{e} & a & \bar{a} & e+a & & \bar{e}+a \\ e & a & \bar{e} & \bar{a} & & \bar{e}+\bar{a} & e+\bar{a} \end{pmatrix};$$

$$\begin{bmatrix} 1\ldots1 & 1\ldots1 & 1\ldots1 & 1\ldots1 & & & & & & \\ e & e & \bar{e} & \bar{e} & a+a^2 & \bar{a}+a^2 & \bar{a}+a^2 & a+a^2 & & \\ a & a^2 & \bar{a} & \bar{a}^2 & \bar{a}+\bar{a}^2 & \bar{a}+a^2 & & & \bar{a}+a^2 & a+a^2 \\ \bar{a}^2 & \bar{a} & a^2 & a & & & a+\bar{a}^2 & a+a^2 & a+\bar{a}^2 & a+a^2 \end{bmatrix};$$

$[XY]$ where

$X =$

$$\begin{pmatrix} 1\ldots1 & 1\ldots1 & 1\ldots1 & 1\ldots1 & & & & & \\ e & e & \bar{e} & \bar{e} & \bar{e}+a & \bar{e}+a & \bar{a}^3+a^5 & e+\bar{a}^2 & \bar{e}+a^3 \\ \bar{a}^2 & a^4 & \bar{a}^2 & a^4 & e+a^3 & e+a^3 & \bar{e}+a^2 & e+a & \bar{e}+a \\ \bar{a}^4 & \bar{a}^2 & a^4 & a^2 & & & & & \end{pmatrix},$$

$Y =$

$$\begin{pmatrix} e+a & e+a & a^3+a^5 & e+a^2 & e+a^3 & & & & & \\ & & & & & \bar{e}+a & e+a & a^3+a^5 & e+\bar{a}^2 & \bar{e}+a^3 \\ e+a^3 & \bar{e}+a^2 & \bar{e}+a^2 & e+\bar{a} & \bar{e}+a & e+\bar{a}^3 & e+a^2 & e+a^2 & e+a & e+a \end{pmatrix}.$$

We note that the order of C in the first, second, and third matrices is respectively 2, 3 and 6.

□

4.3 The Group Z_2 with $\lambda = 6$.

De Launey and Seberry [10] have shown that

THEOREM 4.3.1. A GBRD$(v,4,6;Z_2)$ *exists for* v *an odd prime power greater than* 5 *and for all* $v \in \{v | v \le 40$ *and* $v \ne 28,34,$ *and* $39\}$. □

Remark 4.3.2. Given these designs for $v = 28,34$, and 39, we would have the designs for all $v \ge 5$. Using another theorem of Wilson and Hanani we proved that the designs existed for $v \equiv 0,1 \pmod 5$. Theorem 1.2.14 gives

LEMMA 4.3.3. *Let* $S = \{v \equiv \pm1 \pmod 6\}$ *and* $N = \{1,2,3,\ldots\}$. *Then* $40^N \subseteq \mathbb{B}({}_S N^{25} \cup {}_{25}S^{130})$.

Proof. Let $v_0 = 5$, $K = {}_5N^{25}$. Take $t_0 = 25$ and let U be as defined in Theorem 1.2.14. By Remark 1.2.15, for all $v \ge 130 = 5.25 + 5$, there exist $t \in S$ $(t > t_0)$ and $k \in K$ such that $v = 5t + k$. So $U = \phi$ and, by Theorem 1.2.14,

$$\mathbb{B}({}_5N^{25} \cup {}_{25}S^{130}) \supseteq 130^N.$$

Also applications of Theorem 1.2.13 first with $K = {}_5N^{25}$, $S = \{5,7,8,9,11,12,13,16,17,19,23\}$ and $v_0 = S$, and then with $v_0 = 6$, $K = {}_5N^{25}$ and $S = \{7,8,9,11,12,13,16,17,19,23\}$, deal with all values of v, $40 \le v \le 130$, except with $v = 43,44,49,57,58$, and 85.

These are dealt with below.

v	
43	TD(7,6) + 1
44	Remove all but one point from any two distinct groups of a TD(7,8)
49	TD(7,7)
57	TD(8,7) + 1
58	Remove all but one point from any two distinct groups of a TD(8,9)
85	TD(12,7) + 1

Theorem 2.2. then produces

THEOREM 4.3.4. *A* GBRD$(v,4,6;Z_2)$ *exists for all* $v \ge 5$ *except possibly for* $v = 28,34$, or 39. *The design does not exist for* $v = 4$.

4.4 The group Z_2 with $\lambda > 6$.

De Launey and Seberry [10] have proved

LEMMA 4.4.1. *A* GBRD$(v,4,12;Z_2)$ *exists for* $v \ge 4$.

LEMMA 4.4.2. *A* GBRD$(v,4,18;Z_2)$ *exists for* $v \ge 5$.

Proof. Taking three copies of the designs obtained from Lemma 4.3.3 gives the result for all $v \ge 5$, $v \ne 28,34,39$. Lemmas 1.3.5 and 1.3.6 show that $28,34,39 \in \mathbb{B}(\{5,6,\ldots,15\},3)$; from Theorem 4.3.3, GBRD$(u,4,6;Z_2)$ exists for $u \in \{5,6,\ldots,15\}$. □

THEOREM 4.4.3. *For* $t > 1$, *the necessary conditions*, $t(v-1) \equiv 0 \pmod 3$ *and* $v \geq 5$ *for odd* t, *are sufficient for the existence of a* $GBRD(v,4,2t;Z_2)$, $v \geq 4$, *except possibly when*

i) $t = 3$ *and* $v = 28, 34$, or 39;

ii) $t = 5, 7$, *and* $v = 28$ *or* 34.

Proof. The second constraint only applies in the case $v = 4$, which has already been dealt with in Section 2. The first condition places no restriction on v when $t \equiv 0 \pmod 3$, while if $t \equiv 1,2 \pmod 3$ then $v \equiv 1 \pmod 3$. By Lemmas 4.4.1 and 4.4.2, when $t = 6a+9b$, $a,b \in N$, there exists a $GBRD(v,4,2t;Z_2)$ for all $v \geq 5$, while from Theorem 4.2.1, when $t = 6a+9b+2c$, a,b and $c \in N$, there exists a $GBRD(v,4,2t;Z_2)$ for all $v \equiv 1 \pmod 3$. □

5. The Group Z_3^p, $p \geq 1$.

5.1 The Group Z_3 with $\lambda = 3$

The necessary conditions are that $v \equiv 0, 1 \pmod 4$.

We have only partial results for $v \equiv 0 \pmod 4$. For $v \equiv 1 \pmod 4$, we give a preliminary lemma.

LEMMA 5.1.1. *There exists a* $GBRD(v,4,3,Z_3)$ *whenever* $v \equiv 1 \pmod 4$ *is a prime power*.

Proof. Let x be the generator of the cyclic group of $GF(v)/\{0\}$ where $v = 4p+1$.

$$\text{Let } D_i = \{x_0^i, x_1^{p+i}, x_0^{2p+i}, x_1^{3p+i}\} \quad \text{for } i = 0,1,\ldots,p-1.$$

Consider the set of differences from D_i, noting that $-1 = x^{2p}$.

$$E_i = \{x^i(x^p-1)_1, -x^i(x^p-1)_2, \pm x^i(x^{2p}-1)_0, x^i(x^{3p}-1)_1, -x^i(x^{3p}-1)_2,$$

$$x^{p+i}(x^p-1)_2, -x^{p+i}(x^p-1)_1, \pm x^{p+i}(x^{2p}-1)_0, x^{2p+i}(x^p-1)_1, x^i(x^p-1)_2\}$$

$$i = 0,1,\ldots,p-1.$$

Thus

$$E_i = \{C_i(x^{2p}-1)_0, \pm x^i(x^p-1)_1, \pm x^i(x^{3p}-1)_1, \pm x^i(x^{3p}-1)_2, \pm x^i(x^p-1)_2\}$$

where $C_i = \{x^i, x^{p+i}, x^{2p+i}, x^{3p+i}\}$. We need to show that the elements $\{\pm x^i(x^p-1), \pm x^i(x^{3p}-1)\}$, $i = 0,1,\ldots,p-1$, give exactly one copy of the group. Clearly $\pm x^i(x^p-1) \neq x^i(x^{3p}-1)$. So it remains to show that $\{\pm x^i(x^p-1), \pm x^i(x^{3p}-1)\} \cap \{\pm x^j(x^p-1), \pm x^j(x^{3p}-1)\} = \emptyset$ unless $i = j$ or $i=j+2p$. If $x^i(x^p-1)=\pm x^j(x^p-1)$, then $x^i=\pm x^j$. Thus $i=j$ or $i=j+2p$. If $x^i(x^p-1) = \pm x^j(x^{3p}-1) = \pm x^{j+3p}(1-x^p)$, then $x^i = \pm x^{j+3p}$. Thus $i = j+p$ or $i = j + 3p$; but this is impossible as $i, j \in \{0,1,\ldots,p-1\}$. Hence each group element appears in $\cup E_i$ with each subscript 0,1,2 exactly once.

Thus $D_i (i = 0,1,\ldots,p-1)$ are initial blocks which when developed give the required design.

Example 5.1.2. There exists a GBRD$(9,4,3;Z_3)$. Let x satisfying $x^2 = 2x+2$ be a generator of $GF(3^2)$. Then, developing the following initial blocks gives the result:

$$\{0_0, 1_0, x_0, x+1_1\},\ \{0_0, 1_2, x+2_1, 2x+2_0\} \pmod{3,3;Z_3}$$

or

$$\{x_0, 2x+2_1, 2x_0, x+1_1\},\ \{2x+1_0, 2_1, x+2_0, 1_0\} \pmod{3,3;Z_3}$$

A complete computer search has shown that GBRD$(9,4,3;Z_3)$ supplementary difference sets $(\bmod\ 9, Z_3)$ do not exist.

LEMMA 5.1.3. *There exist* GBRD$(v,4,3;Z_3)$ *for* $v \equiv 1$ or 5 (mod 20).

Proof. We observe that there exist BIBD$(v,5,1)$ for these v. H = GBRD$(5,4,3;Z_3)$ is given explicitly in the next section. Theorem 1.1.1(ii) then gives the result.

LEMMA 5.1.4. *There exist* GBRD$(v,4,3;Z_3)$ *for* $v \equiv 8,12,16,20,24,28$.

There is no GBRD$(4,4,3;Z_3)$.

Proof. Develop the following initial blocks, found by T.Mark Ellison on a VAX.

v = 8 $(\infty,0_0,1_1,3_2)$, $(0_0,1_0,2_2,5_0)$ (mod $7,Z_3$);

v = 12 $(\infty,0_0,1_1,3_2)$, $(0_0,1_0,3_0,7_1)$, $(0_0,1_2,4_0,6_2)$ (mod $11,Z_3$);

v = 16 $(\infty,0_0,1_1,3_2)$, $(0_0,1_0,5_1,6_0)$, $(0_0,2_0,6_2,8_1)$ $(0_0,3_0,7_0,10_1)$ (mod $15,Z_3$);

v = 20 $(\infty,0_0,1_1,5_2)$, $(0_0,2_1,7_2,13_2)$, $(0_0,3_1,8_1,12_0)$, $(0_0,1_0,3_2,16_0)$, $(0_0,1_2,8_0,10_0)$ (mod $19,Z_3$);

v = 24 $(\infty,0_0,1_1,3_2)$, $(0_0,3_1,8_2,13_1)$, $(0_0,4_1,10_1,16_2)$, $(0_0,1_0,7_2,9_1)$, $(0_0,1_2,4_2,15_0)$, $(0_0,2_0,7_0,11_0)$ (mod $23,Z_3$);

v = 28 $(\infty,0_0,1_1,3_2)$, $(0_0,3_1,8_1,12_2)$, $(0_0,5_1,11_2,18_0)$ $(0_0,1_0,5_2,21_1)$, $(0_0,1_2,11_0,14_0)$, $(0_0,2_0,8_0,12_0)$, $(0_0,2_2,9_2,17_1)$ (mod $27,Z_3$).

An easy combinatorial argument establishes the non-existence of a GBRD$(4,3,3,4,3;Z_3)$.

THEOREM 5.1.5. *There exists a* GBRD$(v,4,3;Z_3)$ *for* $v \equiv 1 \pmod 4$.

Proof. The theorem of Wilson (M. Hall and J.H. van Lint, (1974), p 35) shows that it is merely necessary to establish the existence of GBRD$(u,4,3;Z_3)$ for $u \in H_4^4 = \{5,9,13,17,29,33,49,57,89,93,129,137\}$.

All of these except {33,57,93,129} are prime powers and so the designs exist. Theorem 1.1.3, the existence of the design for u = 8, and the equalities

$$33 = 8(5-1) + 1$$
$$57 = 8(8-1) + 1$$
$$129 = 8(17-1) + 1,$$

give the result for all the required values except possibly 93, but it was shown, in Lemma 1.3.4, that

$$93 \in \mathbb{B}\{5,8,9,12\}.$$ □

LEMMA 5.1.6. *Suppose all* $GBRD(4t,4,3;Z_3)$, $t \neq 1$, $t < t_0$ *exist. Then* $GBRD(4t,4,3;Z_3)$ *exist for* $4t \equiv \{8,16,20,36,40,56,60,64,72,76,80,96,100,104,116,120,128,136,140,156\} \pmod{160}$. *In addition,* $GBRD(4t,4,3;Z_3)$ *exist for* $4t \in \{24,28,92,144,192,204,208\}$.

Proof. Since $GBRD(v,4,3;Z_3)$ exist for $v \equiv 1 \pmod 4$ (from Theorem 5.1.5), we have their existence for $8v$ and $5(v-1)+1$, that is for orders $4t \equiv 8 \pmod{32}$ and $16 \pmod{20}$. In addition, if designs exist for orders $4s$, $s < t_0$, they exist for $32s$ and $20s$, that is, for orders $4t \equiv 0 \pmod{32}$ and $0 \pmod{20}$. This gives the first result.

The other results are obtained by noting:

$$24,28 \text{(cf. Lemma 5.1.4)}; 92 = 13(8-1)+1; 144=12\times 12;$$
$$192=12\times 16;\ 204 = 12 \times 17;\ 208 = 16\times 13.$$

PROPOSITION 5.1.7. *If there exist* $GBRD(v,4,3;Z_3)$ *for* $v \in \{32,44,48,52,68,84,88,124,132\}$, *then these designs exist for* $v \equiv 0,1 \pmod 4$, $v \geq 5$.

Proof. From Corollary 1.2.10 $\{4u,4u+1 \mid 4u,4u+1 \geq 5\}$
$$= \mathbb{B}\{5,9,13,17,29,33,49\} \cup \{4u \mid u = 2,3,\dots,8,11,12,13,17,21,22,23,31,33\}.$$
From the previous two lemmas, we have GBRD for all these values except those stated. □

5.2 The Group Z_3 with $\lambda = 6$.

THEOREM 5.2.1. *Let* $v = 4p+3$ *be a prime power. Then there exists a* $GBRD(v,4,6,Z_3)$.

Proof. Let g be a generator of the cyclic group of order $GF(v)/\{0\}$.

Consider the sets $D_i = \{g_0^i, -g_0^i, g_1^{i+1}, -g_1^{i+1}\}$ for $i = 0,1,\ldots,2p$. Each D_i yields differences

$$\pm 2g_0^i, \pm 2g_0^{i+1}, \pm g^i(g+1)_1, \pm g^i(g+1)_2, \pm g^i(g-1)_1, \pm g^i(g-1)_2.$$

Hence, as i runs from 0 to $2p$, we get two copies of the group with each subscript. Thus the D_i can be used as initial blocks to develop the required design. □

THEOREM 5.2.2. *There exists a* $GBRD(v,4,6;Z_3)$ *for all* $v \geq 4$.

Proof. By Hanani's theorem (cf. Hall (1967,p.248)), it is merely necessary to establish the existence of $GBRD(u,4,6;Z_3)$ for all $u \in K_4^2 = \{4,5,6,7,8,9,10,11,12,14,15,18,19,22,23\}$.

We give a $GH(6,Z_3)$, G, found by Rajkundlia (1978). So any four distinct rows give a $GBRD(4,4,6;Z_3)$. Also, we give a $GBRD(5,4,3;Z_3)$, H:

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega^2 & \omega \\ 1 & \omega & 1 & \omega & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$$

By a theorem of Hanani, a $BIBD(v,5,2)$ exists for $v \equiv 1$ or 5 (mod 10). So, combining with H as in Theorem 1.1.1, we have the existence of $GBRD(v,4,6;Z_3)$ for $v \equiv 1$ or 5 (mod 10).

The following is a $GBRD(6,4,6;Z_3)$:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & \vline & 1 & 1 & 1 & 1 & 1 & \vline & 0 & 0 & 0 & 0 & 0 \\
1 & \omega & 0 & 0 & \omega & \vline & 1 & 0 & \omega^2 & \omega^2 & 0 & \vline & 0 & 1 & 1 & 1 & 1 \\
\omega & 1 & \omega & 0 & 0 & \vline & 0 & 1 & 0 & \omega^2 & \omega^2 & \vline & 1 & 0 & 1 & \omega & \omega^2 \\
0 & \omega & 1 & \omega & 0 & \vline & \omega^2 & 0 & 1 & 0 & \omega^2 & \vline & 1 & 1 & 0 & \omega^2 & \omega \\
0 & 0 & \omega & 1 & \omega & \vline & \omega^2 & \omega^2 & 0 & 1 & 0 & \vline & 1 & \omega & \omega^2 & 0 & 1 \\
\omega & 0 & 0 & \omega & 1 & \vline & 0 & \omega^2 & \omega^2 & 0 & 1 & \vline & 1 & \omega^2 & \omega & 1 & 0
\end{pmatrix}$$

Designs for the remaining values can be constructed as indicated in Table 3(Appendix). The starred designs were found on a VAX by T. Mark Ellison.

5.3 The Group Z_3 with $\lambda = 9$.

THEOREM 5.3.1. *There exists a* $GBRD(v,4,9;Z_3)$ *for* $v \equiv 0, 1 \pmod 4$.

Proof. By Hanani's theorem (cf. Hall (1967,p.248)), it is merely necessary to establish the existence of $GBRD(u,4,6;Z_3)$ for all $u \in K_4^1 = \{4,5,8,9,12\}$.

Now a $GH(9,Z_3)$ exists; so any four distinct rows give the result for $u = 4$. Using three copies of the $GBRD(5,4,3;Z_3) = H$ from Theorem 5.2.7 gives the result for 5; for $u \in \{8,9,12\}$, one develops the initial blocks as indicated.

$v = 8$ $(\infty, 1_0, 2_1, 4_1)$, $(\infty, 1_2, 2_2, 4_1)$, $(\infty, 1_0, 2_2, 4_0)$, $(0_0, 1_0, 3_1, 4_1)$, $(0_0, 1_1, 2_0, 6_1)$, $(0_0, 1_1, 3_0, 5_1)$ (mod $7, Z_3$);

$v = 9$ $(0_0, 1_0, 2_0, 3_0)$, $(0_0, 1_1, 2_2, 4_2)$, $(0_0, 1_1, 4_1, 5_0)$, $(0_0, 1_2, 4_0, 6_1)$, $(0_0, 1_2, 3_1, 6_1)$, $(0_0, 2_1, 4_0, 7_2)$ (mod $9, Z_3$);

$v = 12$ $(\infty, 0_0, 1_1, 3_2)$, $(0_0, 1_0, 3_0, 7_1)$, $(0_0, 1_2, 4_0, 6_2)$, three copies of each, (mod $11, Z_3$);

or

$(\infty, 0_0, 1_1, 2_2)$, $(\infty, 0_0, 1_1, 3_2)$, $(\infty, 0_0, 2_1, 1_2)$, $(0_0, 1_0, 4_0, 5_0)$, $(0_0, 1_0, 5_1, 7_1)$, $(0_0, 1_2, 4_2, 7_1)$, $(0_0, 2_0, 4_0, 7_2)$, $(0_0, 2_1, 5_1, 8_2)$, $(0_0, 2_2, 5_0, 7_2)$ (mod $11, Z_3$).

5.4 The Group Z_3 with $\lambda = 3t$, $t > 1$.

THEOREM 5.4.1. *The necessary conditions are sufficient for the existence of a* GBRD$(v,4,3t;Z_3)$ *when* $t > 1$.

Proof. The result is obtained by combining the previous results for the group Z_3 and using multiple copies of the designs for $\lambda = 6$ and 9. □

5.5 The Group $Z_3 \times Z_3$ with $\lambda = 9$.

The necessary condition for the existence of a GBRD$(v,4,9t;Z_3 \times Z_3)$ is that $tv(v-1) \equiv 0 \pmod 4$.

Example 5.5.1. A GBRD$(v,4,9;Z_3 \times Z_3)$ exists for $v = 4,5,9$.

$v = 4$ Use four rows of a $GH(9, Z_3 \times Z_3)$.

$v = 5$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & w & w^2 & a & aw & aw^2 & a^2 & a^2w & a^2w^2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & w^2 & w & a^2 & a^2w^2 & a^2w & 0 & 0 & 0 & a & aw & aw^2 & a^2 & a^2w & a^2w^2 \\ 1 & a & a^2 & 0 & 0 & 0 & w & wa & wa^2 & w^2a^2 & w^2a & w^2 & a^2w^2 & w & a \\ 0 & 0 & 0 & a^2 & a & 1 & w^2 & w^2a^2 & w^2a & aw & a^2w & w & a^2w^2 & aw & 1 \end{pmatrix};$$

$v = 9$ Develop the following initial blocks:

$(0_1, 1_1, 2_w, 3_1)$, $(0_1, 1_a, 3_{a^2w^2}, 4_1)$, $(0_1, 1_{aw^2}, 4_{aw}, 5_w)$

$(0_1, 1_{a^2w}, 3_{aw}, 7_{aw^2})$, $(0_1, 1_a\ 2_{w^2}, 4_a, 6_{w^2})$, $(0_1, 2_{w^2}, 4_{aw^2}, 6_{a^2})$

$(\text{mod } 9,\ Z_3 \times Z_3)$.

□

LEMMA 5.5.2. *A* $GBRD(v,4,9;Z_3 \times Z_3)$ *exists for* $v \equiv 1, 4 \pmod{12}$ *and* $v \equiv 0,1,4,5 \pmod{20}$.

Proof. Use BIBD(v,4,1) and BIBD(v,5,1) with the designs of the previous example. Also use the BIBD(v,5,1) to form PBD({4,5},v-1) to obtain the result for 0,4(mod 20).

5.6 The Group $Z_3 \times Z_3$ with $\lambda = 18$.

LEMMA 5.6.1. *There exist* $GBRD(v,4,18;Z_3 \times Z_3)$ *for* $v \equiv 1 \pmod 3$, $v \equiv 1 \pmod 4$ and $v \equiv 0,1,4,5 \pmod{10}$. *Designs also exist for* $v \in \{4,8,12\}$.

Proof. PBD[{4,5},v] may be obtained for $v \equiv 1 \pmod 3$ or $v \equiv 0,1,4,5 \pmod{10}$ from BIBD(v,4,2) or BIBD(v,5,2). Since a $GBRD(v,4,9;Z_3 \times Z_3)$ exists for v = 4,5, Corollary 1.1.2 may be applied to give designs when $v \equiv 1 \pmod 3$ or $v \equiv 0,1,4,5 \pmod{10}$. A $GBRD(v,4,3;Z_3)$ is known for $v \equiv 1 \pmod 4$ and $v \in \{4,8,12\}$.

Since a $GH(6,Z_3)$ exists, one may apply Theorem 1.1.1 (iii) to obtain the remaining designs. □

This lemma gives designs for $v \in \{4,5,7,8,9,10,11,12,13,14,15,16,17,19,20,21,23\}$. If designs were known for $v = 6,18$, and 23, then Hanani's result (Corollary 1.1.2 (ii) would guarantee the existence of designs for all $v \geq 4$. Using the set of initial values above and Theorem 1.2.14, the question of the existence of the designs is settled in all but 5 extra cases.

THEOREM 5.6.2. *There exist* $GBRD(v,4,18;Z_3 \times Z_3)$ *for all* $v \geq 4$ *except possibly for* $v = 6,18,23,26,27,38,42$ *and* 47. *If there exists a* $GBRD(6,4,18;Z_3 \times Z_3)$, *then designs exist for all* $v \geq 4$ *except possibly for* $v = 18,23,$ *and* 27.

Proof. We apply Theorem 1.2.14 with $v_o = 4$, $t_o = 25$, $K=\{v|4 \leq v \leq 22, v \neq 6, 18\} \cup \{34\}$ and $S = \{v|v \equiv \pm 1 \pmod 6), v \neq 47\} \cup \{45\}$. To do this, it is necessary to find designs on v points where $v \in S$ and $25 \leq v \leq 104$.

Let $V = \{6,18,23,26,27,38,42,47\}$; then we prove that designs exist for all v, where $24 \leq v \leq 104$, except possibly for $v \in V$. An application of Theorem 1.2.13, with $v_o = 4$, $S = \{4,5,7,8,9,11,12,13,16,17,19,20\}$, and $K = \{v|4 \leq v \leq 20, v \neq 6, 18\}$, deals with all values of $v \notin V$ except $v \in \{21,22,28,29,30,31,34,46,50,66,67,70,98,101,102,103,104\}$. For $v \in \{28,50,70,98\}$, there exist $s,t \in K$ such that $v = st$; also, $67 = 11(7-1)+1$, $66 = 5(14-1)+1$, $46 = 5(10-1)+1$, $34 = 11(4-1)+1$, $31 = 10(4-1)+1$, $29 = 7(5-1)+1$, $22 = 7(4-1)+1$ and $21 = 5(5-1)+1$; $PBD[\{7,8,10,11,12,13\},v]$ may be obtained for $v = 104,103,102$, and 101, by removing rows from a $TD[8,13]$. Finally $v = 30$ may be dealt with by using Lemma 5.6.1.

Thus designs exist for all v, $104 \geq v \geq 4$, except possibly for those in V; in particular, with S as defined in the opening paragraph of this proof, designs exist for all $v \in {}_{25}S^{104}$. To apply Theorem 1.2.14, it is necessary to prove that designs exist for all

$v \in U$. Take v_o, t_o, K, and S as defined in the opening paragraph. By Remark 1.2.15, for all $v \geq v_o t_o + k_o = 104$, there exists an f, $4 \leq f \leq 20$, and $t \in {}_{25}S$ such that $v = 4t+f$. If $f \neq 6$ or 18, then $v \notin U$. If $t \equiv -1 \pmod 6$, then $t+2, t-4 \in {}_{25}S$. So $v = 4(t-4)+k$ or $4(t-2)+k$ where $k \in K$, and hence $v \notin U$. If $t = 45$ and $v = 4t+6$ or $4t+18$, then $v = 4.43+14$ or $4.41+34$; hence $v \notin U$. Finally, if $t \equiv 1 \pmod 6$, then $v = 4t+6$ or $4t+18 \equiv 4 \pmod 6$. Therefore $u \leq \{v | v \equiv 1 \pmod 3\}$. But by Lemma 5.6.1. designs exist for all $v \equiv 1 \pmod 3$, $v \geq 4$.

Since designs exist for all $v \in {}_{25}S^{104} \cup U \cup K \cup \{t_o\}$, designs exist for all $v \geq 104$, and indeed for all $v \geq 4$ except possibly for $v \in V$.

Now suppose a $GBRD(6,4,18;Z_3 \times Z_3)$ exists. Then, because $26 = 5(6-1)+1$ and $42 = 6 \times 7$, there would exist a $GBRD(v,4,18;Z_3 \times Z_3)$ for $v = 26,42$. Finally $PBD[\{4,5,8,6\},38]$ and $PBD[\{6,7,5\},47]$ may be obtained from $TD[5,8]$ and $TD[7,7]$. One then obtains designs on 38 and 47 points, completing the proof of the second statement of the theorem. □

5.7 The group $Z_3 \times Z_3$ with $\lambda = 9t$, $t > 2$.

THEOREM 5.7.1. *The necessary conditions are sufficient for the existence of* $GBRD(v,4,9t;Z_3 \times Z_3)$ *for* $t > 2$.

Proof. It t is odd, then $v \equiv 0,1 \pmod 4$ and $v \geq 4$, while if t is even then $v \geq 4$. Suppose t is odd, $t \geq 3$. Consider the case $t = 3$. A $BIBD(v,4,3)$ exists for $v \equiv 0,1 \pmod 4$. Also a $GBRD(4,4,9;Z_3 \times Z_3)$ exists; so one may apply Theorem 1.1.1 (ii) to obtain the designs for all $v \geq 4$, $v \equiv 0,1 \pmod 4$. By Theorem 5.6.2, a $GBRD(v,4,18;Z_3 \times Z_3)$ exists for all $v \equiv 0,1 \pmod 4$. It follows that, if t is odd, $t \geq 3$, designs exists for all $v \equiv 0,1 \pmod 4$, $v \geq 4$.

Now suppose t is even and $t = 2m$, $m > 1$. By Theorem 2.2, there exists a $GBRD(4,4,3m;Z_3)$ for all $m > 1$. By Theorem 5.2.2, there exists a GBRD$(v,4,6;Z_3)$ for all $v \geq 4$. So, applying Theorem 1.1.1 (i), one has GBRD$(v,4,9(2m);Z_3 \times Z_3)$ for all $v \geq 4$. □

5.8 The group Z_3^s, $s > 0$.

THEOREM 5.8.1. *The necessary conditions are sufficient for the existence of a* GBRD$(v,4,3^s t;Z_3^s)$ *for* $t > 1$ *and* $s > 0$, *except possibly when* $s = t = 2$ *and* $v \in \{6,18,23,26,27,38,42,47\}$.

Proof. The result for $s = 1$ follows from Theorem 5.4.1, while that for $s = 2$ follows from Theorems 5.6.2 and 5.7.1. When $s \geq 3$, there exists a GH$(3^{s-1};EA(3^{s-1}))$ where $3^{s-1} > 4$. Since the necessary conditions are sufficient for the existence of GBRD$(v,4,3t;Z_3)$ when $t > 1$, one may apply Theorem 1.1.1 (iii) to give the required designs for $s \geq 3$.
□

6. The Group Z_2^p, $p > 1$.

6.1 The Group $Z_2 \times Z_2$ with $\lambda = 4$.

There is a useful construction when there are n-$\{v;4;\lambda\}$ supplementary difference sets.

LEMMA 6.1.1 *Suppose there exist* n-$\{v;4;\lambda\}$ *s.d.s.* $\{t^{1i},t^{2i},t^{3i},t^{4i}\}$, $i = 1,\ldots,n$, *and* $Z_2 \times Z_2 = \{e,a,b,ab\}$. *Then*

$$\{t^{1i}_e,t^{2i}_e,t^{3i}_e,t^{4i}_e\},\{t^{1i}_e,t^{2i}_a,t^{3i}_b,t^{4i}_{ab}\},\{t^{1i}_e,t^{2i}_b,t^{3i}_{ab},t^{4i}_a\},\{t^{1i}_e,t^{2i}_{ab},t^{3i}_a,t^{4i}_b\},$$
$$i = 1,\ldots,n,$$

are $4n$-$\{v;4;4\lambda\}$ *GBRSDS*(generalized Bhaskar Rao supplementary difference sets).

Example 6.1.2. $(0,1,2,4) \pmod 7$ is a 1-$\{7,4;2\}$ s.d.s. and $(0_e,1_e,2_e,4_e)$, $(0_e,1_a,2_b,4_{ab})$, $(0_e,1_b,2_{ab},4_a)$, $(0_e,1_{ab},2_a,4_b)$ $(\text{mod } 7, Z_2 \times Z_2)$ are GBRSDS and hence can be used as initial blocks to generate a GBRD$(7,4,8;Z_2\times Z_2)$.

LEMMA 6.1.3. *There is no* GBRD$(7,4,4;Z_2\times Z_2)$

Proof. The four inequivalent BIBD(7,14,8,4,4) cannot be signed. This is proved in de Launey and Sarvate [9].

LEMMA 6.1.4. *There exists a* GBRD$(v,4,4;Z_2\times Z_2)$ *when* $v \equiv 1,4 \pmod{12}$.

Proof. There exists a BIBD(v,4,1) for $v \equiv 1,4 \pmod{12}$ which is combined with a GH$(4,Z_2\times Z_2)$ to give the result.

Remark 6.1.5. The problem is as yet unsolved for $v \equiv 7,10 \pmod{12}$. For v = 7, there is no design. For v = 19 a design can be obtained by using the following sets as initial blocks.

$$\text{GBRD}(19,4,4;Z_2\times Z_2)(0_e,1_e,2_a,3_b),(0_e,1_b,6_b,8_b),(0_e,2_{ab},6_a,14_b)$$

$$(0_e,3_e,9_{ab},12_b),(0_e,3_{ab},8_e,12_{ab}),(0_e,4_e,8_a,14_a)$$

$$(\text{mod } 19,Z_2\times Z_2).$$

Remark 6.1.6. By Theorem 1.2.11, if GBRD$(v,4,4;Z_2\times Z_2)$ exist for $v \in \{4,10,19,22,34,43,55,79,199\}$, then the designs exist for all $v \equiv 1 \pmod 3$ except v = 1 or 7.

□

6.2 The group $Z_2\times Z_2$ with $\lambda = 8$.

THEOREM 6.2.1. *The necessary condition* $v \equiv 1 \pmod 3$ *is sufficient for the existence of a* GBRD$(v,4,8;Z_2\times Z_2)$.

Proof. A BIBD(v,4,2) exists for $v \equiv 1 \pmod 3$ and may be combined with a GH$(4,Z_2\times Z_2)$ to get the result.

6.3 The group $Z_2\times Z_2$ with $\lambda = 12$.

THEOREM 6.3.1. GBRD$(v,4,12;Z_2\times Z_2)$ *exist for* $v \equiv 0,1 \pmod 4$.

Proof. A BIBD(v,4,3) exists for $v \equiv 0,1 \pmod 4$ and may be combined with a $GH(4,Z_2\times Z_2)$ to get the result.

As the next two Lemmas show, more structured designs exist.

LEMMA 6.3.2. *Suppose* $v = 4p + 1$ *is a prime power. Then there is a* $GBRD(v,4,12;Z_2\times Z_2)$.

Proof. Let x be a generator of $GF(v)/\{0\}$. Then

$$(x^i, x^{p+i}, x^{2p+i}, x^{3p+i}), \quad i = 0,1,\ldots,p-1$$

are $p - \{v;4;3\}$ supplementary difference sets. By Theorem 6.1.1, we have

$$(x_e^i, x_e^{p+i}, x_e^{2p+i}, x_e^{3p+i}),\ (x_e^i, x_a^{p+i}, x_b^{2p+i}, x_{ab}^{3p+i}),$$

$$(x_e^i, x_b^{p+i}, x_{ab}^{2p+i}, x_a^{3p+i}),\ (x_e^i, x_{ab}^{p+i}, x_a^{2p+i}, x_b^{3p+i})\ \ i = 0,1,\ldots,p-1$$

are $4p - \{v;4;12;Z_2\times Z_2\}$ GBRSDS and can be used as the initial blocks of a $GBRD(v,4,12;Z_2\times Z_2)$.

LEMMA 6.3.3. *Suppose* $v = 4p + 1$ *is a prime power. Then there is a* $GBRD(v+1,4,12;Z_2\times Z_2)$.

Proof. Take x a generator of $GF(v)/\{0\}$ and choose initial blocks

$$(x_e^i, x_e^{p+i}, x_e^{2p+i}, x_e^{3p+i}),\ (x_e^i, x_a^{p+i}, x_b^{2p+i}, x_{ab}^{3p+i}),\ i = 0,1,\ldots,p-1$$

$$(x_e^i, x_b^{p+i}, x_{ab}^{2p+i}, x_a^{3p+i}),\ (x_e^i, x_{ab}^{p+i}, x_a^{2p+i}, x_b^{3p+i})\ \ i = 1,\ldots,p-1$$

$$(\infty_e, 0_e, (x^{3p}-1)_a, (x^p-1)_b),\ (\infty_e, 0_e, (x^{2p}-1)_a, (x^p-1)_{ab}),$$

$$(\infty_e, 0_e, (x^{3p}-1)_b, (x^p-1)_{ab}), (\infty_e, 0_a, (x^{2p}-1)_b, (x^p-1)_{ab}) \pmod{v, Z_2\times Z_2}.$$

To prove the theorem, we observe that the difference $\infty_e, \infty_a, \infty_b, \infty_{ab}$ occurs the correct number of times and, using $x^{2p} = -1$, that the other differences are exactly the differences of the blocks

$$(x_e^0, x_b^p, x_{ab}^{2p}, x_a^{3p}),\ (x_e^0, x_{ab}^p, x_a^{2p}, x_b^{3p})$$

which we have removed from the set of initial blocks used in the previous theorem.

□

LEMMA 6.3.4. $GBRD(v,4,12;Z_2\times Z_2)$ *exist for* $v = 4,7,9,11$.

Proof. These designs can be constructed as indicated.

$v = 4$ $(\infty_e,0_e,1_e,2_e),(\infty_e,0_a,1_b,2_{ab}),(\infty_e,0_b,1_{ab},2_a),(\infty_e,0_{ab},1_a,2_b)$ $(\text{mod } 3,\ Z_2\times Z_2)$

$v = 7$ $(0_e,1_e,2_e,3_e),(0_e,1_a,2_e,3_b),(0_e,1_a,3_e,4_{ab}),(0_e,1_b,3_a,4_b),$ $(0_e,1_b,3_{ab},5_a),(0_e,1_{ab},3_a,5_b)(\text{mod } 7,\ Z_2\times Z_2);$

$v = 9$ $(0_e,1_e,2_e,3_e),(0_e,1_a,2_e,3_b),(0_e,1_a,2_{ab},6_{ab}),(0_e,1_b,4_e,6_a),$ $(0_e,1_{ab},4_a,6_6),(0_e,1_{ab},4_e,6_a),(0_e,1_{ab},4_b,6_e),(0_e,2_b,4_a,6_{ab})$ $(\text{mod } 9,\ Z_2\times Z_2);$

$v = 11$ $(1_e,2_a,4_b,8_{ab}),(2_e,4_a,8_b,5_{ab}),(4_e,8_a,5_b,10_{ab}),(8_e,5_a,10_b,9_{ab}),$ $(5_e,10_a,9_b,7_{ab}),(1_a,2_e,4_e,8_e),(2_e,4_e,8_{ab},5_e),(8_e,5_e,10_b,9_e),$ $(4_e,8_e,5_e,10_e),(5_{ab},7_e,9_a,10_b)(\text{mod } 11,Z_2\times Z_2).$

The designs constructed in this section verify the existence of $GBRD(v,4,12;Z_2\times Z_2)$ in all but three cases.

LEMMA 6.3.5. *Let* $Q = \mathbb{B}(\{4,5,6,7,8,9,10,11,12,14,18,19,22\})$. *Then* $Q \supset \{v \mid v > 23\}$.

Proof. We first note that $31,30,29,28,27,26,25,24 \in Q$. The following table gives the construction.

v	Construction
31	BIBD(31,6,1) exists.
30	Delete a row from BIBD(31,6,1).
29	Delete a row from TD(6,5).
28	Use TD(4,7).
26	Add a suitable row to a TD(5,5).
25	Use TD(5,5).

Thus $K = \mathbb{B}(\{4 < v < 32 | v \neq 15,23,27\})$.

Now we show $Q \supseteq A = \{27 < v \leq 129\}$. A TD(5,t) exists for $t = 7,8,9,11,12,13,16,17,19,25,29$; so a PBD[Q,v] exists for all $v \in A$ except possibly $v \in \{46,47,66,67,96,97,\ldots,103\}$. All TDs used are given by McNeish's theorem except TD(5,12); that design may be obtained from the TD(7,12) which is known to exist (see Drake [13]). A PBD[Q,v] may be constructed for the remaining values of v by deleting rows from a TD(6,8), TD(6,12), TD(6,17), or a TD(6,19). Hence $Q \supseteq A$.

Now take $v_0 = 5$, $t_0 = 25$, $K = \{4 < v < 25 | v \neq 15,23\}$, $S = \{v \equiv \pm 1 (\text{mod } 6)\}$, and U as defined in Theorem 1.2.14. By that theorem, $\mathbb{B}(K \cup {}_{25}S^{129})$ contains all $v \geq 130$ except possibly those of the form $v = 5t+15$ where $t \equiv \pm 1(\text{mod } 6)$. So, since in that case $v \equiv \pm 2(\text{mod } 6)$,

$$\mathbb{B}(K \cup {}_{25}S^{129}) \supseteq {}_{25}S .$$

Finally, applying Theorem 1.2.13 with $v_0 = 4$ and K and S as above, one obtains $PBD[K \cup {}_{25}S^{129},v]$ for all $v > 129$ except when $v = 15+4t$ where $t \equiv \pm 1$ (mod 6) (then $v \equiv \pm 1(\text{mod } 6)$). But these all exist so $\mathbb{B}(K \cup {}_{25}S^{129}) \supseteq \{v > 129\}$, and hence $Q \supseteq \{v>4 | v = 15,23\}$. □

THEOREM 6.3.6. $GBRD(v,4,12;Z_2 \times Z_2)$ *exist for all* $v \geq 4$ *except possibly for* $v = 15$ or 23.

Proof. We observe $19 = 6(4-1)+1$ and $22 = 7(4-1)+1$; so by the previous results of this section $GBRD(v,4,12;Z_2 \times Z_2)$ exist for all $4 \le v < 23$ except $v=15$. Applying the Lemma gives the result for $v > 23$.

□

6.4 The group $Z_2 \times Z_2$ with $\lambda = 24$.

THEOREM 6.4.1. *Let* $v \equiv 3 \pmod 4$ *be a prime power; then there exists a* $GBRD(v,4,24;Z_2 \times Z_2)$.

Proof. Let x be a generator of the cyclic group of $GF(v)/\{0\}$. Then the design is constructed using the initial blocks $(g_e^i, g_e^{i+1}, -g_e^{i+1}, -g_e^i)$ and $(g_e^i, g_a^{i+1}, -g_b^{i+1}, -g_{ab}^i)$, each three times $(i=1,2,\ldots,\frac{1}{2}(v-1))$.

Example 6.4.2. For $v = 7$, we use the blocks

$$(2_e,4_e,3_e,5_e),\ (4_e,1_e,6_e,3_e),\ (1_e,2_e,5_e,6_e)$$

and

$$(2_e,4_a,3_b,5_{ab}),\ (4_e,1_a,6_b,3_{ab}),\ (1_e,2_a,5_b,6_{ab})$$

each three times to form the $GBRD(7,4,24;Z_2 \times Z_2)$.

THEOREM 6.4.3. *There exists* $GBRD(v,4,24;Z_2 \times Z_2)$ *for* $v \ge 4$.

Proof. There is a $GBRD(15,4,24;Z_2 \times Z_2)$ obtained by developing the following blocks obtained by using Lemma 6.1.1 on the 6-{15;4;6}s.d.s.

$$\{0,1,5,10\},\ \{0,2,5,10\},\ \{1,2,4,8\} \text{ five times } \pmod{15}.$$

Use Remark 6.3.4 and the Hanani-Wilson theorems. The only previously uncompleted cases are $v = 15$ (just given) and $v = 23$ obtained from the last theorem.

6.5 The group $Z_2 \times Z_2$ with $\lambda = 4t$.

THEOREM 6.5.1. *The necessary conditions are sufficient for the existence of a* GBRD$(v,4,4t;Z_2\times Z_2)$ *when* $t \geq 4$.

Proof. When $t \equiv 1$ or $2 \pmod 3$, $v \geq 4$ and $v \equiv 1 \pmod 3$; if $t \equiv 0 \pmod 3$, $v \geq 4$. By Theorems 6.3.6 and 6.4.3 designs exist when $t = 2$ or 3 for all $v \equiv 1 (\mathrm{mod}\ 3)$ and $v \geq 4$; so the necessary conditions are sufficient for $t \equiv 1$ or $2 \pmod 3$.

By Theorem 6.3.6 there exist GBRD$(v,4,12;Z_2\times Z_2)$ for $v \in K_4^2/\{15,23\}$. Now a BIBD(15,7,3) exists and a PBD[{7,8,9},23,3] may be obtained from the BIBD(25,9,3). It follows that GBRD$(v,4,36;Z_2\times Z_2)$ exist for all $v \in K^2_4$, and hence for all $v \geq 4$. Combining this result with Theorem 6.4.3, one obtains the required designs for $t \equiv 0 \pmod 3$, $t > 1$.

□

6.6 The group Z_2^p, $p \geq 3$.

For $p \geq 3$ and $t > 1$, the question of existence of GBRD$(v,4,2^p t;Z_2^p)$ is completely decided while for $p = 1$ or 2 and $t > 1$ there remain but a few undecided cases.

THEOREM 6.6.1. *Suppose* $t > 1$ *and* $p \geq 3$. *Then the necessary condition* $t(v-1) \equiv 0 \pmod 3$ *is sufficient for the existence of a* GBRD$(v,4,2^p t;EA(2^p))$.

Proof. Suppose $t = 2$ and $p \geq 3$. Then it is necessary that $v \equiv 1 \pmod 3$. Now there exists a BIBD(v,4,2) for $v \equiv 1 \pmod 3$ and there exists a GH$(2^p;EA(2^p))$ for all $p \geq 1$. Since $2^p > 4$ for $p \geq 3$, Theorem 1.1.1 (iii) may be applied to give the required GBRD$(v,4,2^p t;EA(2^p))$.

Now suppose $t = 3$ and $p \geq 3$. By Theorem 4.3.3 there exist GBRD$(v,4,6;Z_2)$ for all $v \geq 4$ except possibly for $v = 28,34,39,44$, and 58. Hence Theorem 1.1.1 (iii) may be applied as before to obtain the designs for all $v \geq 5$ except for $v = 28,34,39,44$, and 58. Finally there exists a GBRD$(4,4,3.2^p;EA(2^p))$, and hence designs on

$28 = 4\times7$, $34 = 11(4-1)+1$, $44 = 11\times4$, and $58 = 19(4-1)+1$ points. When $v = 39$, a PBD[{4,5,6},v] may be obtained from a TD(6,7), and subsequently, by Corollary 1.1.2, a design on 39 points.

If $3 \nmid t$, then $v \equiv 1 (\text{mod } 3)$. Letting a and b satisfy $t = 2a+3b$, one can obtain a $GBRD(v,4,2^p t;EA(2^p))$ by taking a copies of a $GBRD(v,4)2^{p+1};EA(2^p))$ and b copies of a $GBRD(v,4,2^p.3;EA(2^p))$. If $3|t$, let $t = 3m$ and take m copies of a $GBRD(v,4,2^p.3;EA(2^p))$.

□

Combining Theorems 2.2, 4.4.3, 6.5.2, 6.3.6, and 6.6.1 one obtains

THEOREM 6.6.2. *The necessary conditions:*

(i) $t(v-1) \equiv 0 \pmod 3$

(ii) $p = 1$ *and* t *odd* $\Rightarrow$ $v \geq 5$

are sufficient for the existence of a $GBRD(v,4,2^p t;Z_2{}^p)$ *when* $p \geq 1$ *and* $t > 1$ *except possibly when*

(a) $p = 1$, $t = 3$, *and* $v = 28,34$ *or* 39,

(b) $p = 1$, $t = 5,7$ *and* $v = 28$ *or* 34,

(c) $p = 2$, $t = 3$ *and* $v = 15$ *or* 23.

□

7. The group Z_6.

7.1 The group Z_6 with $\lambda = 6$.

Remark 7.1.1. A $GBRD(4,4,6;Z_6)$ does not exist by a simple combinatorial argument. De Launey and Sarvate have shown [9] that a $GBRD(5,4,6;Z_6)$ does not exist.

THEOREM 7.1.2. $GBRD(v,4,6;Z_6)$ *exist for* $v = 7,9,\ldots,16,18,20$.

Proof. The initial blocks for these designs are given in Table 4 (Appendix).

7.2. The Group Z_6 with $\lambda = 12$.

LEMMA 7.2.1. *Suppose* $v \equiv 4p+1$ *is a prime power. Then there exists a*

$GBRD(v,4,12;Z_6)$.

Proof. Let x be a generator of the cyclic group of order $GF(v)/\{0\}$. Consider the sets $D_i=\{x_3^i,x_1^{p+i},x_0^{2p+i},x_1^{3p+i}\}$, $i = 0,1,\ldots,4p-1$. These have differences

$$x^i(x^p-1)_4,x^i(x^{2p}-1)_3,x^i(x^{3p}-1)_4,x^{p+i}(x^p-1)_5,x^{i+p}(x^{2p}-1)_0,x^{2p+i}(x^p-1)_1,$$
$$x^{2p+i}(x^p-1)_2,$$
$$x^{2p+i}(x^{2p}-1)_3,x^{2p+i}(x^{3p}-1)_2,x^{3p+i}(x^p-1)_1,x^{3p+i}(x^{2p}-1)_0,x^i(x^p-1)_5.$$

We see that every non-zero difference occurs exactly twice with each subscript.

□

LEMMA 7.2.2. *Suppose* $v = 4p+3$ *is a prime power,* $v > 4$. *Then there exists a* $GBRD(v,4,12;Z_6)$.

Proof. Let x be a generator of the cyclic group of order $GF(v)/\{0\}$. Consider the sets $D_i=\{x_3^i,x_0^{2p+i},x_1^{i+1},x_1^{2p+i+1}\}$, $i = 0,1,\ldots,4p$. These have differences

$$\pm x^{i+1}(x^{2p}-1)_0,\pm x^i(x^{2p}-1)_3,x^i(x-1)_4,x^i(x^{2p+1}-1)_4,x^i(x^{2p}-x)_5,x^{2p+i}(1-x)_5,$$
$$x^{2p+i}(x-1)_2,$$

$x^i(x+1)_2,x^i(x+1)_1,x^i(1-x)_1$; so we have two copies of the group with each subscript.

□

THEOREM 7.2.3. *There exists a* $GBRD(v,4,12;Z_6)$ *for* $v \geq 4$.

Proof. Table 5 (Appendix) gives $GBRD(v,4,12;Z_6)$ for $v \in \{4,6,8,10,12,14\}$. Lemmas 7.2.1 and 7.2.2 give all the remaining values of K^2_4 except 15,18,22. Two copies of $GBRD(v,4,6;Z_6)$, given in Table 4, give designs for 15 and 18. Also, $22 = 7(4-1)+1$. Hence $GBRD(v,4,12;Z_6)$ exist for all values of $K_4{}^2$ and we have the result by using Theorem 1.1.2.

7.3 The group Z_6 with $\lambda = 18$.

LEMMA 7.3.1. *There exists a* GBRD$(v,4,18;Z_3)$ *for* $v = 5,6$.

Proof. To establish the lemma we first note that if

$$X = \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \\ \underset{\sim}{x}_3 \\ \vdots \\ \underset{\sim}{x}_v \end{bmatrix}$$

is a matrix with elements from the ring $0+G$, where G is an abelian group, with rows $\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_v$ which have the property that

$\underset{\sim}{x}_1 \cdot \underset{\sim}{x}_2 + \underset{\sim}{x}_1 \cdot \underset{\sim}{x}_3 = 2\lambda G$ while $\underset{\sim}{x}_i \cdot \underset{\sim}{x}_j = \lambda G$ in every other case, then the matrix

$$Y = \begin{bmatrix} \underset{\sim}{x}_1 & \underset{\sim}{x}_1 & \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 & \underset{\sim}{x}_3 & \underset{\sim}{x}_4 \\ \underset{\sim}{x}_3 & \underset{\sim}{x}_4 & \underset{\sim}{x}_2 \\ \underset{\sim}{x}_4 & \underset{\sim}{x}_2 & \underset{\sim}{x}_3 \\ \underset{\sim}{x}_5 & \underset{\sim}{x}_5 & \underset{\sim}{x}_5 \\ \vdots & \vdots & \vdots \\ \underset{\sim}{x}_v & \underset{\sim}{x}_v & \underset{\sim}{x}_v \end{bmatrix} = \begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \\ \\ \\ \\ \vdots \\ \underset{\sim}{y}_v \end{bmatrix}$$

has the property that any pair of distinct rows $\underset{\sim}{y}_i$ and $\underset{\sim}{y}_j$ satisfy

$$\underset{\sim}{y}_i \cdot \underset{\sim}{y}_j = 3\lambda G.$$

We note that the matrices

$$
\left(\begin{array}{ccccc|ccccc|ccccc}
1 & . & . & \bar{1} & 0 & . & \bar{2} & 2 & . & \bar{0} & . & 0 & 0 & 0 & 0 \\
. & . & \bar{1} & 0 & 1 & \bar{2} & 2 & & \bar{0} & & 0 & . & 0 & 2 & 1 \\
0 & 1 & . & . & \bar{1} & \bar{0} & . & 2 & \bar{2} & . & 0 & \bar{0} & . & \bar{1} & \bar{2} \\
\bar{1} & 0 & 1 & & & & \bar{0} & & \bar{2} & 2 & \bar{2} & \bar{1} & 0 & . & 2 \\
 & \bar{1} & \bar{0} & 1 & & 2 & & 0 & & \bar{2} & \bar{1} & \bar{2} & 0 & \bar{1} & . \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & &
\end{array}\right)
$$

and

$$
\left(\begin{array}{cccccccccc}
. & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & . & . & \bar{2} & 1 & \bar{2} & \bar{0} & 0 & 1 \\
\bar{1} & 2 & 2 & \bar{1} & . & . & 0 & \bar{1} & 2 & \bar{0} \\
0 & 0 & 2 & \bar{0} & 0 & \bar{1} & . & . & \bar{2} & 1 \\
1 & \bar{1} & 1 & 2 & \bar{0} & \bar{1} & 0 & \bar{2} & . & .
\end{array}\right)
$$

fulfil the requirements for the matrix X for $v=5,6$. Hence the corresponding matrices Y are $GBRD(v,4,18;Z_6)$ for $v=5,6$. □

THEOREM 7.3.2. *Let* $V = \{8,17,22,23,24,27,32,33,34\}$. *Then* $GBRD(v,4,18;Z_3)$ *exist for all* $v \geq 5$ *except possibly for* $v \in V$.

Proof. We first prove the result for $5 \leq v \leq 130$. Theorem 7.1.2 gives $GBRD(v,4,6;Z_6)$ for $v = 7,9,\ldots,16,18,20$. So $GBRD(v,4,18;Z_6)$ exist for these v. Lemma 7.3.1 gives the designs for $v = 5$ and 6. Finally, by Theorem 4.1.1, there exists a $GBRD(19,4,2;Z_2)$, and hence by Theorem 1.1.1 (iii) a $GBRD(19,4,18;Z_6)$. Thus the designs exist for all $v \in L = \{v \mid 5 \leq v \leq 20,\ v \neq 8,17\}$. So Theorem 1.2.13, with $v_o = 5$, $S = \{5,7,9,11,12,13,16,19,20\}$ and $K = L$, gives, by Corollary 1.1.2, a $GBRD(v,4,18;Z_6)$ whenever $v = 5s+k$, $k \leq s$, $k \in K$ and $s \in S$; an application of the same theorem with $v_o = 6$ and S replaced by $S' = S\backslash\{5,20\}$ gives $GBRD(v,4,18;Z_6)$ whenever $v = 6s+k$, $k \leq s$, $k \in K$, and $s \in S'$. Therefore, to prove the theorem for $5 \leq v \leq 130$, it is sufficient to construct $GBRD(v,4,18;Z_6)$ for

$v \in \{117,80,68,58,57,56,55,53,46,45,44,43,38,37,36,35,31,30,29,28,26,25,21\}$.

But 43,37,31, and 25 are prime powers congruent to 1 (mod 6) ; so the design may be obtained from the GBRD$(v,4,2;Z_2)$. For $v \in \{117,80,55,45,36,35,30\}$, there exist s, t $\in$ K such that v=st. Also, 57 = 7(9-1)+1, 56 = 11(6-1)+1, 53 = 13(5-1)+1, 46 = 9(6-1)+1, 29 = 7(5-1)+1, 26 = 5(6-1)+1, and 21 = 5(5-1)+1. So Theorem 1.1.3 applies with w = 1.

By Lemma 1.3.6, there exists a PBD({7,9,10,..,15},v,3) for v = 68,58. By Lemma 1.3.5, there exists a PBD({11,12},v,3), v = 39,44; by Example 1.3.10, there exists a PBD({7,12},28,3). So, using the designs provided by Theorem 7.1.2, and applying Corollary 1.1.2, we find that there exist GBRD$(v,4,18;Z_6)$ for v = 68,58,44,39, and 28.

Finally we construct a PBD({5,6,7},38) and hence a GBRD$(38,4,18;Z_6)$. Select three distinct non-collinear points in 3 separate groups of a TD(8,7). Discard the remaining points of those groups. The resulting design is the required design.

Now we have just proved that GBRD$(v,4,18;Z_6)$ exists for all $v \in \{5,6,7,9,10,11,12,13,14,15,16,18,19,20,28\} \cup {}_5S^{130}$. So, by Lemma 1.2.16 and Corollary 1.1.2, designs exist for all $v \ge 130$. □

7.4 The group Z_6 with $t > 1$.

THEOREM 7.4.1. *Let* $t > 1$. *There exist* GBRD$(v,4,6t;Z_6)$ *for all* $v \ge 5$, $t > 1$, *except possibly when:*

i) $t = 3$ *and* v = 8,17,22,23,24,27,32,33, *or* 34,

ii) $t = 5$ *and* v = 8,17,24,27,32,33, *or* 34,

iii) $t = 7$ *and* v = 8,17,32,33, *or* 34.

Proof. The result for $t = 2$, and hence for t even, is given by Theorem 7.2.3. The result for $t = 3$ is given by Theorem 7.3.2. Now SBIBD(23,11,5) and SBIBD(27,14,7) exist; so a PBD({10,11},v,5) exists for v = 22 and 23 and a PBB({9,10,11,12,13,14},v,7) exists for v = 22,23,24, and 27. Now Theorem 7.1.1 gives GBRD$(v,4,6;Z_6)$ for $v \in \{9,10,11,12,13,14\}$; so, by Corollary 1.1.2, GBRD$(v,4,6t;Z_6)$ exist for v = 22 and 23 when t = 5 and for v = 22,23,24, and 27 when $t = 7$.

We now consider the case $t = 9$. A GBRD$(v,4,6;Z_2)$ is given by Theorem 4.3.4 for all $v \geq 5$ except for $v = 28,34$, and 39. Applying Corollary 1.1.1 (iii), there exists a GBRD$(v,4,9,6;Z_6)$ for all $v \geq 5$ except possibly for $v = 28,34$, and 39. By Example 1.3.10, there exists a PBD$(\{7,12\},28,3)$ and a PBD$(\{6,7,14,15\},34,3)$, and by Lemma 1.3.5 there exists a PBD$(\{9,10,11,12\},39,3)$. So, by Corollary 1.1.2, noting the existence of GBRD$(v,4,3,6;Z_6)$ for $v = 6,7,9,10,11,12,14,15$, there exist GBRD$(v,4, 9.6;Z_6)$ for all $v \geq 5$. Finally, using the designs for $t = 2$ gives the result for $t > 9$ odd.

□

8. The Group EA(12).

8.1 The group $Z_2 \times Z_2 \times Z_3$ with $\lambda = 12$.

THEOREM 8.1.1. *There exist* GBRD(v,4,12;EA(12)) *for all* $v \geq 4$, $v \equiv 0,1 \pmod 4$.

Proof. By Theorem 2.2, there exists a GBRD(4,4,12;EA(12)). By Lemma 5.1.4, there exist GBRD$(v,4,3;Z_3)$ for $v = 8$ and 12. By Theorem 5.1.5, there exists a GBRD$(v,4,3;Z_3)$ for $v = 5$ and 9. Hence, by Theorem 1.1.1 (ii), there exist GBRD(v,4,12;EA(12)) for $v = 5,8,9,12$. Applying Hanani's theorem cited in Corollary 1.1.2 (i) gives the result.

□

There are more structured designs when $v \equiv 1 \pmod 4$ is a prime power.

THEOREM 8.1.2. *Let* $q = 4p+1$ *be a prime power. Then there exist* GSDS(q,4,12;EA(12)).

Proof. Use sets $(i = 0,1,\ldots,p-1)$

$\{x_{00}^{i}, x_{40}^{p+i}, x_{00}^{2p+i}, x_{40}^{3p+i}\}$, $\{x_{00}^{i}, x_{10}^{p+i}, x_{01}^{2p+i}, x_{11}^{3p+i}\}$,

$\{x_{00}^{i}, x_{41}^{p+i}, x_{31}^{2p+i}, x_{10}^{3p+i}\}$, $\{x_{00}^{i}, x_{11}^{p+i}, x_{30}^{2p+i}, x_{41}^{3p+i}\}$.

□

8.2. The group $Z_2 \times Z_2 \times Z_3$ with $\lambda = 24$.

THEOREM 8.2.1. *If* $v \geq 4$, *then a* GBRD$(v,4,24;Z_2 \times Z_2 \times Z_3)$ *exists.*

Proof. We have already constructed a GBRD$(4,4,12;Z_2 \times Z_2 \times Z_3)$. Take two copies of that design. Now we have the existence of GBRD$(v,4,6;Z_3)$ for all $v \geq 5$; using a GH$(4:Z_2 \times Z_2)$ and Theorem 1.1.2 (iii), we have the complete result.

□

8.3 The group $Z_2 \times Z_2 \times Z_3$ with $\lambda = 36$.

THEOREM 8.3.1. *There exists a* GBRD(v,4,36;EA(12)) *for all* $v \geq 4$.

Proof. By Corollary 1.1.2 (ii), it is sufficient to exhibit designs for $v \in K_4^2 = \{4,5,6,7,8,9,10,11,12,14,15,18,19,22,23\}$. By Theorem 8.1.1, GBRD(v,4,12;EA(12)) exist for $v \equiv 0,1 \pmod 4$, $5 \leq v \leq 23$; by Theorem 2.2, there exists a design for $v = 4$. We give cyclically generated designs in Table 6 (Appendix) for all $v \in K_4^2$ except for $v = 22$. But $22 = 7(4-1)+1$; so, by Theorem 1.1.3, the design on 22 points also exists.

□

The construction for $v = 23$ generalises to give the more structured designs for $q \geq 5$ an odd prime power.

THEOREM 8.3.2. *Let* $q \geq 5$ *be an odd prime power. Then there exists a* GSDS$(q,4,36;Z_2 \times Z_2 \times Z_3)$.

Proof. Let g be a generator of GF(q); then the sets $(g_{oe}^{i}, -g_{ou}^{i}, g_{iu}^{i+1}, -g_{ie}^{i+1})$, $(g_{oe}^{i}, -g_{oe}^{i}, g_{1u}^{i+1}, -g_{1w}^{i+1})$, where $i = 0,1,\ldots,\frac{p-3}{2}$ and $(u,w) \in \{(b,ab), (ab,a), (a,b)\}$, comprise CSDS(q,4,36;EA(12)).

□

Finally, we combine Theorems 8.2.1 and 8.3.1. to produce

THEOREM 8.3.3. *There exists a* GBRD(v,4,12t,EA(12)) *for all* $t > 1$, $v \geq 4$.

Proof. Let $t > 1$; then there exist $a,b \in N$ such that $t = 2a+3b$. Take a copies of a design with $t = 2$ and b copies of a design with $t = 3$. This gives the required design.

□

9. The group $Z_3 \times Z_3 \times Z_2^p$, $p > 0$.

The necessary conditions reduce to:

(i) $v \geq 4$;

(ii) $18 | \lambda$.

We have already dealt with $v = 4$ (see §2).

9.1 $Z_3 \times Z_3 \times Z_2$ with $\lambda = 18$.

LEMMA 9.1.1. *A* GBRD(v,4,18;$Z_2 \times Z_3 \times Z_3$) *exists whenever a* GBRD(v,4,2;Z_2) *exists.*

Proof. Use the GH(9;$Z_3 \times Z_3$) in Theorem 1.1.2 (iii). □

9.2 $Z_3 \times Z_3 \times Z_2$ with $\lambda = 36$.

LEMMA 9.2.1. *There exist* GBRD(v,4,36;EA(18)) *for* $4 \leq v \leq 25$ *except possibly for* $v = 6$ or 23.

Proof. By Lemma 7.1.2, there exist GBRD(v,4,6;Z_6) for $v = 7,9,10,\ldots,16,18,20$; there also exists a GH(4;Z_2). So one may apply Theorem 1.1.1 (iii) to obtain the designs for these v. Similarly, by Theorem 4.1.1, the designs exist for $v \equiv 1 \pmod 6$ a prime power; by Lemma 5.1.4 and Theorem 5.1.5, they exist for

$v = 5,8$, and 17. The design on 4 points is obtained from $GH(4;Z_2)$ and $GH(9;Z_3 \times Z_3)$.

□

LEMMA 9.2.2. *Let* $V = \{6,23,26,27,30,38,42,47\}$. *Then* $\mathbb{B}(\{v|4 \le v \le 22,\ v \ne 6,21\}) \supseteq \{v|v \ge 4$ *and* $v \notin V\}$.

Proof. As a corollary to Theorem 6.5.2, one obtains

$$\{v|4 \le v \le 104,\ v \notin V\} \subseteq \mathbb{B}(\{v|4 \le v \le 20,\ v \ne 6\}) \qquad (*)$$

We now apply Theorem 1.2.14 with $v_o = 4$, $t_o = 25$, $K = \{v|4 \le v \le 22,\ v \ne 6\}$ and $S = \{v|v \equiv \pm 1 (\text{mod } 6),\ v \ne 47\} \cup \{45\}$. By (*), a $PBD(\{v|4 \le v \le 20,\ v \ne 6\},v)$ exists for all $v \in {}_{25}S^{104}$; so they exist for all $v \ge 104$ except possibly for $v \in U$ (as defined in Theorem 1.2.14). Now, by Remark 1.2.15, for all $v \ge 104$ there exists a $t \in {}_{25}S$ and an f, $4 \le f \le 20$, such that $v = v_o t+f$; so, if $v \in U$, then $f = 6$. Now, if $t \equiv -1$ (mod 6), then $t-4 \in S$ and $v = 4(t-4)+22$; if $t \equiv 1$(mod 6), $t > 25$, $t \ne 49$, then $t-2 \in S$ and $v = 4(t-2)+14$. Finally, when $t = 25,45$, or 49, $v = 106$, 194, or 202, and suitable PBDs may be obtained by removing rows from a TD(11,19), TD(10,19), or a TD(10,11).

□

THEOREM 9.2.3. *A* $GBRD(v,4,18t;EA(18))$ *exists for* $t > 1$ *and* $v \ge 5$ *except possibly when*

(i) $t = 2$ *and* $v = 6,23,26,27,30,38,42,$ *or* 47,

(ii) $t = 3$ *or* 7 *and* $v = 28,34,$ *or* 39,

(iii) $t = 5$ *and* $v = 6,23,26,27,28,30,34,38,39,42,$ *or* 47.

Proof. Suppose $t = 2$. By Lemma 9.2.1, there exists a $GBRD(v,4,36;EA(18))$ for all $v \in \{v|4 \le v \le 22,\ v \ne 6\}$. So, by Lemma 9.2.2, these designs exist for all $v \ge 5$ except possibly for $v \in \{6,23,26,27,30,38,42,47\}$. Now suppose that t is even and $t \ge 4$. First suppose $t = 4$. By Theorem 5.2.2, there exist

$GBRD(v,4,6;Z_3)$ for all $v \geq 4$. There also exists a $GBRD(4,4,12;Z_6)$; so one may apply Theorem 1.1.1 (ii) to obtain the result for $t = 4$, $v \geq 4$. Now suppose $t = 6$. Assuming for the moment that the result for $t = 3$ is true, one obtains designs for all v, $4 \leq v < 28$. So designs exist for all $v \in K_2^4$ and hence for all $v \geq 4$. The result for $t = 4$ now gives all the designs for t even, $t \geq 8$.

Now we prove the result for t odd. Suppose $t = 3$. By Theorem 4.3.3, there exist $GBRD(v,4,6;Z_2)$ for all $v \geq 5$ except possibly for $v = 28,34$, or 39. Using $GBRD(4,4,9;Z_3 \times Z_3)$ in Theorem 1.1.1 (iii) gives the result for $t = 3$. The results for $t = 5$ and 7 follow from that for $t = 3$ and t even. Now consider $t = 9$. By Lemma 4.4.2, a $GBRD(v,4,18;Z_2)$ exists for $v \geq 5$. Now use the $GH(9;Z_3 \times Z_3)$ in Theorem 1.1.1 (iii). Finally suppose $t = 11$. Using the designs with $t = 3$ and those with $t = 8$, one has designs for $t = 11$ and $v \geq 5$ except when $v = 28,34,39,44$, or 58. But designs exist when $t = 9$ or 2 and $v = 28,34,39,44$, or 58; so designs exist for $t = 11$ and for all $v \geq 5$. Using these results and that for t even, $t > 2$, one has the result for t odd, $t > 11$.

□

10. The existence of $GBRD(v,4,t|G|;G)$ with $t > 1$.

Let $r \in N$ and define the function $\rho : N \to N \times N$ by $\rho(n) = (r,s)$ where 2^r is the greatest power of 2 dividing n and 3^s is the greatest power of 3 dividing n. We prove the following result.

THEOREM 10.1. *Let* $t > 1$ and G *be the elementary abelian group of order* $|G|$; *let* $\lambda = t|G|$ *and* $v > 4$. *Then the necessary conditions*

$$\lambda \equiv 0 \pmod{|G|},$$

$$\lambda(v-1) \equiv 0 \pmod{3},$$

$$\lambda v(v-1) \equiv 0 \pmod{12},$$

$$|G| \equiv 2 \pmod{4},\ v = 4 \Rightarrow t \text{ is even},$$

are sufficient for the existence of a $GBRD(v,4,t|G|;G)$ *except possibly* *when* $\rho(|G|) = (r,s)$ *and*

(i) $r = 0$, $s = 2$, $t = 2$ *and* $v = 6,18,23,26,27,38,42,47$,

(ii) $s = 0$, $r = 1$, $t = 3$ *and* $v = 28,34,39$,

(iii) $s = 0$, $r = 1$, $t = 5,7$ *and* $v = 28,34$,

(iv) $s = 0$, $r = 2$, $t = 3$ *and* $v = 15,23$,

(v) $s = 1$, $r = 1$,

1) $t = 3$ *and* $v = 8,17,22,23,24,27,32,33,34$,

2) $t = 5$ *and* $v = 8,17,24,27,32,33,34$,

3) $t = 7$ *and* $v = 8,17,32,33,34$,

(vi) $s = 2$, $r = 1$,

1) $t = 2$ *and* $v = 6,23,26,27,30,38,42,47$,

2) $t = 3$ *or* 7 *and* $v = 28,34,39$,

3) $t = 5$ *and* $v = 6,23,26,27,28,30,34,38,39,42,47$,

(vii) $s = 2$, $r = 2$, $t = 3$ *and* $v = 15,23$,

(viii) $s \geq 3$, $r = 1$, $t = 3,5,7$, *and* $v = 34$.

Proof. Because of the existence of GBRD(4,4,h;EA(h)) for $h \equiv \pm 1 \pmod 6$, we need only prove the result for $h = 2^r 3^s$. The exceptions for $r=0$ or for $s=0$ are simply those given in Theorems 6.6.2 and 5.8.1. The exceptions for $s=1$ and $r=1$ are those given in Theorem 7.4.1, while those in the case $s=2$, $r=1$ are given in Theorem 9.2.3.

Now suppose that $s=1$. We show that the conditions are sufficient when $r \geq 2$. The result is true for $r=2$ by Theorem 8.3.3, and hence the result holds for $r \geq 4$. Using Theorem 1.1.1 (iii) the result follows, for $r=3$ and t even, from Theorem 7.4.1. Now for $t = 3$ one obtains the required designs from Theorem 4.3.1 for all $v \geq 5$ except for $v = 28,34,39$. But by Theorem 2.2. the design exists for $v = 4$ and hence for $v = 4\times 7$, $11(4-1)+1$. Finally removing three rows from a TD(6,7) gives a PBD({4,5,6,7},39) which can be combined with the designs just obtained for $v = 4,\ldots,7$ completing the result for $r = 3$, $t = 3$. The result for $t \geq 3$, t odd, now follows from the result for t even. And finally the result for $s=1$, $r \geq 2$, follows, by Theorem 1.1.1 (iii), from the cases $r=2$ and $r=3$.

Now suppose that $s=2$. We show that the theorem is true for $r \geq 3$. The designs for $r \geq 3$ and t even exist by Theorem 5.8.1 and Theorem 1.1.1 (iii). The designs for $r \geq 3$ and $t = 3$ can be constructed from those given in Theorem 5.8.1 for all $v \geq 5$ except $v=28,34,39$. But by Theorem 2.2 the design on four points exists and the sufficiency of the necessary conditions in the case $s=2$, $r \geq 3$, now follows by an argument identical to that given in the previous paragraph.

Now suppose $s=r=2$. The result for t even follows from Theorem 5.8.1. The result for t odd follows from Theorem 6.6.1.

Sufficiency when $s \geq 3$ and $r \geq 2$ follows from that for $s=1$ and $r \geq 2$. Finally the case $r=1$ follows from Theorems 7.4.1 and 9.2.3. □

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APPENDIX

Table 1

v		Construction
9	36	Use a TD(7,5) to form a PBD(36,{8,5}).
10	40	Use a TD(8,5).
11		
12		
13		
14	56	Use a TD(11,5) to form a PBD[{5,12},56].
15	60	Use a TD(12,5) to form a PBD[{5,12},60].
16	64	Use BIBD(64,8,1). (8 is a prime power).
17		
18	72	Use a TD(9,8) to form a PBD[{9,8}, 72].
19		
20	80	Use a TD(16,5).
21		
22		
23		
24	96	Remove a row of the BIBD($2^6+2^3+1,2^3+1,1$).
26	104	Use a TD(13,8).
31		
32	128	Use a TD(16,8).
33	132	
34	136	Use a TD(17,8).
36	144	Use a TD(16,9).
56	224	Use a TD(25,9). Remove one row and add 9 columns.
79	316	Use a TD(25,13). Remove 9 rows from one group and complete.
81	324	Similar to 316 only remove 1 row.
116	464	Use a TD(29,16). □

Table 2

v	Construction
4	
10	
13	SBIBD(3^2 + 3+1,4,1)
16	BIBD(4^2,4,1)
19	
22	
25	BIBD(25,4,1)
28	BIBD(28,4,1)
31	PBD[{4,10},31]
34	
37	BIBD(37,4,1)
40	BIBD(40,4,1)
43	
46	3×15+1 get a GDD on 15 points with k=4 from BIBD(16,4,1)
55	
67	
79	
82	3×27+1 get a GDD on 27 points with k=4 from BIBD(28,4,1)
91	3×28+1 use BIBD(28,4,1) as a GDD on groups of size 1.
139	3×49+1 use BIBD(49,4,1) as a GDD on groups of size 1.
199	

Table 3

GBRD$(v,4,6;Z_3)$

Number of treatments	Construction: develop the initial blocks indicated
4	see text;
*5	$(0_0,1_0,2_1,4_1)$ twice (mod $5,Z_3$) ;
6	see text
*7	$(0_0,1_0,3_1,4_1)$, $(0_0,1_1,2_0,6_1)$, $(0_0,1_1,3_0,5_1)$ (mod $7,Z_3$);
*8	$(\infty,0_0,1_1,3_2)$ twice, $(0_0,1_0,2_2,5_0)$ twice (mod $7,Z_3$);
*9	$(0_0,1_0,2_0,3_1)$, $(0_0,1_1,4_1,5_0)$, $(0_0,1_2,3_0,6_2)$, $(0_0,2_0,4_2,6_1)$ (mod $9,Z_3$);
*10	$(\infty,0_0,1_1,2_2)$, $(\infty,0_0,3_1,1_2)$, $(0_0,1_0,4_0,5_2)$, $(0_0,1_0,3_1,6_1)$, $(0_0,2_0,4_0,6_1)$ (mod $9,Z_3$);
11	$11 \equiv 1$ (mod 10);
*12	$(\infty,0_1,1_1,2_2)$, $(\infty,0_0,2_1,1_2)$, $(0_0,1_0,5_1,7_1)$, $(0_0,1_0,4_0,7_1)$, $(0_0,2_0,5_0,8_2)$, $(0_0,2_1,4_0,8_1)$ (mod $11,Z_3$);
*14	$(\infty,0_0,1_1,2_2)$, $(\infty,0_0,2_1,1_2)$, $(0_0,1_0,5_0,6_0)$, $(0_0,2_0,4_0,8_1)$, $(0_0,2_1,5_1,10_2)$, $(0_0,2_2,6_0,9_1)$, $(0_0,3_0,6_2,10_1)$ (mod $13,Z_3$);
*15	$(0_0,1_0,2_0,3_1)$, $(0_0,1_1,2_0,11_0)$, $(0_0,1_2,6_2,8_0)$, $(0_0,2_2,5_2,9_0)$, $(0_0,2_2,6_1,11_1)$, $(0_0,3_0,6_2,10_2)$, $(0_0,3_1,7_2,10_1)$ (mod $15,Z_3$);
*18	$(\infty,0_0,1_1,2_2)$, $(\infty,0_0,2_1,4_2)$, $(0_0,3_1,7_2,11_0)$, $(0_0,3_1,7_1,12_2)$, $(0_0,1_0,2_0,5_2)$, $(0_0,1_2,7_0,10_2)$, $(0_0,1_2,5_2,11_0)$, $(0_0,2_0,5_0,10_0)$ $(0_0,2_2,8_1,11_1)$ (mod $17,Z_3$);
19,22	$19 = 6(4-1) + 1$; $22 = 7(4-1) + 1$;
23	$(5_0^{\,i},-5_0^{\,i},5_1^{i+1},-5_1^{i+1})$, $i = 0,1,\ldots,10$ (mod $23,Z_3$).

Table 4

GBRD$(v,4,6;Z_6)$

v = 7	$(0_0,1_0,3_1,4_4)$, $(0_0,1_1,3_0,5_4)$, $(0_0,1_2,2_0,3_5)$ (mod $7,Z_6$)
v = 8	no cyclic solution found
v = 9	$(0_0,1_0,2_1,3_3),(0_0,1_3,4_2,5_1),(0_0,1_4,3_4,6_4),(0_0,2_2,4_1,6_5)$ (mod $9,Z_6$)
v = 10	$(\infty_0,0_0,1_1,7_2),(\infty_0,0_3,2_4,6_5),(0_0,1_4,3_1,7_1),(0_0,1_0,2_2,4_2)$, $(0_0,1_3,4_3,5_2)$ (mod $9,Z_6$)
v = 11	$(0_0,1_0,2_1,3_0),(0_0,1_2,5_4,7_3),(0_0,1_3,4_4,5_2),(0_0,2_2,4_0,7_5)$, $(0_0,2_3,5_0,8_2)$(mod $11,Z_6$)
v = 12	$(\infty_0,0_0,1_1,3_2),(\infty_0,0_3,3_4,7_5),(0_0,1_4,2_3,7_4),(0_0,2_4,4_3,7_1)$, $(0_0,1_0,2_2,8_0)$, $(0_0,1_3,3_3,6_2)$ (mod $11,Z_6$)
v = 13	$(0_0,1_0,2_1,3_0),(0_0,1_2,2_5,9_4),(0_0,1_4,5_1,8_4),(0_0,2_2,5_0,8_1)$, $(0_0,2_3,6_3,9_5)$, $(0_0,2_4,5_3,9_2)$ (mod $13,Z_6$)
v = 14	$(\infty_0,0_0,1_1,3_2),(\infty_0,0_3,3_4,12_5),(0_0,2_4,5_2,9_3),(0_0,1_0,2_2,6_0)$, $(0_0,1_3,5_5,8_5)$, $(0_0,1_5,3_5,6_2)$, $(0_0,2_3,6_3,8_2)$ (mod $13,Z_6$)
v = 15	$(0_0,1_0,2_1,3_0),(0_0,1_2,2_5,6_0),(0_0,1_4,6_4,8_0),(0_0,2_3,5_1,10_4)$, $(0_0,2_4,6_3,10_1)$, $(0_0,3_1,7_1,11_3)$, $(0_0,3_2,6_1,12_3)$ (mod $15,Z_6$)
v = 16	$(\infty_0,0_0,1_1,3_2),(\infty_0,0_3,3_4,14_5),(0_0,2_4,6_2,11_5),(0_0,3_4,6_3,10_5)$, $(0_0,1_0,2_3,5_0),(0_0,1_2,6_0,8_2)$, $(0_0,1_5,6_1,8_1)$, $(0_0,2_5,5_5,9_2)$, (mod $15,Z_6$)
v = 17	
v = 18	$(\infty_0,0_0,1_1,3_2),(\infty_0,0_3,3_4,16_5),(0_0,2_4,6_2,10_3),(0_0,3_4,7_1,12_2)$, $(0_0,1_0,7_4,8_0)$, $(0_0,1_3,2_2,14_3)$, $(0_0,2_0,7_2,10_1)$, $(0_0,2_3,5_3,11_0)$, $(0_0,2_5,6_1,11_1)$ (mod $17, Z_6$)
v = 19	

v = 20 $(\infty_0,0_0,1_1,3_2),(\infty_0,0_3,3_4,18_5),(0_0,2_4,6_1,10_2),(0_0,3_4,8_2,14_5),$
$(0_0,4_4,9_1,14_1),(0_0,1_0,8_1,9_0),(0_0,1_2,3_5,13_0),(0_0,1_3,3_3,7_5),$
$(0_0,2_2,5_2,12_0),\ (0_0,2_5,6_5,13_2)$ (mod 19, Z_6)

Table 5

GBRD$(v,4,12;Z_6)$

v = 4	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	1	1	2	2	3	3	4	4	5	5
	0	3	5	0	4	1	3	4	2	5	1	2
	0	3	2	5	5	3	1	2	4	0	4	1

v = 6 $(\infty,1_3,2_1,0_4),(\infty,1_0,3_5,4_2),(\infty,1_0,2_1,0_4),(\infty,1_3,3_5,4_2),$
$(1_0,2_1,3_1,4_0),\ (1_0,2_4,3_4,4_0)$ (mod $5,Z_6$);

v = 8 $(\infty,0_5,1_0,3_0),(\infty,0_3,2_3,3_5),(\infty,0_1,3_4,4_2),(\infty,0_4,4_1,3_2),$
$(1_3,6_0,3_1,4_1),(3_3,4_0,2_1,5_1),(6_3,1_0,4_1,3_1),(4_3,3_0,5_1,2_1)$
(mod $7,Z_6$); or
$(\infty,0_0,1_1,3_2),(\infty,0_0,1_4,3_5),(\infty,0_3,1_1,3_5),(\infty,0_3,1_4,3_2),$
$(0_0,1_0,2_2,5_0),(0_0,1_0,2_5,5_3),(0_0,1_3,2_2,5_3),(0_0,1_3,2_5,5_0)$
(mod $7,Z_6$);

v = 10 $(\infty,0_0,1_0,2_3),(\infty,0_0,2_1,3_3),(\infty,0_4,1_4,2_5),(\infty,0_5,2_2,3_2),$
$(0_0,1_3,4_2,5_1)$ twice, $(0_0,1_4,3_4,6_4)$ twice, $(0_0,2_2,4_1,6_5)$ twice
(mod $9,Z_6$);

v = 12 from GBRD$(12,4,3;Z_3)$ and GH$(4,Z_2)$;

v = 14 $(\infty,0_2,2_0,7_2),(\infty,0_4,5_1,7_3),(\infty,0_3,1_5,4_2),(\infty,0_4,1_1,4_5),$
$(0_0,1_0,3_0,9_0),(0_0,1_0,3_3,9_3),(0_0,1_1,3_5,9_2),(0_0,1_1,3_0,9_5),$
$(0_0,1_2,3_1,9_3),(0_0,1_3,3_4,9_2),(0_0,1_4,3_2,9_4),(0_0,1_4,3_5,9_0),$
$(0_0,1_5,3_1,9_4),\ (0_0,1_5,3_2,9_1)$ (mod $13,Z_6$);

Table 6

GBRD(v,4,36;EA(12)) for $v \in K_4^2$ with $v \equiv 2, 3 \pmod 4$

v = 6 $(1_{1e},2_{0e},3_{0e},4_{1e})$ 3 times, $(1_{1e},2_{0u},3_{0w},4_{1uw}),(\infty_{0e},0_{0e},1_{1u},4_{1w}),$
$(\infty_{0e},0_{0u},2_{2e},3_{2w}),(\infty_{0e},0_{0w},1_{1e},4_{1u}),(\infty_{0e},0_{0u},2_{2w},3_{2uw})$ (mod 5)
where $(u,w) \in \{(b,ab),(ab,a),(a,b)\}$

v = 7 $(0_{0e},1_{0e},3_{1u},4_{1e}),(0_{0e},1_{0u},3_{1w},4_{1uw}),(0_{0e},1_{1e},2_{0e},6_{1u}),$
$(0_{0e},1_{1u},2_{0w},6_{1uw}),(0_{0e},1_{1u},3_{0e},5_{1e}),(0_{0e},1_{1u},3_{0w},5_{1uw})$, (mod 7)
where (u,w) is as above.

v = 10 $(\infty_{0e},0_{0e},1_{1u},2_{2u}),(\infty_{0e},0_{0u},1_{1w},2_{2uw}),(\infty_{0e},0_{0u},3_{1e},1_{2e}),$
$(\infty_{0e},0_{ou},3_{1w},1_{2uw}),(0_{0e},1_{0u},4_{0u},5_{2u}),(0_{0e},1_{0u},4_{0e},5_{2w}),$
$(0_{0e},1_{0e},3_{1e},b_{1u}),(0_{0e},1_{0u},3_{1w},6_{1uw}),(0_{0e},2_{0u},4_{0e},6_{1e}),$
$(0_{0e},2_{0u},4_{0w},6_{1uw})$, (mod 9), (u,w) as before.

v = 11 $(0_{0e},1_{0e},2_{1u},3_{0e}),(0_{0e},1_{0u},2_{1u},3_{0w}),(0_{0e},1_{2e},5_{1e},7_{0e}),$
$(0_{0e},1_{2u},5_{1w},7_{uw}),(0_{0e},1_{0u},4_{1e},5_{2w}),(0_{0e},1_{0u},4_{1w},5_{2uw}),$
$(0_{0e},2_{2u},4_{0u},7_{2w}),(0_{0e},2_{2u},4_{0w},7_{2uw}),(0_{0e},2_{0u},5_{0w},8_{2uw}),$
$(0_{0e},2_{0u},5_{0e},8_{2e})$,(mod 11), (u,w) as before.

v = 14 $(\infty_{0e},0_{0e},1_{1u},2_{2u}),(\infty_{0e},0_{0u},1_{1w},2_{2uw}),(\infty_{0e},0_{0u},2_{1e},1_{2e}),$
$(\infty_{0e},0_{0u},2_{1w},1_{2uw}),(0_{0e},1_{0e},5_{0e},6_{0u}),(0_{0e},1_{0u},5_{0w},6_{0uw}),$
$(0_{0e},2_{0e},4_{0u},8_{1w}),(0_{0e},2_{0u},4_{0w},8_{1uw}),(0_{0e},2_{1e},5_{1e},10_{2u}),$
$(0_{0e},2_{1u},5_{1w},10_{2u}),(0_{0e},2_{2e},6_{0e},9_{1e}),(0_{0e},2_{2u},b_{0w},9_{1uw}),$
$(0_{0e},3_{0u},6_{2e},10_{1w}),(0_{0e},3_{0u},6_{2u},10_{1w})$, (mod 13),(u,w) as before.

v = 15 $(0_{0e},1_{0e},2_{0u},3_{1e}),(0_{0e},1_{0u},2_{0w},3_{1uw}),(0_{0e},1_{1e},2_{0e},11_{0u}),$
$(0_{0e},1_{1u},2_{0w},3_{0uw}),(0_{0e},1_{2u},b_{2w},8_{0u}),(0_{0e},1_{2u},6_{2w},8_{0uw}),$
$(0_{0e},2_{2e},5_{2e},9_{0e}),(0_{0e},2_{2u},5_{2w},9_{0uw}),(0_{0e},2_{2u},6_{1e},11_{1e}),$
$(0_{0e},2_{2u},6_{1w},11_{1uw}),(0_{0e},3_{0u},6_{2e},10_{2e}),(0_{0e},3_{0u},6_{2w},10_{2uw}),$
$(0_{0e},3_{1u},7_{2w},10_{1w}),(0_{0e},3_{1u},7_{2e},10_{1w})$,(mod 15),(u,w) as before.

$v = 18$ $(\infty_{0e}, 0_{0e}, 1_{1u}, 2_{2u})$, $(\infty_{0e}, 0_{0u}, 1_{1w}, 2_{2uw})$, $(\infty_{0e}, 0_{0u}, 2_{1e}, 4_{2e})$, $(\infty_{0e}, 0_{0u}, 2_{1w}, 4_{2uw})$, $(0_{0e}, 3_{1e}, 7_{2e}, 11_{0u})$, $(0_{0e}, 3_{1u}, 7_{2w}, 11_{0e})$, $(0_{0e}, 3_{1u}, 7_{1u}, 12_{2u})$, $(0_{0e}, 3_{1u}, 7_{1w}, 12_{2uw})$, $(0_{0e}, 1_{0e}, 2_{0u}, 5_{2e})$, $(0_{0e}, 1_{0u}, 2_{1w}, 5_{2uw})$, $(0_{0e}, 1_{2e}, 7_{0e}, 10_{2e})$, $(0_{0e}, 1_{2u}, 7_{0w}, 10_{2uw})$, $(0_{0e}, 1_{2u}, 5_{2w}, 11_{0uw})$, $(0_{0e}, 1_{2u}, 5_{2w}, 11_{0uw})$, $(0_{0e}, 2_{0e}, 5_{0e}, 10_{0u})$, $(0_{0e}, 2_{0u}, 5_{0w}, 10_{0uw})$, $(0_{0e}, 2_{2e}, 8_{1e}, 11_{1u})$, $(0_{0e}, 2_{2u}, 8_{1w}, 11_{1uw})$, (mod 17), (u,w) as before.

$v = 23$ $(5^i_{0e}, -5^i_{0u}, 5^{i+1}_{1u}, -5^{i+1}_{1e}), (5^i_{0e}, -5^i_{0e}, 5^{i+1}_{1u}, -5^{i+1}_{1w})$, $i = 0,1,\ldots,10$ and (u,w) as before.