

GENERALIZED BHASKAR RAO DESIGNS

Clement LAM*

Department of Computer Science, Concordia University, Montreal, Quebec, Canada

Jennifer SEBERRY

Department of Applied Mathematics, University of Sydney, New South Wales 2006, Australia

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Abstract: Generalized Bhaskar Rao designs with non-zero elements from an abelian group G are constructed. In particular this paper shows that the necessary conditions are sufficient for the existence of generalized Bhaskar Rao designs with $k=3$ for the following groups: $|G|$ is odd, $G = \mathbb{Z}_2^r$, and $G = \mathbb{Z}_2^r \times H$ where $3 \nmid |H|$ and $r \geq 1$. It also constructs generalized Bhaskar Rao designs with $v=k$, which is equivalent to v rows of a generalized Hadamard matrix of order n where $v \leq n$.

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1. Introduction

Bhaskar Rao designs with elements $0, \pm 1$ have been studied by a number of authors including Bhaskar Rao (1966, 1970), Seberry (1982, 1984), Singh (1982), Sinha (1978), Street (1981), Street and Rodger (1980) and Vyas (1982). Bhaskar Rao (1966) used these designs to construct partially balanced designs and this was improved by Street and Rodger (1980) and Seberry (1984). Another technique for studying partially balanced designs has involved looking at generalized orthogonal matrices which have elements from elementary abelian groups and the element 0. Matrices with group elements as entries have been studied by Berman (1977, 1978), Butson (1962, 1963), Delsarte and Goethals (1969), Drake (1979), Rajkundlia (1978), Seberry (1979, 1980), Shrikhande (1964) and Street (1979). In this paper we give some results about such matrices which are not square. This extends work of Seberry (1982, 1984).

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Suppose we have a matrix X with nonzero elements from an abelian group $G = \{h_1, h_2, \dots, h_g\}$, where $X = h_1 A_1 + h_2 A_2 + \dots + h_g A_g$, with A_1, \dots, A_g $v \times b$ $(0, 1)$ -matrices, and the Hadamard product $A_i * A_j = 0$ for $i \neq j$. A generalized Bhaskar Rao design or GBRD was first defined in Seberry (1982). In this paper we are concerned with the special case where the matrix X satisfies

$$XX^+ = rI + \frac{\lambda G}{g} (J - I), \quad (1.1)$$

and

$$NN^T = (r - \lambda)I + \lambda J, \quad \text{where } N = A_1 + \dots + A_g. \quad (1.2)$$

The term $\lambda G/g$ is $(\lambda/g)(h_1 + \dots + h_g)$ and the product XX^+ is defined by

$$(XX^+)_{ij} = \sum_{k=1}^b X_{ik} X_{jk}^{-1},$$

with the sum taken in the group ring of G over the integers. In other words, N is a $\text{BIBD}(v, b, r, k, \lambda)$ and X is obtained by replacing the 1's of N by elements of G in such a way that the off-diagonal terms of XX^+ are all $(\lambda/g)(h_1 + \dots + h_g)$. Such a matrix X is a $\text{GBRD}_G(v, b, r, k, \lambda)$. Since $\lambda(v-1) = r(k-1)$ and $bk = vr$ we usually use the notation $\text{GBRD}(v, k, \lambda; G)$ and $\text{BIBD}(v, k, \lambda)$.

These matrices are generalizations of generalized weighing matrices (Berman (1977, 1978), Seberry (1982), Theorem 6) has shown how they may be used in the construction of PBIBD's.

A generalized Hadamard matrix $H = \text{GH}(b, G)$ is a matrix of order b with elements from an abelian group $G = \{h_1, \dots, h_g\}$ with the property that if $x = (x_1, \dots, x_b)$ and $y = (y_1, \dots, y_b)$ are two distinct rows of H then

$$\sum_{i=1}^b x_i y_i^{-1} = \frac{b}{g} (h_1 + \dots + h_g).$$

Such matrices have been studied by a number of authors: Butson (1962), Drake (1979), Seberry (1980) and Street (1979). We note that a $\text{GH}(b, G)$ can be regarded as a $\text{GBRD}(b, b, b; G)$ and they exist for b a prime power and other orders (see Street (1979) and Seberry (1980)).

In Seberry (1982), generalized Hadamard matrices were used to construct generalized Bhaskar Rao designs. The construction used only a subset of the rows of a generalized Hadamard matrix. In Section 3 of this paper, we present constructions for a $\text{GBRD}(k, k, |G|; G)$, with $k \leq |G|$. We use these designs in Sections 4 and 5 to prove that the necessary conditions are sufficient for the existence of a generalized Bhaskar Rao design with $k=3$ in the following cases:

$$|G| \text{ is odd}, \quad (1.3)$$

$$G = Z_2^r, \quad (1.4)$$

and

$$G = Z_2^r \times H \quad \text{where } 3 \nmid |H| \text{ and } r \geq 1. \quad (1.5)$$

2. Some construction theorems

The following theorem which we use extensively first appeared in Seberry (1982).

Theorem 2.1. *Suppose there are generalized Bhaskar Rao designs $\text{GBRD}(v, k, \lambda; G)$ and $\text{GBRD}(u, k, \mu; G)$. Further suppose there are $k-1$ mutually orthogonal latin squares of order u . Then there is a generalized Bhaskar Rao design $\text{GBRD}(uv, k, \lambda\mu; G)$.*

In this paper we are concerned with $k=3$. We recall (see Wilson (1974)) that for every order greater than 2, except 6, there are two mutually orthogonal latin squares.

The next two theorems are generalizations of Theorems 3 and 4 of Seberry (1984).

Theorem 2.2. *Suppose we have a $\text{GBRD}(v, k, \lambda; H)$, A , and a $\text{GBRD}(k, j, \mu; G)$, B . Then there exists a $\text{GBRD}(v, j, \lambda\mu; H \times G)$.*

Proof. The new GBRD is found by replacing the j -th non-zero element, say x , of each column of A by x times the j -th row of B . \square

Remark. In particular this means that if H of the theorem is the trivial group consisting only of the identity that A is a BIBD.

Corollary 2.3. *Suppose we have a $\text{BIBD}(v, k, \lambda)$ and a $\text{GBRD}(k, j, \mu; G)$. Then there exists a $\text{GBRD}(v, j, \lambda\mu; G)$.*

Theorem 2.4. *Suppose there exists a $\text{GBRD}(v, k, \lambda; G)$, B , and a $\text{GBRD}(u, k, \lambda; G)$, A , with a subdesign on w treatments, X , which is a $\text{GBRD}(w, k, \lambda; G)$ or $w=0, 1$. Further suppose there exist $k-1$ mutually orthogonal latin squares of order $u-w$. Then there exists a $\text{GBRD}(v', k, \lambda; G)$ where $v' = v(u-w) + w$ with a subdesign on w treatments.*

Proof. As in the proof of Theorem 3 of Seberry (1984) we form $(0, 1)$ matrices $M_{ij}, i=1, \dots, k-1, j=1, \dots, u-w$, from the $k-1$ mutually orthogonal latin squares. These matrices satisfy

$$\sum_{j=1}^{u-w} M_{ij} = J, \tag{2.1}$$

$$\sum_{j=1}^{u-w} M_{ij} M_{ij}^T = (u-w)I, \tag{2.2}$$

and

$$\sum_{j=1}^{u-w} M_{ij} M_{kj}^T = J \text{ for } k \neq i. \tag{2.3}$$

We write

$$C = \begin{bmatrix} I & I & \cdots & I \\ M_{1,1} & M_{1,2} & & M_{1,u-w} \\ \vdots & & & \vdots \\ M_{k-1,1} & M_{k-1,2} & \cdots & M_{k-1,u-w} \end{bmatrix}.$$

We now form D_i , $i = 1, \dots, u - w$, from B and the i th column block of C by replacing the j th nonzero element of B , say g_j , by the matrix g_j times $M_{j,i}$. The zeros of B are replaced by zero matrices of size $u - w$. We also write

$$A = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix},$$

where X is the $\text{GBRD}(w, k, \lambda; G)$. If $w = 0$, then both X and Y are vacuous. If $w = 1$, then X is vacuous. We claim that

$$\left[\begin{array}{c|ccc|ccc} X & Y & Y & \cdots & Y & 0 & \cdots & 0 \\ \hline 0 & Z & & & 0 & & & \\ \vdots & & Z & & & D_1 & \cdots & D_{u-w} \\ \vdots & & & \ddots & & & & \\ 0 & 0 & & & Z & & & \end{array} \right]$$

is a $\text{GBRD}(v(u - w) + w, k, \lambda; G)$. \square

3. Special constructions for GBRD 's with $v = k$

A useful method to construct a $\text{GBRD}(v, k, \lambda; G)$ is to use Corollary 2.3 with a $\text{BIBD}(v, k, 1)$ and a $\text{GBRD}(k, k, \lambda; G)$. We now give two methods for constructing GBRD 's with $v = k$. The first method is for cyclic groups. We let Z_n denote the cyclic group of order n . The elements are written as $0, \dots, n - 1$ and the group operation is written additively. If $R = (r_1, \dots, r_s)$ is a vector of s integers and t is an integer, then $tR = (tr_1, tr_2, \dots, tr_s)$ and $t + R = (t + r_1, t + r_2, \dots, t + r_s)$.

Theorem 3.1. *Suppose there are generalized Bhaskar Rao designs $\text{GBRD}(k, k, n; Z_n)$, A , and $\text{GBRD}(k, k, m; Z_m)$, B . Then there is a $\text{GBRD}(k, k, mn, Z_{mn})$.*

Proof. We form a new matrix C by replacing every A_{ij} in A by a row $A_{ij} + n(B_{i_1}, B_{i_2}, \dots, B_{i_m})$ where the resulting vector is reduced modulo mn . One can easily verify that C is a $\text{GBRD}(k, k, mn, Z_{mn})$. \square

Since generalized Hadamard matrices of all prime orders exist, we have:

Corollary 3.2. *Generalized Bhaskar Rao designs $\text{GBRD}(p, p, p^t; Z_{p^t})$ exist for all prime powers p^t .*

Proof. Use the existence of $\text{GBRD}(p, p, p; Z_p)$ and Theorem 3.1 recursively with $n=p$ and $m=p^{t-1}$. \square

One should note that generalized Hadamard matrices of prime power orders also exist, but they are for elementary abelian p -groups.

Theorem 3.3. *Suppose there are generalized Bhaskar Rao designs $\text{GBRD}(k, k, n; G)$, A , and $\text{GBRD}(k, k, m; H)$, B . Then there is a $\text{GBRD}(k, k, nm; G \times H)$.*

Proof. We form a new matrix C by replacing every A_{ij} of A by the vector of ordered pairs $[(A_{ij}, B_{i1}), (A_{ij}, B_{i2}), \dots, (A_{ij}, B_{im})]$. Again C is the required $\text{GBRD}(k, k, nm; G \times H)$. \square

Corollary 3.4. *A generalized Bhaskar Rao design $\text{GBRD}(k, k, |G|; G)$ exists if for every prime p dividing $|G|$,*

$$k \leq \begin{cases} p^r & \text{if the Sylow } p\text{-subgroup of } G \text{ is elementary abelian} \\ & \text{and has order } p^r, \text{ and} \\ p & \text{if the Sylow } p\text{-subgroup is not elementary abelian.} \end{cases} \quad (3.1)$$

Proof. The group G is a direct product of its Sylow p -subgroups. If a Sylow p -subgroup H is elementary abelian, then we use the $\text{GBRD}(p^r, p^r, p^r; H)$ obtained from a generalized Hadamard matrix. Otherwise, we build up a $\text{GBRD}(p, p, |H|; H)$ using $\text{GBRD}(p, p, p^t; Z_{p^t})$'s. If k satisfies (3.1), then a $\text{GBRD}(k, k, |H|; H)$ exists for all Sylow p -subgroups and hence it exists for G . \square

In the special case when $k=3$, we have:

Corollary 3.5. *A generalized Bhaskar Rao design $\text{GBRD}(3, 3, |G|; G)$ exists if the Sylow 2-subgroup of G is trivial or is elementary abelian of order at least 4.*

4. Existence of generalized Bhaskar Rao designs with $k=3$

With reference to the definition of a generalized Bhaskar Rao design, it is clear that the necessary conditions for the existence of a $\text{GBRD}(v, 3, \lambda; G)$ are

$$\lambda \equiv 0 \pmod{g}, \quad \text{where } g = |G|, \quad (4.1)$$

$$\lambda(v-1) \equiv 0 \pmod{2}, \quad (4.2)$$

$$\lambda v(v-1) \equiv 0 \pmod{6}. \quad (4.3)$$

In fact, (4.2) and (4.3) are necessary and sufficient conditions for the existence of a $\text{BIBD}(v, 3, \lambda)$ (Hall (1967), Theorem 15.4.5). Condition (4.1) is from (1.1) because the off diagonal entries cannot contain fractions. If $G = Z_2$, Seberry (1984) proved

an extra necessary condition that $b \equiv 0 \pmod{4}$. The following result establishes a similar result when $G = Z_2 \times H$.

Proposition 4.1. *If a generalized Bhaskar Rao design $\text{GBRD}(v, k, \lambda; H \times K)$, A , exists then a $\text{GBRD}(v, k, \lambda; K)$ exists.*

Proof. Write the elements of $H \times K$ as ordered pairs and form a new matrix B from A by replacing every occurrence of an element (x, y) with $(1, y)$ where 1 is the identity element of H . This matrix B is a $\text{GBRD}(v, k, \lambda; K)$. \square

Corollary 4.2. *A necessary condition for the existence of a $\text{GBRD}(v, 3, \lambda; H \times Z_2)$ is that*

$$b \equiv 0 \pmod{4}. \quad (4.4)$$

Proof. If a $\text{GBRD}(v, 3, \lambda; H \times Z_2)$ exists then a $\text{GBRD}(v, 3, \lambda; Z_2)$ exists and b , the number of columns of the matrix, is unchanged. The result now follows from the corresponding result for $G = Z_2$. \square

Lemma 4.3. *If a generalized Bhaskar Rao design $\text{GBRD}(v, 3, \lambda; Z_3)$, A , exists then a $\text{GBRD}(v, 3, \lambda n; Z_{3n})$ exists when n is odd.*

Proof. Let ω be the generator of Z_3 and let ξ be the generator for Z_{3n} . Suppose that ω^a, ω^b and ω^c are the three non-zero entries of a column of A . We form a new matrix by replacing the entry ω^a by a row $(\xi^a, \xi^a, \dots, \xi^a)$, then entry ω^b by a row $(\xi^b, \xi^{b+3}, \xi^{b+6}, \dots, \xi^{b+3(n-1)})$, the entry ω^c by $(\xi^c, \xi^{c-3}, \xi^{c-6}, \dots, \xi^{c-3(n-1)})$ and the zero entries by a row of n zeros. We claim that the resulting matrix is a $\text{GBRD}(v, 3, \lambda n; Z_{3n})$. The condition n odd is needed in the proof because the contribution of ω^{b-c} is now replaced by $\xi^{b-c}(1 + \xi^6 + \xi^{12} + \dots + \xi^{6(n-1)})$ which becomes $\xi^{b-c}(1 + \xi^3 + \xi^6 + \dots + \xi^{3(n-1)})$ when $\text{g.c.d.}(3n, 2) = 1$. \square

Theorem 4.4. *The necessary conditions (4.1) to (4.3) are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; G)$ when g is odd.*

Proof. We consider first the case when 3 does not divide g . From Corollary 3.5, a $\text{GBRD}(3, 3, g; G)$ exists. We now show that the conditions (4.1) to (4.3) imply the existence of a $\text{BIBD}(v, 3, \lambda/g)$. From (4.1), λ/g is an integer. Since $\text{g.c.d.}(g, 6) = 1$, (4.2) and (4.3) imply $(\lambda/g)(v-1) \equiv 0 \pmod{2}$ and $(\lambda/g)v(v-1) \equiv 0 \pmod{6}$. Hence a $\text{BIBD}(v, 3, \lambda/g)$ exists and now Theorem 2.2 implies that a $\text{GBRD}(v, 3, \lambda; G)$ exists.

For the remaining cases, we use Theorem 2.3 to build a $\text{GBRD}(v, 3, \lambda; G)$ from one where $G = Z_3$ (the existence of these is settled in Seberry (1982)).

Let us consider $G = Z_{3^{t+1}}$ where $t \geq 1$. We start from a $\text{GBRD}(v, 3, \lambda/3^t; Z_3)$. If the parameters of a $\text{GBRD}(v, 3, \lambda; Z_{3^{t+1}})$ satisfy (4.1) to (4.3), then the parameters of a $\text{GBRD}(v, 3, \lambda/3^t; Z_3)$ satisfy the corresponding set of necessary conditions and

thus the GBRD exists by Seberry (1982), Theorem 5. Now, Lemma 4.3 implies that $\text{GBRD}(v, 3, \lambda; Z_{3^{t+1}})$ exists.

For a general abelian group G with 3 dividing g , we let $G = Z_{3^t} \times H$ for some integer t , using the Fundamental Theorem of abelian groups. We let $h = |H|$ and we note that a $\text{GBRD}(v, 3, \lambda/h; Z_{3^t})$ exists because the corresponding conditions (4.1) to (4.3) are satisfied. Now a $\text{GBRD}(v, 3, \lambda; G)$ is constructed using Theorem 2.2 with a $\text{GBRD}(3, 3, h; H)$ constructed via Corollary 3.5. \square

In a similar manner, we have:

Theorem 4.5. *The necessary conditions (4.1) to (4.4) are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; G)$ when $G = Z_2 \times H$ where $|H| = h$ and $\text{g.c.d.}(h, 6) = 1$.*

Proof. The $\text{GBRD}(v, 3, \lambda; G)$ is constructed from a $\text{GBRD}(v, 3, \lambda/h; Z_2)$ and a $\text{GBRD}(3, 3, h; H)$. The conditions (4.1) to (4.4) imply that the necessary conditions for the existence of a $\text{GBRD}(v, 3, \lambda/h; Z_2)$ are satisfied (Seberry (1984), Theorem 15). \square

5. Case $G = Z_2 \times Z_2$

In this section, we shall show that the necessary conditions (4.1) to (4.3) are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; Z_2 \times Z_2)$.

We note that a $\text{GBRD}(3, 3, 4; Z_2 \times Z_2)$ exists by Corollary 3.5. Suppose we write the elements of $Z_2 \times Z_2$ as $1, a, b$ and ab . Then a $\text{GBRD}(4, 3, 4; Z_2 \times Z_2)$ is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & a & b & 0 & b & ab \\ 1 & a & 0 & ab & ab & b & 0 & 1 \\ 1 & b & a & 0 & a & ab & 1 & 0 \end{bmatrix}, \tag{5.1}$$

and a $\text{GBRD}(6, 3, 4; Z_2 \times Z_2)$ is

$$\left[\begin{array}{ccc|ccc|ccc|ccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & a & b & ab & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & a & b & ab & 0 & 0 & a & b & ab & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & a & 0 & 0 & b & ab & 0 & 1 & 0 & 0 & b & ab & 0 & b & ab & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & b & 0 & a & 0 & ab & 0 & 1 & 0 & ab & 0 & a & 0 & a & ba & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & ab & 0 & a & b & 0 & 0 & 1 & 0 & b & a & b & 0 & a & ab \end{array} \right]. \tag{5.2}$$

These two designs are used to build up other GBRD's. We shall establish the main result of this section (Theorem 5.4) by a sequence of lemmas.

Lemma 5.1. *The necessary conditions (4.1) to (4.3) are sufficient for the existence of a $\text{GBRD}(v, 3, 4; Z_2 \times Z_2)$.*

Proof. We consider first the case where v is odd. We construct a $\text{GBRD}(v, 3, 4; \mathbb{Z}_2 \times \mathbb{Z}_2)$ using Theorem 2.2 with a $\text{BIBD}(v, 3, 1)$ and a $\text{GBRD}(3, 3, 4; \mathbb{Z}_2 \times \mathbb{Z}_2)$. The fact that a $\text{BIBD}(v, 3, 1)$ exists is a result of conditions (4.1) to (4.3).

When v is even, we note that the necessary conditions imply that $v \equiv 0$ or $4 \pmod{6}$. We shall prove the result by induction on v . Suppose v' is the smallest value for which the necessary condition is not sufficient. Then v' modulo 18 is one of the six values: 0, 4, 6, 10, 12 or 16. We shall construct a GBRD for this value v' using Theorem 2.3. For $v' = 18t$, we use $v = 6t$, $u = 3$ and $w = 0$. For $v' = 18t + 4$, we use $v = 6t + 1$, $u = 4$ and $w = 1$. For $v' = 18t + 6$, we use $v = 6$, $u = 3t + 1$ and $w = 0$. For $v' = 18t + 10$, we use $v = 3$, $u = 6t + 4$ and $w = 1$. For $v' = 18t + 12$, we use $v = 3$, $u = 6t + 4$ and $w = 0$. Finally, for $v' = 18t + 16$, we use $v = 3$, $u = 6t + 6$ and $w = 1$. Note that the existence of $\text{GBRD}(4, 3, 4; \mathbb{Z}_2 \times \mathbb{Z}_2)$ and $\text{GBRD}(6, 3, 4; \mathbb{Z}_2 \times \mathbb{Z}_2)$ provides the starting points for the induction. \square

Thus we have shown that when $v \equiv 0$ or $1 \pmod{3}$ and $\lambda = 4$, a GBRD over $\mathbb{Z}_2 \times \mathbb{Z}_2$ exists. By repeating blocks when necessary, we have:

Lemma 5.2. *When $v \equiv 0$ or $1 \pmod{3}$, the necessary conditions are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; \mathbb{Z}_2 \times \mathbb{Z}_2)$.*

We now consider $v \equiv 2 \pmod{3}$, which only occurs when $\lambda \equiv 0 \pmod{3}$. The only difficult case is $v \equiv 2 \pmod{4}$ and $\lambda = 12$. This case is handled by the following construction.

Construction 5.3. *If a $\text{GBRD}(t, 3, 12; \mathbb{Z}_2 \times \mathbb{Z}_2)$ exists, then a $\text{GBRD}(4t + 2, 3, 12; \mathbb{Z}_2 \times \mathbb{Z}_2)$ exists.*

Proof. We start with the matrix

$$\left[\begin{array}{cccccc|cccccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]. \quad (5.3)$$

We next form a 6×48 matrix which we write as

$$\begin{bmatrix} J & 0 & A_1 & A_2 & A_3 & A_4 \\ 0 & J & A_1 & A_2 & A_3 & A_4 \end{bmatrix}^T, \quad (5.4)$$

from (5.3) by replacing the first 1 of every column by the first row of a $\text{GBRD}(3, 3, 4; \mathbb{Z}_2 \times \mathbb{Z}_2)$, the second 1 of every column by the second row of the GBRD and the third 1 by the third row of the GBRD . We observe that for the matrix in (5.4), the inner product of the first 2 rows is 0; the inner product of one of the

first two rows with one of the bottom 4 rows is $3(1 + a + b + ab)$ and the inner product of two of the bottom 4 rows is $2(1 + a + b + ab)$.

Next we form a matrix W

$$W^T = \begin{bmatrix} J & 0 & A_1 & A_2 & A_3 & A_4 \\ 0 & J & A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & R_1 & R_2 & R_3 & R_4 \end{bmatrix}^T, \tag{5.5}$$

where R_1 to R_4 are the 4 rows of a $\text{GBRD}(4, 3, 4; \mathbb{Z}_2 \times \mathbb{Z}_2)$. Now, the inner product of two rows of W^T is either 0 or $3(1 + a + b + ab)$. We let W_1, \dots, W_6 be the 6 rows of W^T .

We now construct the first of a $\text{GBRD}(4 + 2, 3, 12; \mathbb{Z}_2 \times \mathbb{Z}_2)$ as follows:

C_1	W_1	\dots	W_1
C_2	W_2		W_2
C_3			
\vdots	0	0	0
C_6			
0	W_3	0	0
	\vdots	0	0
	W_6	0	0
0	0	\ddots	0
0	0	0	W_3
			\vdots
			W_6

(5.6)

t blocks
each with 4 rows

where $C_1 \dots C_6$ are the six rows of a $\text{GBRD}(6, 3, 12; \mathbb{Z}_2 \times \mathbb{Z}_2)$. We observe that for the submatrix in (5.6), the inner product of the first or second row with any other row is $3(1 + a + b + ab)$ and so is the inner product of any two rows from the same block. The inner product of two rows from different blocks is zero.

We next use part of the construction in Theorem 2.3. The matrices D_1, \dots, D_4 are constructed using the 4×4 matrices M_{ij} , $i=1, 2$, $j=1, \dots, 4$, and the $\text{GBRD}(t, 3, 12; \mathbb{Z}_2 \times \mathbb{Z}_2)$. The matrices D_1, \dots, D_4 together give t row blocks where the inner product of rows across blocks is $3(1 + a + b + ab)$ and the inner product of rows from the same block is zero. Thus, they complement (5.6) to form a $\text{GBRD}(4t + 2, 3, 12; \mathbb{Z}_2 \times \mathbb{Z}_2)$. \square

Theorem 5.4. *The necessary conditions (4.1) to (4.3) are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; \mathbb{Z}_2 \times \mathbb{Z}_2)$.*

Proof. Using Lemma 5.2, we only have to consider the case $v \equiv 2 \pmod{3}$ and $\lambda \equiv 0 \pmod{12}$. We observe first that when $\lambda = 12$, GBRD 's with $v \equiv 0 \pmod{4}$ can be constructed, using Theorem 2.2, from a $\text{GBRD}(4, 3, 4; \mathbb{Z}_2 \times \mathbb{Z}_2)$ and a $\text{BIBD}(v, 4, 3)$.

The existence of a BIBD($v, 4, 3$) follows from Hanani (1961). The case $\lambda = 12$, and $v \equiv 2 \pmod{4}$ can be proved by using Construction 5.3 and the observation that the GBRD's with $v = 6$ and 10 both exist by Lemma 5.2. Thus, when $\lambda = 12$, GBRD's exist for every v . By repeating blocks if necessary, they exist for every v whenever $\lambda \equiv 0 \pmod{12}$. \square

6. Case $G = Z_2 \times Z_2 \times Z_2$

Using the method of developing initial blocks presented in Seberry (1982) we have:

Lemma 6.1. *There exists a GBRD($2p+1, 3, 24; Z_2 \times Z_2 \times Z_2$) for every $p \geq 1$.*

Proof. We develop the following blocks modulo $2p+1$:

$$\begin{aligned} & (0_{000}, j_{000}, 2p+1-j_{000}), \\ & (0_{001}, j_{010}, 2p+1-j_{011}), \quad (0_{100}, j_{101}, 2p+1-j_{001}), \quad (0_{110}, j_{100}, 2p+1-j_{010}), \\ & (0_{110}, j_{001}, 2p+1-j_{111}), \quad (0_{111}, j_{010}, 2p+1-j_{101}), \quad (0_{111}, j_{011}, 2p+1-j_{100}), \\ & (0_{101}, j_{011}, 2p+1-j_{110}), \end{aligned}$$

$j = 1, \dots, p$, in each case the subscript is from the group $Z_2 \times Z_2 \times Z_2$. \square

Lemma 6.2. *There exists a GBRD($2p+2, 3, 24; Z_2 \times Z_2 \times Z_2$) for every $p \geq 4$.*

Proof. We develop the same blocks as in the previous lemma except that the block $(0_{101}, j_{011}, 2p+1-j_{110})$ is developed from $j = 5, \dots, p$. We add the extra blocks

$$\begin{aligned} & (\infty_{000}, 0_{000}, 1_{110}), \quad (\infty_{000}, 0_{000}, 2p_{011}), \quad (\infty_{000}, 0_{000}, 2p-1_{101}), \\ & (\infty_{000}, 0_{001}, 2_{111}), \quad (\infty_{000}, 0_{001}, 2p-1_{010}), \quad (\infty_{000}, 0_{001}, 2p-3_{100}), \\ & (\infty_{000}, 0_{010}, 3_{100}), \quad (\infty_{000}, 0_{100}, 2p-2_{111}), \quad (\infty_{000}, 0_{011}, 2p-5_{110}), \\ & (\infty_{000}, 0_{011}, 4_{101}), \quad (\infty_{000}, 0_{101}, 2p-3_{110}), \quad (\infty_{000}, 0_{010}, 2p-7_{111}). \quad \square \end{aligned}$$

Theorem 6.3. *The conditions $\lambda \equiv 0 \pmod{8}$, $\lambda v(v-1) \equiv 0 \pmod{3}$ are necessary and sufficient for the existence of a GBRD($v, 3, \lambda; Z_2 \times Z_2 \times Z_2$).*

Proof. The necessary conditions follow from the properties of block designs. The case for $\lambda \equiv 0 \pmod{24}$ is covered by Lemmas 6.1 and 6.2 except for $v = 4, 6, 8$. In the next paragraphs we show GBRD($u, 3, 8; Z_2 \times Z_2 \times Z_2$), $u = 4, 6$, exist and so three copies gives the result. We note that a GBRD($8, 3, 6; Z_2$), A , exists and so does a GBRD($3, 3, 4; Z_2 \times Z_2$), B (= three rows of a GH($4, Z_2 \times Z_2$)), so replacing the i th non-zero entry of each column of A , x_i , by $(x_i b_{i1}, x_i b_{i2}, x_i b_{i3}, x_i b_{i4})$ where

$(b_{i1}, b_{i2}, b_{i3}, b_{i4})$ is the i th row of B and each zero of B by $(0, 0, 0, 0)$ we have the result for $v = 8$.

So we restrict ourselves to $\lambda \equiv 8$ and $16 \pmod{24}$. The necessary condition in both these cases is that $v(v-1) \equiv 0 \pmod{3}$ and so every design with $\lambda \equiv 16 \pmod{24}$ is in fact two copies of a design on the same number, v , of treatments with $\lambda \equiv 8 \pmod{24}$.

The necessary condition for the existence of a $\text{GBRD}(v, 3, 8; Z_2 \times Z_2 \times Z_2)$ is $v \equiv 0, 1 \pmod{3}$. Now by Seberry (1984) there exists a $\text{BRD}(u, 3, 2; Z_2)$ for $u(u-1) \equiv 0 \pmod{12}$ and hence since there exists a $\text{GH}(4, Z_2 \times Z_2)$ (or taking three rows, a $\text{GBRD}(3, 3, 4; Z_2 \times Z_2)$) there exists a $\text{GBRD}(u, 3, 8; Z_2 \times Z_2 \times Z_2)$ for $u \equiv 0, 1, 4, 9 \pmod{12}$. There exists a $\text{BIBD}(u, 3, 1)$ for $u \equiv 0, 1 \pmod{6}$ and, by Corollary 2.3, since there exists a $\text{GH}(8, Z_2 \times Z_2 \times Z_2)$ (or taking three rows, a $\text{GBRD}(3, 3, 8; Z_2 \times Z_2 \times Z_2)$) there exists the required GBRD for $u \equiv 0, 1 \pmod{6}$.

It remains to establish the existence of $\text{GBRD}(v, 3, 8; Z_2 \times Z_2 \times Z_2)$ for $v \equiv 3, 10 \pmod{12}$. Now the design exists for $v = 3$ and $v = 6t + 1$ so by Theorem 2.4 for orders $(6t+1)(3-1) + 1 = 12t + 3$ giving the result for $v \equiv 3 \pmod{12}$. The designs exist for $6t+1, 6p, 12u+4, 3$ and 4 hence, by use of Theorem 2.4, and noting that

$$3(12u+4-1) + 1 = 36u + 10 \equiv 10 \pmod{36},$$

$$(6t+1)(4-1) + 1 = 18t + 4 \equiv 22 \pmod{36},$$

$$3(6p-1) + 1 = 18p - 2 \equiv 34 \pmod{36},$$

we have the result for $v \equiv 10 \pmod{12}$. \square

7. The main results and application

We are now in a position to establish the promised results (1.4) and (1.5).

Theorem 7.1. *The necessary conditions (4.1) to (4.3) are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; Z_2^r)$ where $r \geq 1$ is an integer.*

Proof. The case $r = 1$ was established in (Seberry (1984)). The cases $r = 2$ and 3 have been established above in Theorems 5.4 and 6.3. It is sufficient to note that every positive integer ≥ 2 can be written as $2a + 3b$, where a, b are non-negative integers to see that the result follows from repeated applications of Theorem 2.2. \square

Theorem 7.2. *The necessary conditions (4.1) to (4.3) are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; Z_2^r \times H)$ where $r \geq 1$ is an integer, $|H| = h$ and $\text{g.c.d.}(h, 6) = 1$.*

Proof. The case $r = 1$ is proved in Theorem 4.5. For $r > 1$ we construct the required GBRD from a $\text{GBRD}(3, 3, 2^{r-2}h; Z_2^{r-2} \times H)$ and a $\text{GBRD}(v, 3, \lambda/(2^{r-2}h); Z_2 \times Z_2)$.

We remark that the difficulty with $r=2$ is the requirement that $b \equiv 0 \pmod{4}$. \square

Finally using the results of Bhaskar Rao (1966, 1970), Street and Rodger (1980) and Seberry (1982) where each element of the $\text{GBRD}(v, k, \lambda; G)$ is replaced by its $|G| \times |G|$ right regular matrix representation we are able to establish.

Theorem 7.3. *Suppose $|G|$ is odd and $\lambda = t|G|$, $\lambda(v-1) \equiv 0 \pmod{2}$, $\lambda v(v-1) \equiv 0 \pmod{6}$ then there is a regular group divisible design with two association classes and parameters*

$$\begin{aligned} v^* &= |G|v, & b^* &= |G|\lambda v(v-1)/6, & r^* &= \lambda(v-1)/2, \\ k^* &= 3, & \lambda_1^* &= 0, & \lambda_2^* &= t, & m^* &= v, & n^* &= |G|. \end{aligned}$$

Similarly we have:

Theorem 7.4. *Suppose $\lambda = gt$ where either $g=2^s$ or $g=2^sh$ with $s \geq 1$ and $\text{g.c.d}(3, h) = 1$. Further suppose*

$$\lambda v(v-1) \equiv 0 \pmod{24}.$$

Then there is a regular group divisible design with two association classes and parameters

$$\begin{aligned} v^* &= gv, & b^* &= g\lambda v(v-1)/6, & r^* &= \lambda(v-1)/2, \\ k^* &= 3, & \lambda_1^* &= 0, & \lambda_2^* &= t, & m^* &= v, & n^* &= g. \end{aligned}$$

These last two results are not as comprehensive as the equivalent results of Hanani but have arisen in a quite different way.

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