Journal of Statistical Planning and Inference 10 (1984) 69–82
North-Holland

REGULAR GROUP DIVISIBLE DESIGNS AND BHASKAR RAO DESIGNS WITH BLOCK SIZE THREE

Jennifer SEBERRY*

Department of Applied Mathematics, University of Sydney, Australia, N.S.W. 2006

Received July 1980; revised manuscript received 5 December 1982
Recommended by N.M. Singhi and J. Srivastava

Abstract: Some recursive constructions are given for Bhaskar Rao designs. Using examples of these designs found by Shyam J. Singh, Rakesh Vyas and new ones given here we show the necessary conditions $\lambda \equiv 0 \pmod 2$, $\lambda v(v-1) \equiv 0 \pmod{24}$ are sufficient for the existence of Bhaskar Rao designs with one association class and block size 3. This result is used with a result of Street and Rodger to obtain regular partially balanced block designs with $2v$ treatments, block size 3, $\lambda_1 = 0$, group size 2 and v groups.

AMS Subject Classifications: Primary 05B99; Secondary 05B05, 05B30, 62K10.

Key words and phrases: Partially balanced incomplete block designs; Regular group divisible designs; Bhaskar Rao designs.

Bhaskar Rao designs with elements $0, \pm 1$ have been studied by a number of authors including Bhaskar Rao (1966, 1970), Singh (1982), Sinha (1978), Street (1981), Street and Rodger (1980), Vyas (1982) and Seberry (1978). Bhaskar Rao (1966) used these designs to construct partially balanced designs and this was improved by Street and Rodger (1980). In this paper we show the necessary conditions are sufficient for Bhaskar Rao designs with one association class when $k=3$.

1. Preliminaries

For the definitions of regular group divisible designs, partially balanced incomplete block designs and mutually orthogonal latin squares we refer the reader to Raghavarao (1970).

Suppose $X = A - B$, where A and B are $v \times b$ (0, 1) matrices, and the Hadamard product of A and B, $A * B$, is zero. Then X is a *Bhaskar Rao design* or *BRD*, if

(i) $XX^{\mathrm{T}} = rI + \sum_{i=1}^{m} c_i B_i$,

* This paper was partly written while visiting the Department of Statistics, University of Indore, Indore.

(ii) $N = A + B$ satisfies $NN^{\mathrm{T}} = rI + \sum_{i=1}^{m} \lambda_i B_i$ (that is, N is the incidence matrix of a PBIBD(m)).

Such a matrix will be denoted by $\mathrm{BRD}(v, b, r, k; \lambda_1, \ldots, \lambda_m; c_1, \ldots, c_m)$. In this paper we shall only be concerned with $c = 0$, $m = 1$ and $B_1 = J - I$ in this case N is the incidence matrix of a PBIBD(1) that is a BIBD. So the equations become

(i) $XX^{\mathrm{T}} = rI$,

(ii) $NN^{\mathrm{T}} = (r - \lambda)I + \lambda J$,

and X is a $\mathrm{BRD}(v, b, r, k; \lambda)$. Since $\lambda(v-1) = r(k-1)$ and $bk = vr$ we sometimes use the notation $\mathrm{BRD}(v, k, \lambda)$.

These designs have been considered by several authors. They are a generalization of weighing matrices (Geramita and Seberry (1979)) and useful in the construction of PBIBD.

First we give a necessary condition which is slightly stronger than that given in Bhaskar Rao (1970) and less comprehensive than that given in Street and Rodger (1980):

Theorem 1. *A Bhaskar Rao design* $W = \mathrm{BRD}(v, k, \lambda)$ *can only exist if the equations*

(i) $x_3 + 3x_5 + 6x_7 + \cdots + ((k^2-1)/8)x_k = b(k-1)/8$ *for k odd,*

(ii) $-x_0 + 3x_4 + 8x_6 + \cdots + ((k^2-4)/4)x_k = b(k-4)/4$ *for k even,*

have integral solutions. In particular for $k = 3$ a Bhaskar Rao design can only exist if $4 \mid b$. For $k = 4$ no restriction is obtained.

Proof. Since W is a BRD, $WW^{\mathrm{T}} = rI$. Suppose the column sum of the i-th column is s_i then $(1, \ldots, 1)W = (s_1, \ldots, s_b)$ and so

$$(1, \ldots, 1)WW^{\mathrm{T}}(1, \ldots, 1)^{\mathrm{T}} = \sum_{i=1}^{b} s_i^2 = (1, \ldots, 1)rI_v(1, \ldots, 1)^{\mathrm{T}} = vr. \tag{1}$$

If k is odd the column sums can only be $\pm 1, \pm 3, \ldots, \pm k$ and if k is even the sums can only be $0, \pm 2, \pm 4, \ldots, \pm k$. Hence, if there are x_i columns with column sum $\pm i$, we have respectively, using (1),

$$\left.\begin{aligned} &x_1 + 9x_3 + 25x_5 + \cdots + k^2 x_k = vr, \\ &x_1 + x_3 + x_5 + \cdots + x_k = b \end{aligned}\right\} \text{ for } k \text{ odd,}$$

and

$$\left.\begin{aligned} &4x_2 + 16x_4 + \cdots + k^2 x_k = vr, \\ &x_0 + x_2 + x_4 + \cdots + x_k = b \end{aligned}\right\} \text{ for } k \text{ even,}$$

giving, since $vr = bk$ for a BRD

$$\begin{aligned} &8x_3 + 24x_5 + \cdots + (k^2-1)x_k = vr - b = b(k-1) && \text{for } k \text{ odd,} \\ &-4x_0 + 12x_4 + \cdots + (k^2-4)x_k = b(k-4) && \text{for } k \text{ even.} \end{aligned} \tag{2}$$

Hence for $k = 3$ we have $8x_3 = 2b$ or $4 \mid b$ and for $k = 4$ we have $x_0 = 3x_4$ as required.

2. Some replication theorems

We first recall (see Wilson (1974)) that for every order $n \geq 3$ except 6 there are two mutually orthogonal latin squares and for any $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where p_i are distinct primes, the number of mutually orthogonal latin squares is at least $\min_i (p_i^{r_i} - 1)$.

Mutually orthogonal latin squares may be used to form *auxiliary matrices* (for example in Glynn (1978, pp. 168–9), Wallis (1971), Wallis (1973)) in the following fashion: Let $M_1, M_2, \ldots, M_n$ be n mutually orthogonal latin squares of order t on the symbols $x_1, x_2, \ldots, x_t$. Define

$$M_0 = \begin{bmatrix} x_1 & x_2 & \cdots & x_t \\ x_1 & x_2 & & x_t \\ \vdots & & & \vdots \\ x_1 & x_2 & \cdots & x_t \end{bmatrix}^{\mathrm{T}},$$

so that M_0 is pairwise orthogonal with $M_1, \ldots, M_n$ but M_0 is not itself a Latin square. Now define

$$(M_{ij})_{ab} = \begin{cases} 1 & (M_i)_{bj} = x_a, \\ 0 & \text{otherwise.} \end{cases}$$

We first observe that each M_{ij} is a permutation matrix since for fixed ij, x_a occurs once in each row and column of M_i, $1 \leq i \leq k$, and $M_{0j} = I$, $1 \leq j \leq t$. For fixed ab,

$$\sum_{j=1}^{t} (M_{ij})_{ab} = \text{number of times the element } x_a \text{ occurs in row } b \text{ of } M_i$$
$$= 1.$$

So

$$\sum_{j=1}^{t} M_{aj} = J, \quad 1 \leq a \leq n. \tag{3}$$

Since each M_{ij}, $0 \leq i \leq n$, $1 \leq j \leq t$, is a permutation matrix,

$$\sum_{j=1}^{t} M_{aj} M_{aj}^{\mathrm{T}} = tI, \quad 0 \leq a \leq t. \tag{4}$$

Now consider for fixed cd

$$\sum_{j=1}^{t} \sum_{i=1}^{t} (M_{cj})_{ai} (M_{dj})_{bi}$$
$$= \sum_{i=1}^{t} \sum_{j=1}^{t} (M_{cj})_{ai} (M_{dj})_{bi}$$
$$= \sum_{i=1}^{t} (\text{number of times } \{\text{the element } x_a \text{ occurs in row } i \text{ of } M_c\} \text{ and } \{\text{the element } x_b \text{ occurs in row } i \text{ of } M_d\})$$
$$= 1 \quad (\text{since } M_c \text{ and } M_d \text{ are mutually orthogonal}).$$

Thus we have

$$\sum_{j=1}^{t} M_{aj} M_{bj}^{\mathrm{T}} = J, \quad a \neq b,\ 0 \leq a,\ b \leq t. \tag{5}$$

Matrices which satisfy (3), (4), (5), are auxiliary matrices.

Example 1. Let

$$M_1 = \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix}, \quad M_2 = \begin{pmatrix} a & b & c & d \\ c & d & a & b \\ d & c & b & a \\ b & a & d & c \end{pmatrix}, \quad M_3 = \begin{pmatrix} a & b & c & d \\ d & c & b & a \\ b & a & d & c \\ c & d & a & b \end{pmatrix}$$

be three mutually orthogonal latin squares of order 4 on the symbols $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = d$. Define M_{ij}, $1 \leq i \leq k$, by

$$(M_{ij})_{ab} = \begin{cases} 1 & (M_i)_{bj} = x_a, \\ 0 & \text{otherwise.} \end{cases}$$

So M_{ij}, $0 \leq i \leq 3$ and $1 \leq j \leq 4$ can be written

$$\begin{array}{cccc|cccc|cccc|cccc}
1 & & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & & & 1 & & & & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & & & & 1 & & & & 1 & \\
 & & & 1 & & & & 1 & & & & 1 & & & & 1 \\
\hline
1 & & & & & 1 & & & & & 1 & & & & & 1 \\
 & 1 & & & 1 & & & & & & & 1 & & & 1 & \\
 & & 1 & & & & & 1 & 1 & & & & & 1 & & \\
 & & & 1 & & & 1 & & & 1 & & & 1 & & & \\
\hline
1 & & & & & & & 1 & & 1 & & & & & 1 & \\
 & & & 1 & 1 & & & & & & 1 & & & 1 & & \\
 & 1 & & & & & 1 & & 1 & & & & & & & 1 \\
 & & 1 & & & 1 & & & & & & 1 & 1 & & & \\
\hline
1 & & & & & & 1 & & & & & 1 & & 1 & & \\
 & & 1 & & 1 & & & & & 1 & & & & & & 1 \\
 & & & 1 & & 1 & & & 1 & & & & & & 1 & \\
 & 1 & & & & & & 1 & & & 1 & & 1 & & &
\end{array}$$

Theorem 2. *Suppose there are Bhaskar Rao designs with parameters* BRD$(v, b, r, k; \lambda)$ *and* BRD$(u, a, s, k; \lambda)$. *Further suppose there are $k-1$ mutually orthogonal latin squares of order u then there is a Bhaskar Rao design with parameters* BRD$(uv, bu^2 + au, ur + s, k; \lambda)$.

Proof. Form the auxiliary matrices M_{ij}, $i = 1, \ldots, k-1$, $j = 1, \ldots, u$, from the mutually orthogonal latin squares. Write

$$C = \begin{bmatrix} I & I & \cdots & I \\ M_{11} & M_{12} & & M_{1u} \\ \vdots & & & \vdots \\ M_{k-1,1} & M_{k-1,2} & \cdots & M_{k-1,u} \end{bmatrix}.$$

Write A for the BRD$(v, b, r, k; \lambda)$ and B for the BRD$(u, a, s, k; \lambda)$. We now form D_i, $i=1, \ldots, u$, by replacing the first non-zero element in each column of A by $\pm$ the first element (I) in the i-th column of C, the second non-zero element in each column of A by $\pm$ the second element (M_{1i}) in the i-th column of $C, \ldots,$ the k-th non-zero element in each column of A by $\pm$ the k-th element ($M_{k-1,i}$) in the i-th column of $C \pm$ in each case according as the element replaced is ± 1. We now claim the matrix

$$E = [\underbrace{B \oplus B \oplus \cdots \oplus B}_{v \text{ copies}} \vdots D_1 \vdots D_2 \vdots \cdots \vdots D_u]$$

is a BRD$(uv, bu^2 + av, ur + s, k; \lambda)$.

It is easy to check the first four parameters for E are correct. Also, replacing all -1 in B by 1, it is easy to check that λ is the inner product. It only remains to show that in E the inner product of each pair of rows is zero in the $F = [D_1 \vdots D_2 \vdots \cdots \vdots D_u]$ part of E. We see that column k and rows g and h of D_i will contribute, when taken over $i = 1, \ldots, u$, exactly J or $-J$ according as the contribution from A is $+1$ or -1. Hence, since the rows of A are orthogonal, the rows of F will be orthogonal and we have the result.

Example 2. We use the BRD(4, 3, 2) for both designs, A and B of the theorem to from a BRD(16, 3, 2). Hence

$$A = B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & - \\ 1 & - & 0 & 1 \\ 1 & 1 & - & 0 \end{bmatrix}.$$

With M_{1i}, M_{2i}, $i = 1, 2, 3, 4$ as in Example 1,

$$C = \begin{pmatrix} I & I & I & I \\ M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \end{pmatrix}$$

and hence

$$D_i = \begin{bmatrix} 0 & I & I & I \\ I & 0 & M_{1i} & -M_{1i} \\ M_{1i} & -M_{1i} & 0 & M_{2i} \\ M_{2i} & M_{2i} & -M_{2i} & 0 \end{bmatrix}.$$

So

$$[A \oplus A \oplus A \oplus A \vdots D_1 \vdots D_2 \vdots D_3 \vdots D_4]$$

is the required matrix.

We now give a number of important construction results:

Theorem 3. *Suppose there exists a* BRD$(v, b, r, k; \lambda)$, B, *and a* BRD$(u, a, s, k; \lambda)$, A. *Further suppose there exist* $\geq k-1$ *mutually orthogonal latin squares of order* $u-1$. *Then there exists a* BRD$(v(u-1)+1, b(u-1)^2+av, vs, k; \lambda)$.

Proof. Write

$$A = \begin{pmatrix} x \\ E \end{pmatrix}$$

where x is the first row of A. Form the auxiliary matrices M_{ij}, $i=1,\dots,k-1$, $j=1,\dots,u-1$, of size $u-1$ from the mutually orthogonal latin squares. Write

$$C = \begin{bmatrix} I & I & \cdots & I \\ M_{11} & M_{12} & & M_{1,u-1} \\ \vdots & & & \vdots \\ M_{k-1,1} & M_{k-1,2} & \cdots & M_{k-1,u-1} \end{bmatrix}.$$

We now form D_i, $i=1,\dots,u-1$, by replacing the first non-zero element in each column of B by $\pm$ the first element (I) in the i-th column of C, the second non-zero element in each colunm of B by $\pm$ the second element (M_{1i}) in the i-th column of $C, \dots,$ the k-th non-zero element in each column of B by $\pm$ the k-th element $(M_{k-1,i})$ in the i-th column of C, $\pm$ in each case according as the element replaced as ± 1.

We now claim

$$\left(\begin{array}{c|cccc} x \quad x \quad \cdots \quad x & 0 & 0 & \cdots & 0 \\ \underbrace{E \oplus E \oplus \cdots \oplus E}_{v \text{ copies}} & D_1 & D_2 & \cdots & D_{u-1} \end{array}\right)$$

is a BRD$(v(u-1)+1, b(u-1)^2+av, vs, k, \lambda)$.

Example 3. We use the BRD(4, 3, 2),

$$A = B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & - \\ 1 & - & 0 & 1 \\ 1 & 1 & - & 0 \end{pmatrix}$$

for both designs of the theorem to form a BRD(13, 3, 2). Suitable M_{ij}, $i, j = 1, 2, 3$, are

$$M_{11} = M_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M_{12} = M_{23} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$M_{13} = M_{22} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$C=\begin{pmatrix} I & I & I \\ M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \end{pmatrix}.$$

Hence

$$D_1=\begin{bmatrix} 0 & I & I & I \\ I & 0 & M_{11} & -M_{11} \\ M_{11} & -M_{11} & 0 & M_{21} \\ M_{21} & M_{21} & -M_{21} & 0 \end{bmatrix},$$

$$D_2=\begin{bmatrix} 0 & I & I & I \\ I & 0 & M_{12} & -M_{12} \\ M_{12} & -M_{12} & 0 & M_{22} \\ M_{22} & M_{22} & -M_{22} & 0 \end{bmatrix}$$

and

$$D_3=\begin{bmatrix} 0 & I & I & I \\ I & 0 & M_{13} & -M_{13} \\ M_{13} & -M_{13} & 0 & M_{23} \\ M_{23} & M_{23} & -M_{23} & 0 \end{bmatrix}.$$

So

$$\left[\begin{array}{cccccccccccccccc|c}
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0\ 0 \cdots 0 \\ \hline
1 & 0 & 1 & - & & & & & & & & & & & & & \\
1 & - & 0 & 1 & & & & & & & & & & & & & \\
1 & 1 & - & 0 & & & & & & & & & & & & & \\
 & & & & 1 & 0 & 1 & - & & & & & & & & & \\
 & & & & 1 & - & 0 & 1 & & & & & & & & & \\
 & & & & 1 & 1 & - & 0 & & & & & & & & & D_1\ D_2\ D_3 \\
 & & & & & & & & 1 & 0 & 1 & - & & & & & \\
 & & & & & & & & 1 & - & 0 & 1 & & & & & \\
 & & & & & & & & 1 & 1 & - & 0 & & & & & \\
 & & & & & & & & & & & & 1 & 0 & 1 & - & \\
 & & & & & & & & & & & & 1 & - & 0 & 1 & \\
 & & & & & & & & & & & & 1 & 1 & - & 0 &
\end{array}\right],$$

where

$$X=\begin{pmatrix} 1 & 0 & 1 & - \\ 1 & - & 0 & 1 \\ 1 & 1 & - & 0 \end{pmatrix},$$

is the required matrix.

Theorem 4. *Suppose we have a* BIBD(v,b,r,k,λ) *and a* BRD$(k,a,s,j;\mu)$. *Then there exists a* BRD$(v,ab,rs,j;\lambda\mu)$.

Proof. Let B be the BIBD and W the BRD. Replace the j-th non-zero element of each column of B by the j-th row of W to obtain the result.

Corollary 5. *Suppose* $v \equiv 1$ *or* 4 (mod 12) *then there exists a* BRD(v, 3, 2). *Suppose* $v \equiv 1$ (mod 3) *then there exists a* BRD(v, 3, 4).

Proof. In the theorem we let the BRD be the BRD(4, 3, 2). Now by the theorems of Hanani (see Hall (1967)) a BIBD($v, v(v-1)/12, (v-1)/3, 4, 1$) exists whenever $v \equiv 1$ or 4 (mod 12) and a BIBD($v, v(v-1)/6, 2(v-1)/3, 4, 2$) exists whenever $v \equiv 1$ (mod 3). The results now follow by using the theorem.

Theorem 6. *Suppose there exists a* BRD(v, b, r, k; $4t$), $4t$ *is the order of an Hadamard matrix and there exist* $k-1$ *mutually orthogonal latin squares of order* k. *Then there exists a* BRD($kv, 4tv + k^2b, kr + 4t, k$; $4t$).

Proof. We form auxiliary matrices of order k, M_{ij}, and the matrices

$$D_1, D_2, \ldots, D_k$$

as in the proofs of earlier theorems. Let E be the $k \times 4t$ matrix obtained from the first k rows of an Hadamard matrix. Then $[E \oplus E \oplus \cdots \oplus E_{v\,\text{copies}} \,|\, D_1\, D_2 \cdots D_k]$ is the required matrix.

Corollary 7. *Suppose there exists a* BRD(v, b, r, k; 4) *then there exists a* BRD($kv, 4v + k^2b, kr + 4, k$; 4) *whenever* k *is a prime power.*

A *generalized Hadamard matrix* $H = \mathrm{GH}(g, G)$ is a matrix of order g with elements from an abelian group G with the property that if $\boldsymbol{a} = (a_1, \ldots, a_g)$ and $\boldsymbol{b} = (b_1, \ldots, b_g)$ are any two distinct rows of H then $\bigcup_{i=1}^{g} a_i b_i^{-1} = \lambda G$, that is the totality of elements $a_i b_i^{-1}$, $i = 1, \ldots, g$, is λ copies of the group G. Such matrices have been studied by a number of authors, e.g. Butson (1962), Drake (1979), Seberry (1980), Street (1979).

The next theorem was discovered after conversations with Mr. Dinesh G. Sarvate which led to a generalization of a previous result.

Theorem 8. *Suppose* $v \equiv 1$ *or* 3 (mod 6). *Suppose there exist three rows of a* GH($2r, F \times Z_2$) $= G$, $|F| = r$. *Further suppose there exists a* BRD($2pr$, 3, 2) $= B$ *where* p *is odd. Then there exists a* BRD($2prv$, 3, 2).

Proof. For $v \equiv 1$ or 3 (mod 6), by Steiner's Theorem there exists a BIBD($v, v(v-1)/6, (v-1)/3, 3, 1$) $= A$.

We define $\boldsymbol{m} \cdot \boldsymbol{n}^{-1}$, $\boldsymbol{m} = (m_1, \ldots, m_s)$, $\boldsymbol{n} = (n_1, \ldots, n_s)$, by $\{m_i n_i^{-1} : i = 1, \ldots, s\}$, where the inverse is in the group upon which the elements are defined.

Write

$$G = \begin{pmatrix} \boldsymbol{e} & \boldsymbol{e} \\ \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & -\boldsymbol{c} \end{pmatrix}, \quad \begin{array}{l} \boldsymbol{a} = (a_1, \ldots, a_r), \\ \boldsymbol{b} = (b_1, \ldots, b_r), \\ \boldsymbol{c} = (c_1, \ldots, c_r), \end{array}$$

for the first three rows of the $GH(2r, F\times Z_2)$ where $|F|=r$ and $Z_2=\{1,-1\}$ the cyclic group of order 2. Then

$$\bigcup_{i=1}^{r}(a_i\cup b_i)=\bigcup_{i=1}^{r}(-a_i\cup -b_i)=\bigcup_{i=1}^{r}(c_i\cup -c_i)=\boldsymbol{a}\cdot \boldsymbol{c}^{-1}\cup \boldsymbol{b}\cdot -\boldsymbol{c}^{-1}=F\times Z_2.$$

Now let w, x, y, z be the vectors $(1,2,\dots,p), (p+1,p+2,\dots,2p), (p,p-1,\dots,2,1)$, and $(2p,2p-1,\dots,p+1)$, p odd, respectively. Then with the inverse in the additive cyclic group of order$2p$, we have

$$\boldsymbol{w}\cdot \boldsymbol{y}^{-1}=\boldsymbol{x}\cdot \boldsymbol{z}^{-1}=\{p+1,p+3,\dots,p-1\}=\text{evens},$$

$$\boldsymbol{w}\cdot \boldsymbol{z}^{-1}=\boldsymbol{x}\cdot \boldsymbol{y}^{-1}=\{1,2,\dots,2p-1\}=\text{odds}.$$

Hence, writing $H=\{1,2,\dots,2p\}$ we have

$$H=\boldsymbol{w}\cdot \boldsymbol{y}^{-1}\cup \boldsymbol{x}\cdot \boldsymbol{y}^{-1}=\boldsymbol{w}\cdot \boldsymbol{z}^{-1}\cup \boldsymbol{x}\cdot \boldsymbol{z}^{-1}.$$

Define

$$M=\begin{pmatrix} \boldsymbol{e},\boldsymbol{\varepsilon} & \boldsymbol{e},\boldsymbol{\varepsilon} & \boldsymbol{e},\boldsymbol{\varepsilon} & \boldsymbol{e},\boldsymbol{\varepsilon} \\ \boldsymbol{a},\boldsymbol{w} & \boldsymbol{b},\boldsymbol{w} & -\boldsymbol{a},\boldsymbol{x} & -\boldsymbol{b},\boldsymbol{x} \\ \boldsymbol{c},\boldsymbol{y} & -\boldsymbol{c},\boldsymbol{z} & -\boldsymbol{c},\boldsymbol{y} & \boldsymbol{c},\boldsymbol{z} \end{pmatrix}=(m_{ij})$$

where $\boldsymbol{e},\boldsymbol{\varepsilon}$ represents pr copies of the unit element of $H\times(F\times Z_2)$ and, for example, $\boldsymbol{b},\boldsymbol{x}$ is the vector $(b_1,\boldsymbol{x}),(b_2,\boldsymbol{x}),\dots,(b_r,\boldsymbol{x})$ or $(b_1,p+1),(b_1,p+2),\dots,(b_1,2p),(b_2,p+1),\dots,(b_r,2p)$, that is pr elements of the group $H\times(F\times Z_2)$. We observe that if $\boldsymbol{p}=(p_1,\dots,p_{4pr})$ and $\boldsymbol{q}=(q_1,\dots,q_{4pr})$ are two distinct rows of M then $\boldsymbol{p}\cdot \boldsymbol{q}^{-1}=H\times(F\times Z_2)$.

We now form N from M by ensuring the elements of Z_2 in M are written as ± 1, so that the inner product $\boldsymbol{p}^*\cdot \boldsymbol{q}^{*-1}$ of two distinct rows of N is 0.

We form a new matrices $C_1,\dots,C_{4pr}$ from A by replacing the j-th non-zero element in each column of A by m_{ij} to obtain C_i. We now replace each element of $C_1,\dots,C_{4pr}$ by its $2pr\times 2pr$ matrix representation to form $(0,1,-1)$ matrices $D_1,\dots,D_{4pr}$.

Then with $B=\text{BRD}(2pr,3,2)$

$$(\underbrace{B\oplus B\oplus\cdots\oplus B\oplus B}_{v\text{ copies}}\mid D_1\ \cdots\ D_{4pr})$$

is the required matrix.

Example 4. Let $\boldsymbol{w}=(1,2,3)$, $\boldsymbol{x}=(4,5,6)$, $\boldsymbol{y}=(3,2,1)$, $\boldsymbol{z}=(6,5,4)$. So $\boldsymbol{w}\cdot \boldsymbol{y}^{-1}=\boldsymbol{x}\cdot \boldsymbol{z}^{-1}=\{4,6,2\}$ and $\boldsymbol{w}\cdot \boldsymbol{z}^{-1}=\boldsymbol{x}\cdot \boldsymbol{y}^{-1}=\{1,3,5\}$. Now

$$G=\begin{pmatrix} e & e & e & e \\ e & a & b & ab \\ e & b & ab & a \end{pmatrix}=\begin{pmatrix} e & e & e & e \\ e & -e & b & -b \\ e & b & -b & -e \end{pmatrix}$$

are the first three rows of $GH(4, Z_2\times Z_2)$, $r=2$. We now define the matrix M of size $3\times 4pr$ (3×24 here) by, $M=[E\ F]=(m_{ij})$,

$$E=\begin{pmatrix} e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 \\ e,1 & e,2 & e,3 & a,1 & a,2 & a,3 & b,1 & b,2 & b,3 & ab,1 & ab,2 & ab,3 \\ e,3 & e,2 & e,1 & b,3 & b,2 & b,1 & ab,6 & ab,5 & ab,4 & a,6 & a,5 & a,4 \end{pmatrix},$$

$$F=\begin{pmatrix} e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 & e,1 \\ e,4 & e,5 & e,6 & a,4 & a,5 & a,6 & b,4 & b,5 & b,6 & ab,4 & ab,5 & ab,6 \\ ab,3 & ab,2 & ab,1 & a,3 & a,2 & a,1 & e,6 & e,5 & e,4 & b,6 & b,5 & b,4 \end{pmatrix}.$$

Now the j-th non-zero element of A is replaced by m_{ij} to form C_i.

We replace the elements of C_i by their matrix representations:

$$1\to\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}=T, \quad i\to T^i,$$

$$e\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad b\to\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad a\to-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad ab\to-\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

so

$$ab,4\to-\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\times T^4,\quad \text{etc.},$$

giving matrices $D_1,\ldots,D_{24}$ with $2prv=12v$ rows.

Now, with $B=\mathrm{BRD}(12,3,2)$,

$$[I_v\times B\,|\,D_1\ \cdots\ D_{24}]$$

is the required matrix.

Corollary 9. *Suppose* $v\equiv 1$ *or* 3 (mod 6). *There exist three rows of a* $GH(2r, F\times Z_2)$ *for* $r=2^t, 6, 10$. *There exist* BRD($2pr$, 3, 2) *for* $2pr=4, 16, 12, 24$ (*Singh* (1982)). *Hence there exist*

$$\mathrm{BRD}(4v,3,2),\quad \mathrm{BRD}(12v,3,2),\quad \mathrm{BRD}(24v,3,2),\quad \mathrm{BRD}(16v,3,2),$$

thus we have BRD(u, 3, 2) *for* $u\equiv 4$ *or* 12 (mod 24), 16 *or* 48 (mod 96), 12 *or* 36 (mod 72), 24 *or* 72 (mod 144).

Corollary 10. *There exist* BRD(u, 3, 2) *for all* $u\equiv 0$ (mod 12).

Proof. The existence result follows $u\not\equiv 0$, 48, 96, 120 (mod 144) from the previous corollary. Also from the previous corollary the existence of BRD(u, 3, 2) for $u\equiv 48$ (mod 96) gives the result for $u\equiv 240$ (mod 288). Now BRD(12, 3, 2), BRD(24, 3, 2) and BRD($12t+4$, 3, 2) exist for t integer, so by Theorem 3 we have BRD($144t+48$, 3, 2) and BRD($288t+96$, 3, 2) giving the result for $u\not\equiv 0, 120$ (mod 144). We now observe that three rows of a GH($2^s, F\times Z_2$), $s\geq 2$, can always be found. Now all $\mathrm{BRD}(2^s p = 12t+4, 3, 2)$, p odd exist, that is every $\mathrm{BRD}(24s+16, 3, 2)$ exists. We have with $v = 3$ that every $\mathrm{BRD}(u, 3, 2)$ with $u \equiv 48 \pmod{72}$ exists, giving the result for $u \equiv 120 \pmod{144}$. The result for $u \equiv 0 \pmod{144}$ follows by induction after first observing that a $\mathrm{BRD}(12, 3, 2)$ exists.

Theorem 11. *There exist* $\mathrm{BRD}(u, 3, 2)$ *for all* $u \equiv 9 \pmod{12}$.

We use i to represent the matrix T^i of order $2n+1$, n odd and $-$ for the zero matrix. The notation a, b is used for $T^a + T^b$, $\bar{a}, b$ for $-T^a + T^b$ and $a, \bar{b}$ for $T^a - T^b$.

Let

$$C = [X_1 \quad X_2 \quad Y_1 \quad Y_2 \quad Z_1 \quad Z_2]$$

where

$$X_1 = \left(\begin{array}{cc|cccccc} 0 & 0 & \bar{1}, \overline{2n} & \bar{2}, \overline{2n-1} & 3, 2n-2 & 4, 2n-3 & \cdots & n, n+1 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ n & n+2 & - & - & - & - & \cdots & - \end{array}\right),$$

$$X_2 = \begin{pmatrix} \bar{1}, 2n & 2, \overline{2n}-1 & 3, \overline{2n-2} & \bar{4}, 2n-3 & \cdots & n, \overline{n+1} \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ - & - & - & - & \cdots & - \end{pmatrix}$$

when $n \equiv 3 \pmod 4$ and

$$X_1 = \left(\begin{array}{cc|cccccc} 0 & 0 & \bar{1}, \bar{2}n & \bar{2}, \overline{2n-1} & 3, 2n-2 & 4, 2n-3 & \cdots & \bar{n}, \overline{n+1} \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ n & n+2 & - & - & - & - & \cdots & - \end{array}\right),$$

$$X_2 = \begin{pmatrix} \bar{1}, 2n & 2, \bar{2}n-1 & 3, \overline{2n}-2 & \bar{4}, 2n-3 & \cdots & \bar{n}, n+1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ - & - & - & - & \cdots & - \end{pmatrix}$$

when $n \equiv 1 \pmod 4$ (note the signs occur in pairs),

$$Y_1 = \begin{pmatrix} - & - & - & & \cdots & - \\ \bar{1}, \bar{2}n & 2, \overline{2n-1} & 3, 2n-2 & 4, 2n-3 & \cdots & n, n+1 \\ n & n & n & n & \cdots & n \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} - & - & - & - & \cdots & - \\ \bar{1}, 2n & 2, \bar{2}n-1 & 3, \overline{2n-2} & \bar{4}, 2n-3 & \cdots & n, \overline{n+1} \\ n+1 & n+1 & n+1 & n+1 & \cdots & n+1 \end{pmatrix},$$

when $n \equiv 3 \pmod 4$ and

$$Y_1 = \begin{pmatrix} - & - & - & - & \cdots & - \\ \bar{1}, \overline{2n} & \bar{2}, \bar{2}n-1 & 3, 2n-2 & 4, 2n-3 & \cdots & \bar{n}, \overline{n+1} \\ n & n & n & n & \cdots & n \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} - & - & - & - & \cdots & - \\ \bar{1}, 2n & 2, \overline{2n-1} & 3, \overline{2n-2} & \bar{4}, 2n-3 & \cdots & \bar{n}, n+1 \\ n & n & n & n & \cdots & n \end{pmatrix}$$

when $n \equiv 1 \pmod 4$ (again the signs occur in pairs).

We use the following notation to describe the vectors we use:

$$-w_i=(\overline{i+1},\overline{2n-i}\quad \overline{i+2},\overline{2n-1-i}\quad \overline{i+3},\overline{2n-2-i}\quad \overline{i+4},\overline{2n-3-i}),$$

so

$$w_i=(i+1,2n+1-i\quad i+2,2n-1-i\quad i+3,2n-2-i\quad i+4,2n-3-i),$$

$$-u_i=(i,\overline{2n-i+1}\quad \overline{i+1},\overline{2n-i}\quad i+2,2n-1-i\quad \overline{i+3},\overline{2n-2-i}),$$

$$-v_i=(\bar{i},2n-i+1\quad i+1,\overline{2n-i}\quad \overline{i+2},2n-1-i\quad \overline{i+3},2n-2-i),$$

$$-t_i=(\bar{i},2n-i+1\quad \overline{i+1},2n-i\quad \overline{i+2},2n-1-i\quad \overline{i+3},2n-2-i).$$

The last row of $Z=Z_1Z_2$ when $n\equiv 3 \pmod 4$ follows the pattern

$$Z_1:\quad [1,2n\quad -w_1\quad w_5\quad -w_9\cdots n-1,n+2\quad n,n+1],\quad n\equiv 7 \pmod 8,$$

$$Z_1:\quad [1,2n\quad -w_1\quad w_5\quad -w_9\cdots \overline{n-1},\overline{n+2}\quad n,\overline{n+1}],\quad n\equiv 3 \pmod 8,$$

$$Z_2:\quad [1,\overline{2n}\quad 2,\overline{2n-1}\quad 3,\overline{2n-2}\quad -t_4\quad t_8\quad -t_{12}\cdots -t_{n-3}],\quad n\equiv 7 \pmod 8,$$

$$Z_2:\quad [1,\overline{2n}\quad 2,\overline{2n-1}\quad 3,\overline{2n-2}\quad -t_4\quad t_8\quad -t_{12}\cdots t_{n-3}],\quad n\equiv 3 \pmod 8,$$

where there are basic repetitions four at a time. The last row of $Z=Z_1Z_2$ when $n\equiv 1 \pmod 4$ follows the pattern

$$Z_1:\quad [-u_1\quad u_5\quad -u_9\cdots \bar{n},n+1],\quad n\equiv 1 \pmod 8,$$

$$Z_1:\quad [-u_1\quad u_5\quad -u_9\cdots n,n+1],\quad n\equiv 5 \pmod 8,$$

$$Z_2:\quad [-v_1\quad v_5\quad -v_9\cdots \bar{n},n+1],\quad n\equiv 1 \pmod 8,$$

$$Z_2:\quad [-v_1\quad v_5\quad -v_9\cdots n,n+1],\quad n\equiv 5 \pmod 8.$$

where there are basic repetitions four at a time. Now

$$Z_1=\begin{pmatrix} n+1 & n+1 & \cdots & n+1 \\ - & - & \cdots & - \\ \text{last row of } Z_1 \end{pmatrix}\quad\text{and}\quad Z_2=\begin{pmatrix} n-1 & n-1 & \cdots & n-1 \\ - & - & \cdots & - \\ \text{last row of } Z_2 \end{pmatrix}.$$

It is now possible to check that C is a BRD$(3(2n+1),3,2)$ for n odd. That is we have constructed BRD$(u,3,2)$ for $u\equiv 9 \pmod{12}$.

3. General results on BRD

Theorem 12. *The condition* $v(v-1)\equiv 0 \pmod{12}$ *is necessary and sufficient for the existence of a* BRD$(v,3,2)$ *with one association class.*

Proof. The necessary conditions for the existence of a BIBD$(v,v(v-1)/3,v-1,3,2)$ require that $v(v-1)/3$ is an integer, and that the number of blocks for a BRD is divisible by 4 gives us $v(v-1)\equiv 0 \pmod{12}$ is a necessary condition.

Corollary 5 gives the existence result for $v\equiv 1$ or $4 \pmod{12}$, Corollary 10 gives the result for $v\equiv 0 \pmod{12}$ and Theorem 11 gives the result for $v\equiv 9 \pmod{12}$.

Theorem 13. *The condition* $v(v-1)\equiv 0 \pmod 3$ *is necessary and sufficient for the existence of a* BRD$(v,3,4)$ *with one association class.*

Proof. The necessary conditions for the existence of a BIBD require that $2v(v-1)/3$ must be an integer. Hence $v(v-1)\equiv 0 \pmod 3$ is a necessary condition. Now the number of blocks must be divisible by 4 for a BRD but this is clearly satisfied if $2v(v-1)/3$ is integral.

From Corollary 5 we have that BRD$(v,3,4)$ exist whenever $v\equiv 1 \pmod 3$.

A result of Street and Rodger (1980) shows that a BRD$(v,3,4)$ can always be obtained from a BIBD$(v, v(v-1)/6, (v-1)/2, 3, 1)$. Hence we have the result for $v\equiv 1$ or $3 \pmod 6$.

We use Corollary 7 with $k=3$ to see that since every design wih $6t+4$ treatments exists then every design $3(6t+4)$ treatments exists. That is we have the result for $v\equiv 12$ or $30 \pmod{36}$. The fact that every BRD with $6t+1$, $6t+3$ or $6t+4$ treatments exists can be used in Theorem 2 with the BRD$(6,20,10,3,4)$ of Vyas (1982) to obtain the result for $v\equiv 6$, 18 or $24 \pmod{36}$. Finally we see that if every design with $6t_0$ treatments exists for $t_0<t$ then Theorem 2 can be used with BRD with $v=6$ to obtain the result for $v\equiv 0 \pmod{36}$. This completes the proof.

Theorem 14. *The condition* $v(v-1)\equiv 0 \pmod 4$ *is necessary and sufficient for the existence of a* BRD$(v,3,6)$ *with one association class.*

Proof. The condition that the number of blocks is divisible by 4 gives the necessary condition that $v(v-1)\equiv 0 \pmod 4$. Now by a Theorem of Hanani all BIBD exist with $k=4$, and in particular all BIBD$(v, v(v-1)/4, v-1, 4, 3)$ exist whenever $v(v-1)\equiv 0 \pmod 4$. We use these BIBDs with the BRD$(4,3,2)$ in Theorem 4 to obtain the result.

Together these results give us:

Theorem 15. *The conditions* $\lambda\equiv 0 \pmod 2$, $\lambda v(v-1)\equiv 0 \pmod{24}$ *are necessary and sufficient for the existence of a* BRD$(v,3,\lambda)$ *with one association class.*

Proof. The only case not considered is for $12\,|\,\lambda$. Now there exist BIBD$(v, v(v-1)/2, 2(v-1), 4, 6)$ for $v(v-1)\equiv 0 \pmod 2$. Theorem 4 and the BRD$(4,3,2)$ can now be used to get BRD$(v,3,12)$.

4. Construction of PBIBD(2)

We now use the following Theorem given in Street and Rodger (1980) to construct PBIBD(2).

Theorem 16. *Suppose there exists a* BRD(v,k,λ). *Then there exists a regular group divisible design with two association classes and parameters* $(2v, 2b, r, k, \lambda_1 = 0, \lambda_2 = \lambda/2,\ m = v,\ n = 2)$.

Hence we have part of a theorem of Hanani on PBIBD by a different method:

Theorem 17. *Suppose* $\lambda v(v-1) \equiv 0 \pmod{12}$. *Then there exist regular group divisible designs with two association classes and parameters*

$$v^* = 2v, \quad b^* = \frac{2\lambda v(v-1)}{3}, \quad r^* = \lambda(v-1),$$

$$k^* = 3, \quad \lambda_1^* = 0, \quad \lambda_2^* = \lambda, \quad m^* = v, \quad n^* = 2.$$

References

Bhaskar Rao, M. (1966). Group divisible family of PBIB designs. *J. Indian Stat. Assoc.* 4, 14–28.

Bhaskar Rao, M. (1970). Balanced orthogonal designs and their applications in the construction of some BIB and group divisible designs. *Sankhya Ser. A* 32, 439–448.

Butson, A.T. (1962). Generalized Hadamard matrices. *Proc. Amer. Math. Soc.* 13, 894–898.

Drake, D.A. (1979). Partial λ-geometries and generalized Hadamard matrices over groups. *Canad. J. Math.* 31, 617–627.

Geramita, A.V. and J. Seberry (1979). *Orthogonal designs; Quadratic Forms and Hadamard Matrices.* Marcel Dekker, New York-Basel.

Glynn, D. (1978). Finite Projective Planes and Related Combinatorial Systems. Ph.D. thesis, University of Adelaide.

Hall, M. Jr. (1967). *Combinatorial Mathematics.* Blaisdell, Waltham, MA.

Raghavarao, D. (1970). *Construction and Combinatorial Problems in Design of Experiments.* Wiley, New York.

Seberry, J. (1978). A class of group divisible designs. *Ars Combinatoria* 6, 151–152.

Seberry, J. (1980). A construction for generalized Hadamard matrices. *J. Statist. Plann. Inference* 4, 365–368.

Singh, S.J. (1982). Some Bhaskar Rao designs and applications for $k=3$, $\lambda=2$. *University of Indore J. Science* 7, 8–15.

Sinha, K. (1978). Partially balanced incomplete block designs and partially balanced weighing designs. *Ars Combinatoria* 6, 91–96.

Street, D.J. (1979). Generalized Hadamard matrices, orthogonal arrays and F-squares. *Ars Combinatoria* 8, 131–141.

Street, J. (1981). Bhaskar Rao designs from cyclotomy. *J. Austral. Math. Soc.* 29(A), 425–430.

Street, D.J. and C.A. Rodger (1980). Some results on Bhaskar Rao designs. In: R.W. Robinson, G.W. Southern and W.D. Wallis, Eds., *Combinatorial Mathematics VII*, Lecture Notes in Mathematics No. 829. Springer, Berlin-Heidelberg-New York, 238–245.

Vyas, R. (1982). Some Bhaskar Rao designs and applications for $k=3$, $\lambda=4$. *University of Indore J. Science*, 7, 16–25.

Wallis, J. (1971). Some results on configurations. *J. Austral. Math. Soc.* 12, 378–384.

Wallis, J. (1973). Kronecker products and BIBD. *J. Combinatorial Theory Ser. A* 14, 248–252.

Wilson, R.M. (1974). A few more squares. In: *Proceedings of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing.* Congressus Numerantium XXI. Utilitas Math., Winnipeg, 675–680.