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SOME FAMILIES OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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Bhaskar Rao designs with elements from abelian groups are defined and it is shown how such designs can be used to obtain group divisible partially balanced incomplete block designs with group size g, where g is the order of the abelian group.

This paper studies the group Z_3 and shows, using recursive constructions given here, that the necessary conditions are sufficient for the existence of generalized Bhaskar Rao designs. These designs are then used to obtain families of partially balanced designs.

1. INTRODUCTION

Bhaskar Rao designs with elements 0, ± 1 have been studied by a number of authors including Bhaskar Rao [1,2], Seberry [18], Singh [21], Sinha [22], Street [24], Street and Rodger [25] and Vyas [26]. Bhaskar Rao [1] used these designs to construct partially balanced designs and this was improved by Street and Rodger [25]. Another technique for studying partially balanced designs has involved looking at generalized orthogonal matrices which have elements from elementary abelian groups together with the element 0. Matrices with group elements as entries have been studied by Berman [3,4], Butson [5,6], Delsarte and Goethals [7], Drake [9], Rajkundlia [15], Seberry [16,17], Shrikhande [20], and Street [23].

Suppose we have a matrix W with elements from an elementary abelian group $G = \{h_1, h_2, \ldots, h_g\}$, where $W = h_1A_1 + h_2A_2 + \ldots + h_gA_g$, with $A_1, \ldots, A_g$ $v \times b$ $(0,1)$ matrices, and the Hadamard product $A_i * A_j$, $i \neq j$ is zero. Suppose $(a_{i1}, \ldots, a_{ib})$ and $(b_{j1}, \ldots, b_{jb})$ are the i^{th} and j^{th} rows of W then we define WW^+ by $(WW^+)_{ij} = (a_{i1}, \ldots, a_{ib}).(b_{j1}^{-1}, \ldots, b_{jb}^{-1})$, with $.$ the scalar product. Then W is a *generalized Bhaskar Rao design or GBRD* if

(i) $WW^+ = rI + \sum_{i=1}^{m} (c_iG)B_i$

(ii) $N = A_1 + \ldots + A_g$ satisfies $NN^T = rI + \sum_{i=1}^{m} \lambda_i B_i$,

that is, N is the incidence matrix of a PBIBD(m), and (c_iG) gives the number of times a complete copy of the group G occurs.

Such a matrix will be denoted by $GBRD_G(v,b,r,k;\lambda_1,\dots,\lambda_m;c_1,\dots,c_m)$. In this paper we shall only be concerned with $m = 1$, $c = \lambda/g$ and $B_1 = J-I$. In this case N is the incidence matrix of a PBIBD(1), that is a BIBD, so the equations become:

$$\text{(i)}\quad WW^+ = rI + \frac{\lambda G}{g}(J-I)$$

$$\text{(ii)}\quad NN^T = (r-\lambda)I + \lambda J.$$

So that W is a $GBRD_G(v,b,r,k,\lambda)$. Since $\lambda(v-1) = r(k-1)$ and $bk = vr$ we sometimes use the notation $GBRD(v,k,\lambda;G)$.

These matrices are generalizations of generalized weighing matrices (Berman [3,4], Seberry [16]) and we will show how they may be used in the construction of PBIBD.

<u>Example</u>. A $GBRD(5,3,3;Z_3)$ is

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & \omega & \omega^2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & \omega^2 & \omega & 0 & \omega & \omega^2 & 0 & 1 \\ 0 & 1 & 0 & \omega^2 & 0 & \omega & 1 & 0 & \omega & \omega \\ 0 & 0 & 1 & 0 & \omega^2 & \omega & 0 & \omega^2 & 1 & \omega^2 \end{pmatrix}.$$

Note, with $G = Z_3$,

$$WW^+ = 6I + G(J-I)$$

and N, obtained by replacing each non-zero element of W by 1, satisfies

$$NN^T = 3I + 3J.$$

<u>Example</u>. A $GBRD(7,3,3;Z_3)$ is obtained from the BIBD(7,3,1) and a $GH(3,Z_3)$ or $GBRD(3,3,3,Z_3)$ (see Section 2).

We use the following notation for initial blocks of a GBRD. We say $(a_\alpha,b_\beta,\dots,c_\gamma)$ is an initial block, when the latin letters are developed mod n and the greek subscripts are the elements of the group, which will be placed in the incidence matrix in the position indicated by the latin letter. That is in the $(i,a-1+i)$th position of the incidence matrix we place α, in the $(i,b-1+i)$th position we place β and so on.

For example the initial block $(1_0,2_0,4_2)(\text{mod } 7,\ Z_3)$ gives the blocks

$(1_0,2_0,4_2)$
$(2_0,3_0,5_2)$
$(3_0,4_0,6_2)$
$(4_0,5_0,0_2)$ where the subscripts are from Z_3,
$(5_0,6_0,1_2)$
$(6_0,0_0,2_2)$
$(0_0,1_0,3_2)$

and the incidence matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & \omega^2 \\ \omega^2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & \omega^2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 \end{bmatrix}$$

We form the difference table of an initial block $(a_\alpha,b_\beta,\ldots,c_\gamma)$ by placing in the position headed by x_δ and by row y_η the element $(x-y)_{\delta\eta^{-1}}$ where $(x-y)$ is mod n and $\delta\eta^{-1}$ is in the abelian group.

A set of initial blocks will be said to form a *GBR difference set* (if there is one initial block) or *GBR supplementary difference sets* (if more than one) if in the totality of elements

$$(x-y)_{\delta\eta^{-1}} (\text{mod } n,\ G)$$

each non-zero element a_g, a(mod n), $g \in G$, occurs $\lambda/|G|$ times.

Example. $(1_0,2_0,4_2)$, $(1_0,2_1,4_1)$, $(1_0,2_2,4_0)$ are GBR supplementary difference sets (mod 7, Z_3) with parameters 3-{7;3;3}. This follows since the difference tables are

$(x-y)_{\delta\eta^{-1}}$	1_0	2_0	4_2
1_0		1_0	3_2
2_0	6_0		2_2
4_2	4_1	5_1	

	1_0	2_1	4_1
1_0		1_1	3_1
2_1	6_2		2_0
4_1	4_2	5_0	

	1_0	2_2	4_0
1_0		1_2	3_0
2_2	6_1		2_1
4_0	4_0	5_2	

so each non-zero element a_g, $a \in \{1,2,\ldots,6\}$, $g \in \{0,1,2\}$ occurs $\lambda/|G| = 3/3 =$ once.

The incidence matrices of these GBR supplementary difference sets are

$$\left[\begin{array}{ccccccc|ccccccc|ccccccc}
0 & 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 1 \\
\omega^2 & 0 & 0 & 0 & 1 & 1 & 0 & \omega & 0 & 0 & 0 & 1 & \omega & 0 & 1 & 0 & 0 & 0 & 1 & \omega^2 & 0 \\
0 & \omega^2 & 0 & 0 & 0 & 1 & 1 & 0 & \omega & 0 & 0 & 0 & 1 & \omega & 0 & 1 & 0 & 0 & 0 & 1 & \omega^2 \\
1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 1 & 0 & 0 & 0
\end{array}\right]$$

which is a $GBRD(7,3,3;Z_3)$.

Example. The following initial blocks (mod 11, Z_3) give a $GBRD(11,3,3;Z_3)$:

$(1_0,2_2,7_2)$, $(1_0,4_2,8_0)$, $(1_0,3_0,5_2)$, $(1_0,2_1,3_1)$, $(1_0,4_0,7_1)$.

Example. The following initial blocks (mod 7, Z_3) give a $GBRD(8,3,6;Z_3)$:

$(\infty_0,1_0,6_0)$, $(\infty_0,2_1,5_1)$, $(\infty_0,3_2,4_2)$, $(0_0,2_0,6_0)$,
$(0_0,2_1,6_2)$ twice, $(0_0,2_2,6_1)$ twice.

2. SOME CONSTRUCTION THEOREMS

Theorem 1. *Suppose there are generalized Bhaskar Rao designs* $GBRD(v,k,\lambda;G)$ *and* $GBRD(u,k,\mu;G)$. *Further suppose there are* $k-1$ *mutually orthogonal latin squares of order* u. *Then there is a generalized Bhaskar Rao design* $GBRD(uv,k,\lambda\mu;G)$.

Proof. Proceed as in the proof of Theorem 2 of Seberry [18]. We first use the mutually orthogonal latin squares of order u, $M_1,M_2,\ldots,M_{k-1}$ to form $(0,1)$ matrices M_{ij} which satisfy

$$\sum_{j=1}^{u} M_{ij}M_{ij}^T = mI, \quad \sum_{j=1}^{u} M_{ij} = J, \quad \sum_{j=1}^{u} M_{ij}M_{kj}^T = J, \quad k \neq i.$$

Write

$$C = \begin{bmatrix} I & I & \cdots & I \\ M_{11} & M_{12} & \cdots & M_{1u} \\ \vdots & \vdots & & \vdots \\ M_{k-1,1} & M_{k-1,2} & \cdots & M_{k-1,u} \end{bmatrix} = (C_{ij}).$$

Write A for the $GBRD(v,k,\lambda;G)$ and B for the $GBRD(u,k,\mu;G)$. We now form D_i, $i = 1,\ldots,u$, by replacing the first non-zero element in each column of A, g say, by gC_{1i}, and the m^{th} non-zero element in each column of A, h say, by hC_{mi}. We now claim the matrix is the required design

$$[\underbrace{B \oplus B \oplus \ldots \oplus B}_{v \text{ copies}} \vdots D_1 \vdots D_2 \vdots \ldots \vdots D_u].$$

Also we have, as in the case of BRD with elements ±1,

Theorem 2. *Suppose we have a* BIBD(v,k,λ) *and a* GBRD$(k,j,\mu;G)$. *Then there exists a* GBRD$(v,j,\lambda\mu;G)$.

Example. There exists a BIBD$(v,5,1)$ for every v satisfying $v-1 \equiv 0 \pmod 4$ and $v(v-1) \equiv 0 \pmod{20}$. We have exhibited a GBRD$(5,3,3;Z_3)$. Hence there exist GBRD$(v,3,3;Z_3)$ for these v, i.e. $v \equiv 1$ or $5 \pmod{20}$.

We note that *generalized Hadamard matrices* GH$(h|G|, G)$ can be regarded as GBRD$(h|G|, h|G|, h|G|, G)$ and hence used in the above theorems since they exist for $h|G|$ a prime power and other orders (see Street [23] and Seberry [17]).

Theorem 3. *There exists a* GBRD$(2p+1,3,3;Z_3)$ *for every integer* p.

Proof. The p initial blocks

$$(0_0,1_1,2p_1),\ (0_0,2_1,(2p-1)_1),\ \ldots,\ (0_0,p_1,(p+1)_1)$$

give the required design when developed $(\bmod\ 2p+1,\ Z_3)$.

Theorem 4. *There exists a* GBRD$(2p+2,3,6;Z_3)$ *for every integer* p.

Proof. The following initial blocks $(\bmod\ 2p+1, Z_3)$ give the required design:

$$(\infty_0,1_0,2p_0),\quad (\infty_0,2_1,(2p-1)_1),\quad (\infty_0,3_2,(2p-2)_2),\quad (0_0,1_1,2p_1),$$
$$(0_0,2_1,(2p-1)_1),\quad (0_0,3_1,(2p-2)_1),\quad (0_0,1_1,3_2),\quad (0_0,1_2,3_1)$$

and each of the following twice

$$(0_0,4_1,(2p-3)_1),\quad (0_0,5_1,(2p-4)_1)\ \ldots\ (0_0,p_1,(p+1)_1).$$

3. EXISTENCE THEOREMS

If we consider GBRD$(v,b,r,3,\lambda;Z_3)$ then we have as necessary conditions

(1) $\lambda \equiv 0 \pmod 3$,

(2) $\lambda(v-1) \equiv 0 \pmod 2$,

(3) $\lambda v(v-1) \equiv 0 \pmod 6$.

If $\lambda \equiv 0 \pmod 6$ we have to find GBRD$(v,tv(v-1),3t(v-1),3,6t;Z_3)$.

If v is even then we have three cases:

(a) $v \equiv 4 \pmod 6$ then we are considering GBRD($v = 6p+4, 3tv(2p+1), 9t(2p+1), 3, 6t, Z_3$). Now every BIBD($6p+4, v(2p+1), 3(2p+1), 3, 2$) exists and using the GH($3, Z_3$) we see every GBRD($6p+4, 3, 6; Z_3$) exists, giving the result;

(b) $v \equiv 0 \pmod 6$ then we are considering GBRD($v = 6p, 6pt(6p-1), 3t(6p-1), 3, 6t; Z_3$). Every BIBD($6p, 2p(6p-1), 6p-1, 3, 2$) exists and so, as in (a) we have every GBRD($6p, 3, 6t; Z_3$);

(c) $v \equiv 2 \pmod 6$ then we are considering GBRD($v = 6p+2, 2t(3p+1)(6p+1), 3t(6p+1), 3, 6t, Z_3$). These all exist by Theorem 4.

If v is odd, and $v \equiv 1$ or $3 \pmod 6$ then every BIBD($v, 3, 1$) exists and so all GBRD($v, 3, 6t; Z_3$) exist. When $v \equiv 5 \pmod 6$, since GBRD($v, 3, 3; Z_3$) exists by Theorem 3, we have the result.

Similarly if $\lambda \equiv 3 \pmod 6$ we wish to establish the existence of a GBRD($v, 3, 6t+3; Z_3$). This design has as a necessary condition that v is odd.

Now, as above, we observe that if $v \equiv 1$ or $3 \pmod 6$ every BIBD($v, 3, 1$) exists and so all GBRD($v, 3, 3(2t+1); Z_3$) exist. The case for $v \equiv 5 \pmod 6$ is established by Theorem 3.

Summarizing we have:

Theorem 5. *The necessary conditions* $\lambda \equiv 0 \pmod 3$, $\lambda(v-1) \equiv 0 \pmod 2$, $\lambda v(v-1) \equiv 0 \pmod 6$, *are sufficient for the existence of a* GBRD($v, 3, \lambda; Z_3$).

We note the similarity between this theorem and those of Hanani (see Hall [12]) and indeed with the elements of Z_3 replaced by the element 1 the theorem is Hanani's theorem when $\lambda \equiv 0 \pmod 3$.

4. APPLICATIONS

As in Street and Rodger [25] we have the following:

Theorem 6. *Let* $W = h_1A_1 + h_2A_2 + \ldots + h_gA_g$ *be a* $GBRD_G(v,b,r,k;\lambda_1,\ldots,\lambda_m;c_1,\ldots,c_m)$, *where* $G = \{h_1,\ldots,h_g\}$ *is an abelian group of order* g. *Then, with* $P_1,\ldots,P_g$ *the permutation matrix representation of* G *by* $g \times g$ *matrices,*

$$M = A_1 \times P_1 + A_2 \times P_2 + \ldots + A_g \times P_g$$

is the incidence matrix of a PBIBD *with at most* $2m+1$ *associated classes and parameters* $v^* = vg$, $b^* = bg$, $r^* = r$, $k^* = k$,

$$\lambda_i^* = \begin{cases} \frac{1}{g}(\lambda_i+c_i) & 1 \le i \le m \\ 0 & i = m+1 \\ \frac{1}{g}(\lambda_i-c_i) & m+2 \le i \le 2m+1 \end{cases}$$

Corollary 7. *Let* W *be a* $GBRD(v,g,\lambda;Z_g)$. *Then there exists a regular group divisible design with parameters* $v^* = vg$, $b^* = bg$, $r^* = r$, $k^* = k$, $\lambda_1 = 0$, $\lambda_2 = \lambda/g$ *with* v *groups of size* g.

Corollary 8. *Suppose there exists a* $GBRD(v,3,\lambda;Z_3)$. *Then there exists a regular group divisible design with two association classes and parameters* $v^* = 3v$, $b^* = 3b$, $r^* = r$, $k^* = 3$, $\lambda_1^* = 0$, $\lambda_2^* = \lambda/3$, $m^* = v$, $n^* = 3$.

Hence we have Hanani's theorem but by a different approach:

Theorem 9. $\lambda(v-1) \equiv 0 \pmod 2$ *is a necessary and sufficient condition for the existence of a regular group divisible design with two association classes with* $k^* = 3$, $\lambda_1^* = 0$, $\lambda_2^* = \lambda$, $m^* = v$, $n^* = 3$.

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