The Skew - Weighing Matrix Conjecture

Jennifer Seberry*

SUMMARY :

We review the history of the skew-weighing matrix conjecture and show that there exist skew-symmetric weighing matrices $W(21.2^t, k)$ for all $k=0,1, \ldots\ldots, 21.2^t - 1$, $t \geqslant 4$ a positive integer. Hence there exist orthogonal designs of type $1(1,k)$ for all $k=0,1, \ldots, 21.2^t - 1$, $t \geqslant 4$ a positive integer, in order 21.2^t.

1. Introduction :

An orthogonal design X of order n and type $(s_1, s_2, \ldots, s_t)$ on the commuting variables $0, x_1, \ldots, x_t$ is a square matrix with entries from $\{0, \pm x_1, \ldots, \pm x_t\}$ with the properties that

(i) the element $\pm x_i$ occurs s_i times in each row and column,

(ii) the inner product of distinct rows is zero. Hence X satisfies

$$XX^T = \sum_{i=1}^{t} s_i x_i^2 I_n$$

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where I_n is the identity matrix.

A weighing matrix $W = W(n,k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies. $WW^T = kI_n$,
and W is equivalent to an orthogonal design of order n and type (k). The number k is called the weight of W. Weighing matrices have long been studied because of their use in weighing experiments as first studied by Hotelling (1944) and latter by Raghavarao (1960) and others.

In Wallis (1972) it was conjectured that if $n \equiv 0 \pmod 4$ then weighing matrices $W(n, k)$ exist for every $k=1,2 \ldots, n$. Call this Conjecture A. This was proved true for orders $n=2^t$, t a positive integer in Gerammita, Pullman, Wallis (1974). Later the conjecture was made stronger by Seberry until it appeared in the following form :

Conjecture B : There exist skew-symmetric weighing matrices $W(n,k)$ for every $k=1,2 \ldots, n-1$ when $n \equiv 0 \pmod 8$. Equivalently, every orthogonal design $(1,k)$, $k=1,2, \ldots, n-1$ exists when $n \equiv 0 \pmod 8$.

This conjecture was established for $n=2^t.3$, $2^t.5$, $2^t.9$ by Geramita and Wallis (1974, 1975), for $n=2^t.7$ by Eades and Wallis (1976) and for $n=2^{t+1}.15$ by Seberry (1980), $t \geqslant 3$, an integer. In this paper we will establish the conjecture for $n=2^{t+1}.21$.

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2. Preliminary Results And Notaion :

We make extensive use of the book Geramita and Seberry (1979). We quote the following theorems, giving their reference from the aforementioned book, that we use :

Lemma 1 : (Lemma 4.11). If there exists an orthogonal design in order n of type $(s_1, s_2, ..., s_u)$ then there exists an orthogonal design in order $2n$ of type $(s_1, s_1, es_2, ..., es_u)$, $e=1$ or 2.

Lemma 2 : (Lemma 4.4). If A is an orthogonal design of order n and type $(u_1, ..., u_s)$ on the variables $x_1, ..., x_s$, then there is an orthogonal design of order n and types $(u_1, ..., u_i+u_j, ..., u_s)$ and $(u_1, ..., u_2, ..., u_{j-1}, u_{j+1}, ..., u_s)$ on the $s-1$ variables $x_1, ..., x_j, ..., x_s$.

Example :

$$\begin{bmatrix} x & y & z & w \\ -y & x & w & -z \\ -z & -w & x & y \\ -w & z & -y & x \end{bmatrix}$$

is an orthogonal design of type (1, 1, 1, 1) in order 4. Using Lemma 2 we see

$$\begin{bmatrix} x & y & z & z \\ -y & x & z & -z \\ -z & -z & x & y \\ -z & z & -y & x \end{bmatrix} \text{ and } \begin{bmatrix} x & y & o & w \\ -y & x & w & o \\ o & -w & x & y \\ -w & o & -y & x \end{bmatrix}$$

are orthogonal designs of types (1,1,2) and (1,1,1) respectively. Lemma 1 tells us there are otherogonal designs of types (1,1,2,2,2,), (1,1,2,4), (2,2,2,2,) and (2,2,2,) in order 8.

Lemma 3 (Corollary 5.2) : If all orthogonal designs $(1, k)$, $k=1, ..., n-1$, exist in order n, then all orthogonal designs $(1, j)$, $j=1, ..., 2n-1$, exist in order $2n$.

Example : Using the orthogonal design of type (1, 1, 1, 1) in order 4 with Lemma 2 we see that orthogonal designs of types (1, 1), (1, 2) and (1, 3) exist in order 4. Lemma 3 is now used to see that orthogonal designs of types $(1, k)$, $k=1, 2, ..., 7$ exist in order 8. By repeated use of Lemma 3 we see that orthogonal designs of types$(1, j)$, $j=1, 2, ..., 2^t-1$, t a positive integer, exist in every order 2^t.

Theorem 4 (Theorems 2.19 and 2.20) : suppose $n \equiv 0 \pmod 4$. Then the existence of a $W(n, n-1)$ implies the existence of a skew-symmetric $W(n, n-1)$. The existence of a skew-symmetric $W(n, k)$ is equivalent to the existence of an orthogonal design of types $(1,k)$ in order n.

Theorem 5 (Proposition 3.54 and Theorem 2.20) : An orthogonal design of type $(1, k)$ can only exist in order $n \equiv 4 \pmod 8$ if k is the sum of three squares. An orthogonal design of type $(1, n-2)$ can only exist in order $n \equiv 4 \pmod 8$ if $n-2$ is the sum of two squares.

Theorem 6 : Orthogonal designs of types $(1, k)$ exist for $k=1, ..., n-1$ in orders $n=2^t, 2^{t+3}.3, 2^{t+3}.5, 2^{t+3}.7, 2^{t+3}.9, 2^{t+4}.15$ $t \geqslant 0$ an integer.

Theorem 7 : (Theorem 4.49) If there exist four circulant matrices A_1, A_2, A_3, A_4 of order n satisfying.

$$\sum_{i=1}^{4} A_i A_i^T = fI$$

where f is the quadratic form $\sum_{j=1}^{s} t_j x_j^2$, then there is an orthogonal design of order $4n$ and type $(t_1, t_2, \ldots, t_s)$.

Theorem 8 : (Theorems 4.124 and 4.41). Let q be a prime power then there is a circulant $W(q^2+q+1, q^2)$. Let $p \equiv 1 \pmod 4$ then there are two circulant symmetric matrices R, S of order $\frac{1}{2}(p+1)$ satisfying

$$RR^T + SS^T = PI,$$

and a symmetric matrix P, with zero diagonal satisfying $PP^T = p^I{}_{p+1}$ that is a symmetric $W(p+1, p)$.

Corollary 9 : There exists a circulant $W(21, 16)$ and $W(7, 4)$. There exist two circulant symmetric matrices R, S of order $n=41, 21$ and 7 satisfying $RR^T + SS^T = (2n-1) I_n$. There exist symmetric $W(6, 5) = X$ and $W(14, 13) = Y$ with zero diagonal.

We use the following notation :

I is the identity matrix with order taken from the context,
J is the matrix of ones with order taken from the context;
A is the back-circulant matrix with first row $b\ a\ \bar{a}$;
B is the back-circulant matrix with first row $b\ a\ a\ \bar{a}\ a\ \bar{a}\ \bar{a}$;
W_7 is the back-circulant matrix with first row $\bar{1}\ 1\ 1\ 0\ 1\ 0\ 0$;
W_{21} is the back-circulant matrix with first row the same as the circulant $W(21,16)$;
R_n, S_n are circulant symmetric matrices of order $n=7, 21, 41$ satisfying $S_n S_n^T + R_n R_n^T = (2n-1) I_n$;
X is the symmetric $W(6, 5)$;
Y is the symmetric $W(14, 13)$;
U, V, Z are the circulant symmetric matrices with first rows $0\ 1\ 1\ -\ -1$, $1, 0\ 1\ -\ 1\ 1\ -\ 1$, $0\ -\ 1\ 1\ 1\ 1\ -$ respectively, $U^2 + V^2 + Z^2 = 19I - J$.

3. Results In Orders Divisible By 21 :

We recall from Theorem 5 that orthogonal designs of type $(1, k)$ can only exist in order 84 if k is the sum of three squares. We see $84-2 = 9^2 + 1^2$ so the other condition is satisfied. Hence we have $(1, k)$ can not exist for $k = 4^a(8b+7)$, i.e. $k \in \{7, 15, 23, 28, 31, 39, 47, 55, 60, 63, 71, 79\}$.

Theorem 10 : Orthogonal designs of types $(1, k)$ exist in order 84 for $k \in \{x / x = a^2+b^2+c^2, x \neq 42, 61, 62, 67, 70, 73\text{–}78, 82\}$.

Proof : Theorem 6 and Lemma 8.36 of Geramita and Seberry (1979) say that all orthogonal designs of type $(1, k)$ exist in order 40 and 44 for $k \leq 38$ the sum of three integer squares. We have by direct sum, $A \oplus B$, of the OD of type $(1, k)$ in order 40, A and the OD of type $(1, k)$ in order 44, B, the result in order 84.

Table H.4, Table H.2, Theorem 4.139 (xvi) and the proof of Theorem 4.139 of Geramita and Seberry (1979) give $(1, k)$ in order 84 for $k \in \{40, 41, 43, 44, 48, 51, 52, 54, 58, 59, 66, 68\}$.

Using Table H.6 which gives 2-complementary sequences giving ODs of type (4, 16) in orders $2m$, $m \geqslant 10$ in Lemma 4.118 we obtain designs of types (1, 1, 16, 64) in every order $4p$, $p \geqslant 21$. In particular, we obtain designs of types (1, 64), (1, 80) and (1, 81) in order 84.

From corrollary 9 there is a circulant W (21, 16) = W and two circulant symmetric matrices, R, S satisfying

$RR^T + SS^T = 41\ I$. Hence xI, yW, zR, zS satisfy

$(xI)(xI)^T + (yW)(yW)^T + (zR)(zR)^T + (zS)(zS)^T = (x^2 + 16y^2 + 41z^2)\ I,$
and so by Theorem 7 we have an orthogonal design of type (1, 16, 41), and thus (1, 57), in order 84.

We note from Geramita and Seberry (1979) that the following ODs exist : (i) (1, 2, 3, 6) in order 12; (ii) (1, 1, 9, 9) in 28; (iii) (1, 1, 13, 13) in 28; (iv) (1, 6, 18) in 28. We replace the variables of these matrices by other matrices as follows : (i) aI_7, $b(J-2I)$, cI, the back circulant matrix with first row $d\ b\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}$ to an OD of type (1, 3, 6, 50) in order 84; (ii-a) cI, dI, circulant matrix with first row $0\ b\ b$, back-circulant matrix with first row $\bar{a}\ b\ \bar{b}$ get an OD of type (1, 1, 9, 36) in order 84; (ii-b) and (iii) back-circulant matix with first row $a\ b\ \bar{b}$, circulant matrices with first rows $0\ b\ b$, $\bar{c}\ c\ c$, $o\ c\ c$ to get an ODs of types (1, 4, 45) and (1, 4, 65); (iv) aI, bJ, the back-circulant matrix with first row $c\ b\ \bar{b}$ to get an OD of type (1, 18, 54) in order 84. These give ODs $(1, k)$ for $k \in \{45, 46, 49, 50, 53, 56, 65, 69, 72\}$ in order 84.

This completes the result.

Theorem 11 : There exist W $(84, k)$ for all $1 \leqslant k \leqslant 84$ except possibly $k=71, 79$.

Proof : Combining the results of the previous theorem, nothing that $(1, j)$ gives both a $W(84, j)$ and a $W(84, j+1)$, with the results of Lemma 8.46 of Germaita and Seberry gives the result for all values except W (84, 63) and W (84, 77).

We note that an OD of type (7, 7, 14) exists in order 28 and replacing the variables by cJ, the circulant matrix with first row $a\ c\ \bar{c}$ and the back-circulant matrix with first row $b\ c\ \bar{c}$ respectively, gives an OD of type (7, 14, 63) in order 84 giving the W (84, 63) and W (84, 77) required.

Theorem 12 : There exists an OD of type $(1, k)$ in order 168 for $k \in \{1, \ldots\ldots, 154, 156, 157, 160, \ldots, 163, 165, \ldots, 167\}$.

Proof : Using Theorem 6 we have the result immediately for $(1, k)$ $k \in \{(1, \ldots, 71\}$ as $168 = 72 + 96$ and $(1, k)$ exist in both these orders.

We use the existance of W = W $(84, k)$, from Theorem 11, in $\begin{bmatrix} aI & W \\ W^T & -aI \end{bmatrix}$ to obtain $(1, k)$ in 168 for $k \in \{72, \ldots, 78, 80, \ldots, 84\}$. Using Lemmas 1, 2 and Theorem 10 we get the result for $k \in \{86, ..., 93, 96, ..., 109, 112, ..., 119, 128, ..., 133, 136, ..., 139, 144, 145, 160, ..., 163, 166, 167\}$.

Suppose there exists an OD of type $(1, j)$ in order 28 then by Lemma 1 we have a design of type $(1, 1, j, j)$ in order 56.

Define K=J—I and A to be the back-circulant matrix with first row $b\ a\ \bar{a}$. Then replacing the variables of the $(1, 1, j, j)$ design by (i) cI, dI, A, a(J—2I) and (ii) A, aK, cK, c(J-2I), we obtain designs of types $(1, 1, j, 5j)$ and $(1, 4, 5j)$ respectively in order 168. Now from Geramita and Seberry (1979), Appendix C, we have ODs of type $(1, j)$ in order 28 for $j \in \{17, 18, 19, 22, 24, 25, 26, 27\}$ giving designs $(1, k)$ in 168 for $k \in \{85, 94, 95, 110, 111, 120, 121, 124, 125, 126, 134, 135, 150, 151, 156, 157\}$.

There exist orthogonal designs of types (1, 1, 1, 25) and (1, 3, 24) in order 28. Hence, using Lemma 1, ODs of types (1, 1, 1, 25, 25,) and (1, 3, 24, 24,) exist in order 56. Replacing the variables by cI, dI, eI, A, a(J—2I) and cI, dI,A, a(J-2I) respectively, we obtain ODs of types (1, 1, 1, 25, 125) and (1, 3, 24, 120) in order 168. Hence ODs of types $(1, k)$ for $k \in \{123, 127, 147, 152\}$ exist in order 168.

The following ODs exist in order 24 : (1, 1, 1, ,1, 1, 19), (1, 1, 1, 1, 10, 10) (1, 1, 1, 1, 1, 5, 5, 9), (1, 1, 1, ,1, 1, ,1, 9, 9,), (1, 1, 1, 4, 17), (1, 2, 2, 3, 16) and (1, 1, 1, 1, 5, 15). We replace the variables by the matrices as follows :

(i) aI, bI, cI, eI, dI, $f\mathrm{W}_7$; (ii) aI, $b\mathrm{W}_7$, $c\mathrm{W}_7$, $d\mathrm{W}_7$, $e\mathrm{R}_7$, $e\mathrm{S}_7$; (iii) cI, dI, fI+a(J-I), fI-a (J-aI) B, $e\mathrm{R}_7$, $e\mathrm{S}_7$, B; (iv) B, a(I+U), a(I+V), a(I+Z), cI, dI, $e\mathrm{R}_7$, $e\mathrm{S}_7$; (v) cI, dI, a(J-I), a (J-2I), B; (vi) cI, a(J-I), a (J-2I), dI, B; (vii) cI, dI, eI, fI, a (J-2I), B. This gives us ODs in order 168 of the following types. (1, 1, 1, 1, 1, 76), (1, 4, 4, 4, 130), (1, 1, 2, 65, 10, 72), (1,1,1, 27, 117), (1, 1, 17, 136), (1, 3, 16, 122), (1, 1, 1, ,1 15, 125).

Hence we have orthogonal designs of types $(1, k)$ for $k \in \{79, 122, 140, ..., 143, 146, 148, 149, 153, 154\}$, in order 168.

The proof of Lemma 8.46 of Geramita and Seberry (1979) gives a back-circulant matrix A_1, and three circulant matrices, A_2, A_3, A_4, all of order 21 which can be used to replace four of the variables ofa (1, 1, 1, 1, 1, 1, 1, 1,) in order 8 while the others are replaced by $c\mathrm{R}_{41}$, $c\mathrm{S}_{41}$, $d\mathrm{R}_{41}$, $d\mathrm{S}_{41}$ to give an OD of type (1, 83, 41, 41,) in 168 that is a (1,165) in 168. This completes the proof.

Theorem 13 : All orthogonal designs of type $(1, k)$, $0 \leqslant k \leqslant n—1$, exist in order $n=n^{t+4}$. 21, t a positive integer.

Proof : From the previous theorem and Lemmas 1 and 2 we have all orthogonal designs of type $(1, k)$ in order 336 except for $k \in$ (310, 311, 316, 317, 318, 319, 328, 329). Theorem 5.1 of Geramita and Seberry (1979) says that the existence of an orthogonal design of type (a, b) in order n gives an OD of type $(a, a, 2a, b, b, 2b)$ in order $4nn$. Hence, using the (1,26) and (1,27) designs in order 28, there are ODs of type (1, 1, 2, 26, 26, 52) and (1, 1, 2, 27, 27, 54), in order 112. Replacing the variables by A, a (J-I), cI, dJ, d (J-2I), d (J-2I) in both cases we get ODs of types (1, 2, 4, 312) and (1, 2, 4, 324) in order 336. So we have solutions for $k \in \{316, 318, 328\}$.

Using the (1, 1, 13, 13) and (1, 27) designs in order 28 with Lemmas 1 and 2 we obtain ODs of types (1, 1, 2, 26, 26,) and (1,1, 27, 27) in order 56. Replacing the variables by aI. bX, cI, dI+eX, dI—eX and aI, bX, cI+dX, cI—dX respectively we obtain designs of types (1, 2, 5, 52, 260) and (1, 5, 54, 270) giving (1, 317) ,(1, 319), (1, 329) in order 336.

We use the theorem quoted earlier to show an OD of type (1, 11) in order 12 may be used to construct an OD of type (1, 1, 2, 11, 11, 22) in order 48. Replacing the matrices by cI, dI, eI, a (J–2I), B, B gives an OD of type (1, 1, 2, 33, 275) in order 336 giving a (1, 310) and (1, 311).

Hence we have shown all orthogonal designs of type (1, k), k=0, 1, ..., 335 exist in order 336. We now use Lemma 3 to obtain the result of the theorem.

4. GENERAL RESULTS :

We note that orthogonal designs of types (1, k), $k \in \{1, ..., 71\}$ exist in order 72, 96 and 120. Hence orthogonal designs (1, k) for $k \in \{1, ..., 71\}$ exist in every order 24n, $n \geqslant 3$ and orthogonal designs of type (1, j), $j \in \{1, ..., 143\}$ exist in every order 48n, $n \geqslant 3$.

Since ODs of types (1, k) for $k \in \{1, ..., 95\}$ exist in orders 96, 120, 144, 168 they exist in every order 24n, $n \geqslant 4$ and designs of type (1, j), $j \in \{1, ..., 191\}$ exist in every order 48n, $n \geqslant 4$. Summarizing ,and using Seberry (1980) we have :

Theorem 14 ; There exist orthogonal designs of types (1, k) in orders 48n with

(a) $n \geqslant 3$, $k \in \{1, ..., 143\}$,

(b) $n \geqslant 4$, $k \in \{1, ..., 191\}$,

(c) $n \geqslant 5$, $k \in \{1, ..., 211, 216, ..., 219, 221, 223, 226\ 227, 231\text{-}239\}$.

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Department of Applied Mathematics,
University of Sydney, Australia.