

HIGHER DIMENSIONAL ORTHOGONAL DESIGNS AND HADAMARD MATRICES

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1. INTRODUCTION. When n^2 elements are given they can be arranged in the form of a square, similarly when n^g elements ($g \geq 3$ an integer) are given they can be arranged in the form of a *g-dimensional cube of side n* (in short a *g-cube*). The position of the elements can be indicated by g suffixes. Suppose there is a set of elements with g suffixes such that,

$$a_{ijk\dots r}$$

n^g in number, arranged in g sets of 2-dimensional square matrices in a space of g dimensions and forming a g -dimensional cube of side n . The elements which have all suffixes the same, except one say i , lie in the same row (line), those which have all suffixes the same, save two say i and j , lie in the same 2-dimensional layer parallel to a coordinate axes (plane or face) ... those which have only one in common lie a $(g-1)$ -dimensional layer.

In [2] it is pointed out that it is possible to define orthogonality for higher dimensional matrices in many ways.

Intuitively we see that each two-dimensional matrix within the n -dimensional matrix could have orthogonal row vectors (we call this 'propriety $(2,2,\dots,2)$); or perhaps each pair of two-dimensional layers

$$A^j = \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \tilde{a}_t \end{bmatrix} \quad \text{and} \quad B^j = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_t \end{bmatrix}$$

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could have $A \cdot B = \text{tr}(AB^T) = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_t \cdot b_t = 0$ (note if the row vectors in this direction had been orthogonal we would have had $a_i \cdot b_i = 0$ for each i) (we call this propriety (...3...)); or perhaps each pair of three-dimensional layers

$$\alpha = \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^t \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} B^1 \\ B^2 \\ \vdots \\ B^t \end{bmatrix}$$

could have $\alpha \cdot \beta = A^1 \cdot B^1 + \dots + A^t \cdot B^t = 0$ (note that if the 2-dimensional matrices had been orthogonal we would have had $A^j \cdot B^j = 0$ for each j); and so on.

We say an n -dimensional matrix is orthogonal of *propriety* (d_1, \dots, d_n) with $2 \leq d_i \leq n$ where d_i indicates that in the i^{th} direction (i.e the i th coordinate) the $d_i - 1^{\text{st}}$, d_i^{th} , $d_i + 1^{\text{st}}$, ..., $(n-1)^{\text{st}}$ dimensional layers are orthogonal but the $d_i - 2^{\text{nd}}$ layer is not orthogonal. $d_i = \infty$ means not even the $(n-1)^{\text{st}}$ layers are orthogonal.

The Paley cube of size $(q+1)^n$ constructed in [2] for $q \equiv 3 \pmod{4}$ a prime power has propriety $(\infty, \infty, \dots, \infty)$ but if the 2-dimensional layer of all ones is removed in one direction the remaining n -dimensional matrix has all 2-dimensional layers in that direction orthogonal.

An n -cube orthogonal design, $D = [d_{ijk\dots}]$, of propriety

(d_1, d_2, \dots, d_n) , side d and type $(s_1, s_2, \dots, s_t)^n$ on the commuting variables x_1, x_2, \dots, x_t has entries from the set $\{0, \pm x_1, \dots, \pm x_j, \dots, \pm x_t\}$ where $\pm x_i$ occurs s_i times in each row and column of each 2-dimensional layer and in which e_j -dimensional layer, $d_i - 1 \leq e_j \leq n - 1$, in the i^{th} direction is orthogonal.

Shlichta [4] found n -dimensional Hadamard matrices of size $(2^t)^n$ and propriety $(2, 2, \dots, 2)$. In [2] the concept of higher dimensional Williamson matrices there is a 3-dimensional Hadamard matrix of size $(4t)^3$ and propriety $(2, 2, 2)$. The existence of a higher dimensional orthogonal design of type $(1, 1, 1, 1)^n$ was used in [3] to extend this result to obtain t -dimensional Hadamard matrices of size $(4t)^t$ and propriety $(2, 2, \dots, 2)$. It was also shown in [3] that if k is the order of a 2-dimensional Hadamard matrix which can be formed by using an abelian group difference set then there is a higher dimensional Hadamard matrix of type $(k)^k$ and propriety $(2, 2, \dots, 2)$. In [5] it is shown that there are n -dimensional orthogonal designs of type $(2^t, 2^t)^n$, side 2^t , $t \geq 0$ and propriety $(2, 2, \dots, 2)$ and in [3] of type $(q)^{\frac{1}{2}(q+1)}$, $q \equiv 1 \pmod{4}$ a prime power, with propriety $(2, 2, \dots, 2)$.

2. Higher Dimensional Anti-Amicable Hadamard Matrices

Let two matrices $A = (a_{ijk\dots v})$ and $B = (b_{ijk\dots v})$ of dimension t and side n be called *anti-amicable* if

$$(i) \sum_{y=1} a_{ijk\dots x.y..v} b_{ijk\dots x.y..v} = 0 ,$$

where the p th coordinate varies and all other coordinates are fixed, and if

$$(ii) \sum_{y=1} a_{ijk\dots x.y..v} b_{ijk\dots z.y..v} + a_{ijk\dots z.y..v} b_{ijk\dots x.y..v} = 0$$

$x \neq z$, where the p th coordinate varies, the q th coordinate takes two values x and z , and all other coordinates are fixed.

In particular for side 2 and dimension 2 two matrices are anti-amicable if

$$(i) \begin{aligned} a_{11}b_{11} + a_{12}b_{12} &= 0 , \\ a_{21}b_{21} + a_{22}b_{22} &= 0 , \end{aligned}$$

$$(ii) a_{11}b_{21} + a_{12}b_{22} + a_{21}b_{11} + a_{22}b_{12} = 0 .$$

Example 1.

$$A = \begin{bmatrix} - & 1 \\ 1 & 1 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$$

are anti-amicable Hadamard matrices.

Example 2.



are anti-amicable Hadamard matrices of side 2 dimension 3 and propriety (2,2,2).

LEMMA 1. *There are anti-amicable Hadamard matrices of side 2, dimension n and propriety (2,2,...,2).*

Proof. Define $A = (a_{ijw})$, $B = (b_{ijw})$ where w is a vector of length $n-2$ and the subscripts ij can appear in any of the $\binom{n}{2}$ subscripts of a and b but for convenience we put them first, and $s = \text{sum of the elements of } w + i + j$

$$a_{ijw} = \begin{cases} (-1)^{s/2+1} & s \text{ even} \\ (-1)^{(s+1)/2} & s \text{ odd} \end{cases} \quad b_{ijw} = \begin{cases} (-1)^{s/2} & s \text{ even} \\ (-1)^{(s+1)/2} & s \text{ odd} \end{cases}$$

First we establish that A and B are both Hadamard matrices of side, dimension n and propriety $(2,2,\dots,2)$. Clearly for any face of A $a_{ijw} = a_{jiw}$ and $a_{iiw} = -a_{jjw}$, and similarly for B so we have Hadamard matrices of propriety $(2,2,\dots,2)$. We now establish the anti-amicability. We consider $t = \text{sum of elements of } w$. Now for $t = 0 \pmod{4}$.

$$A = a_{11w}b_{11w} + a_{12w}b_{12w} = (-1)^2 (-1) + (-1)^2 (-1)^2 = 0$$

$$B = a_{21w}b_{21w} + a_{22w}b_{22w} = (-1)^2 (-1)^2 + (-1)(-1)^2 = 0$$

$$\begin{aligned} C &= a_{11w}b_{21w} + a_{12w}b_{22w} + a_{21w}b_{11w} + a_{22w}b_{12w} \\ &= (-1)^2 (-1)^2 + (-1)^2 (-1)^2 + (-1)^2 (-1) + (-1)(-1)^2 = 0 ; \end{aligned}$$

for $t \equiv 1 \pmod{4}$

$$A = (-1)^2 (-1)^2 + (-1) (-1)^2 = 0$$

$$B = (-1)(-1)^2 + (-1)(-1) = 0$$

$$C = (-1)^2 (-1)^2 + (-1)(-1) + (-1)(-1)^2 + (-1)(-1)^2 = 0 ;$$

for $t \equiv 2 \pmod{4}$

$$A = (-1)(-1)^2 + (-1)(-1) = 0$$

$$B = (-1)(-1) + (-1)^2 (-1) = 0$$

$$C = (-1)(-1) + (-1)(-1) + (-1)(-1)^2 + (-1)^2 (-1) = 0 ;$$

and for $t \equiv 3 \pmod{4}$

$$A = (-1)(-1) + (-1)^2 (-1) = 0$$

$$B = (-1)^2 (-1) + (-1)^2 (-1)^2 = 0$$

$$C = (-1)(-1) + (-1)^2 (-1)^2 + (-1)^2 (-1) + (-1)^2 (-1) = 0 . \square$$

3. An Interesting Non-Orthogonal n-Dimensional Matrix

We now construct, in a manner similar to Hammer and Seberry [2], a cube, like the Paley cube described therein, which has all but two of the two-dimensional faces in any (and every) direction an Hadamard matrix.

Let $q \equiv 1 \pmod{4}$ be a prime power and $z_0 = 0, z_1 = 1, \dots, z_{q-1}$ be the elements of $GF(q)$, the Galois Field. We define

$$P_{ij\dots r} = \begin{cases} 0 & \text{if all the subscripts are } q, \\ +1 & \text{if any, but not all, the subscripts are } q, \\ \chi(x_i + x_j + \dots + x_r) & \text{otherwise,} \end{cases}$$

where each subscript runs from 0 to $q-1$, and

$$\chi(z) = \begin{cases} 0 & z = 0, \\ +1 & \text{if } z \text{ is a square in } GF(q), \\ -1 & \text{otherwise.} \end{cases}$$

Call the q -dimensional cube, $[P_{ij\dots r}]$, P . Using the same reasoning as in [2] we see that P is a q -dimensional weighing matrix of side $q+1$, $q \equiv 1 \pmod{4}$ a prime power, with exactly one zero in each row and with propriety $(\infty, \infty, \dots, \infty)$ for each two-dimensional face of this cube (except one which contains q^2+q ones) in any (and every) direction parallel to an axis is a weighing matrix $W(q+1, q)$.

Let E be the positions where P is zero. Let A and B be anti-amicable Hadamard matrices of dimension n . Then $A \times E + B \times p$ is an q -dimensional orthogonal matrix and we have:

THEOREM 2. *Let $q \equiv 1 \pmod{4}$ be a prime power then there is an almost Hadamard q -dimensional cube of side $2(q+1)$ and propriety $(\infty, \infty, \dots, \infty)$ which has two faces in each dimension mostly ones and every other face an Hadamard matrix.*

4. Higher Dimensional Amicable Hadamard Matrices and Orthogonal Designs

Two Hadamard matrices (or orthogonal designs) X, Y of size r are said to be *amicable* if

$$XY^t = YX^t$$

and *anti-amicable* if

$$XY^t + YX^t = 0.$$

Such terms are defined and used because they can be used to replace variables and form bigger matrices. For example, if X, Y are amicable and

$$XX^t + 3YY^t = 4nI_n$$

then

$$Z = \begin{bmatrix} X & Y & Y & Y \\ -Y & X & Y & -Y \\ -Y & -Y & X & Y \\ -Y & Y & -Y & X \end{bmatrix}$$

satisfies

$$ZZ^t = 4nI_{4n}.$$

If X, Y are anti-amicable and

$$XX^t + 5YY^t = 6nI_n$$

then

$$Z = \begin{bmatrix} X & Y & Y & Y & Y & Y \\ Y & X & Y & -Y & -Y & Y \\ Y & Y & X & Y & -Y & -Y \\ Y & -Y & Y & X & Y & -Y \\ Y & -Y & -Y & Y & X & Y \\ Y & Y & -Y & -Y & Y & X \end{bmatrix}$$

satisfies

$$ZZ^t = 6nI_{6n}.$$

In section 2 we saw that anti-amicable Hadamard matrices of side 2 exist in every dimension 2 and prove most useful. It is obvious therefore, especially because of their extreme usefulness in finding large Hadamard matrices, to look for higher dimensional amicable orthogonal designs and Hadamard matrices. We say, two n -dimensional orthogonal designs $H = (h_{ab\dots c})$ and $G = (g_{ij\dots k})$ (where we will write $h_{ab\dots c} = h_{abv}$ and $g_{ij\dots k} = g_{ijw}$ with v and w representing the other $n-2$ subscripts which could be in any of the n subscript positions as i, j and a, b are only written first for convenience) are *property amicable* of side m if

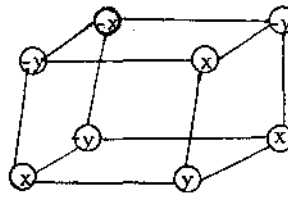
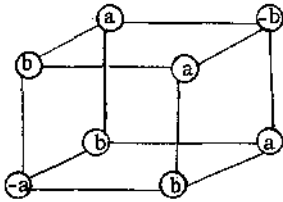
$$\sum_{j=1}^m g_{ijw} h_{kvw} = \sum_{j=1}^m g_{kvw} h_{ijw} \quad (1)$$

for every vector w and every pair i, k . If (1) is true except for some subscripts we will say H and G are *amicable except for those subscripts*.

THEOREM 3. *There are no proper amicable orthogonal designs both of type $(1,1)$ in three dimensions.*

Proof. A complete search shows none exist.

Example 3. X_{ab} and Y_{xy} are amicable except for the top and bottom faces.



In [5] we showed that $H = (h_{ij\dots k})$ defined by

$$h_{ij\dots k} = \begin{cases} (-1)^{\frac{1}{2}w+1} a & w \equiv 0 \pmod{2}, \\ (-1)^{\frac{1}{2}(w-1)} b & w \equiv 1 \pmod{2} \end{cases}$$

was an orthogonal design of dimension n and type $(1,1)^n$, side 2 and propriety $(2,2,\dots,2)$.

5. Amicable n-dimensional orthogonal designs both of type $(1,1)^n$.

Define $[g_{ij\dots sk}]$, with $w = i+j+\dots+k+s$, to have elements which are the commuting variables x, y by

$$g_{ij\dots ks} = \begin{cases} (-1)^{\frac{1}{2}w-s} x & w \text{ even,} \\ (-1)^{\frac{1}{2}(w+1)-s} y & w \text{ odd.} \end{cases}$$

We now establish this is an orthogonal design by considering

$$g_{00x}g_{01x} + g_{10x}g_{11x} \quad (2)$$

and

$$g_{00x}g_{10x} + g_{01x}g_{11x} \quad (3)$$

where x is a constant vector of $n-2$ subscripts. For convenience we put the two varying constant first but, of course, we are really checking them in each of $n(n-1)$ positions. Suppose $v = \text{sum of the subscripts in } x$. There are two main cases for s constant (i.e. s is included in x) and for s one of the first two subscripts in (2) and (3).

First consider $s(\text{constant}) = 0, 1$ with $v \equiv 0, 1, 2, 3 \pmod{4}$:

(i) $v \equiv 0 \pmod{4}$ then (2) and (3) both become

$$(-1)^{-s} x (-1)^{1-s} y + (-1)^{1-s} y (-1)^{1-s} x = 0 ;$$

(ii) $v \equiv 1 \pmod{4}$ then (2) and (3) both become

$$(-1)^{1-s} y (-1)^{1-s} x + (-1)^{1-s} x (-1)^{2-s} y = 0 ;$$

(iii) $v \equiv 2 \pmod{4}$ then (2) and (3) both become

$$(-1)^{1-s} x (-1)^{2-s} y + (-1)^{2-s} y (-1)^{2-s} x = 0 ;$$

(iv) $v \equiv 3 \pmod{4}$ then (2) and (3) both become

$$(-1)^{2-s} y (-1)^{-s} x + (-1)^{-s} x (-1)^{1-s} y = 0 .$$

The case for s one of the first two subscripts is similar.

Hence each face of this n -cube is a 2-dimensional orthogonal design and this a proper n -dimensional orthogonal design of type $(1,1)^n$.

Let $[h_{ij\dots k}]$ and $[g_{ij\dots k}]$ defined above orthogonal designs of type $(1,1)^n$, then amicability may be checked by considering whether

$$h_{00x} g_{10x} + h_{01x} g_{11x} = h_{10x} g_{00x} + h_{11x} g_{01x} \quad (4)$$

where x is a constant vector of $n-2$ subscripts and we have written the two varying constants first (but we are really checking them in each of $n(n-1)$ positions). Let $v = \text{sum of the subscripts in } x$. We have two main cases for s constant (i.e. s included in x) and s one of the first two subscripts in (4).

We first consider $s(\text{constant}) = 0$ or 1 with $v \equiv 0, 1, 2, 3 \pmod{4}$

We only show the case for $v \equiv 0 \pmod{4}$ as the other cases are similar:

the left hand side of (4) becomes

$$-a(-1)^{1-s} x + b(-1)^{1-s} y = (-1)^{1-s} (-ax+by) \quad (5)$$

while the right hand side side of (4) becomes

$$b(-1)^{-s} y + a(-1)^{1-s} x = (-1)^{1-s} (ax-by) . \quad (6)$$

So (4) is never satisfied for constant s and we never have amicability in this case.

On the other hand, for side 2 and s not constant, without loss of generality we can express equation (1) as requiring

$$\begin{aligned} &g_{0j\dots s}h_{0j\dots(s+1)} + g_{1j\dots s}h_{1j\dots(s+1)} \\ &= g_{0j\dots(s+1)}h_{0j\dots s} + g_{1j\dots(s+1)}h_{1j\dots s} \end{aligned} \quad (7)$$

We illustrate this case for $v \equiv 0 \pmod{4}$ only as the other cases are similar and (7) becomes

$$(-1)^{-s}x_b + (-1)^{1-s}y_a = (-1)^{1-s-1}y_{(-a)} + (-1)^{1-s-1}x_b .$$

So we do have amicability when s is not constant.

LEMMA 4. *There exist n -dimensional orthogonal designs of type $(1,1)^n$ and side 2 which are amicable except when one distinguished coordinate (the last) is constant.*

On the other hand we see that if the variables are replaced by $+1$ (or -1) then (5) and (6) are satisfied and so we have

LEMMA 5. *There exist amicable n -dimensional Hadamard matrices of side 2 and both of propriety $(2,2,\dots,2)$.*

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