Vol 829, Lecture Notes in Mathematics, Springer-Verlag, Berlin - Heidelberg - New York, 1980

HIGHER DIMENSIONAL ORTHOGONAL DESIGNS AND HADAMARD MATRICES

JENNIFER SEBERRY

We construct n-dimensional orthogonal designs of type $(1,1)^n$, side 2 and propriety $(2,2,\ldots,2)$. These are then used to show that orthogonal designs of type $(2^t,2^t)^n$, side 2^{t+1} and propriety $(2,2,\ldots,2)$ exist.

1. INTRODUCTION

In [2] it is pointed out that it is possible to define orthogonality for higher dimensional matrices in many ways.

Intuitively we see that each two-dimensional matrix withing the n-dimensional matrix could have orthogonal row vectors (we call this propriety $(2,2,\ldots,2)$); or perhaps each pair of two-dimensional layers

$$A^j = \begin{pmatrix} \underset{\sim}{a}_1 \\ \underset{\sim}{a}_2 \\ \vdots \\ \underset{\sim}{a}_t \end{pmatrix} \quad \text{and} \quad B^j = \begin{pmatrix} \underset{\sim}{b}_1 \\ \underset{\sim}{b}_2 \\ \vdots \\ \underset{\sim}{b}_t \end{pmatrix}$$

could have $A \cdot B = tr(AB^T) = \underset{\sim}{a}_1 \cdot \underset{\sim}{b}_1 + \underset{\sim}{a}_2 \cdot \underset{\sim}{b}_2 + \ldots + \underset{\sim}{a}_t \cdot \underset{\sim}{b}_t = 0$ (note if the row vectors in this direction had been orthogonal we would have had $\underset{\sim}{a}_i \cdot \underset{\sim}{b}_i = 0$ for each i) (we call this propriety $(\ldots,3,\ldots)$); or perhaps each pair of three-dimensional layers

$$\alpha = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^t \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} B^1 \\ B^2 \\ \vdots \\ B^t \end{pmatrix}$$

could have $\alpha \cdot \beta = A^1 \cdot B^1 + \ldots + A^t \cdot B^t = 0$ (note that if the 2-dimensional matrices had been orthogonal we would have had $A^j \cdot B^j = 0$ for each j); and so on.

We say an n-dimensional matrix is orthogonal of *propriety* $(d_1,\ldots,d_n)$ with $2 \le d_i \le n$ where d_i indicates that in the i^{th} direction (i.e., the i^{th} coordinate) the $d_i-1^{st}, d_i{}^{th}, d_i+1^{st},\ldots,(n-1)^{st}$ dimensional layers are orthogonal but the d_i-2^{nd} layer is not orthogonal. $d_i = \infty$ means not even the $(n-1)^{st}$ layers are orthogonal.

The Paley cube of size $(q+1)^n$ constructed in [2] for $q \equiv 3 \pmod 4$ a prime power has propriety $(\infty,\infty,\ldots,\infty)$ but if the 2-dimensional layer of all ones is removed in one direction the remaining n-dimensional matrix has all 2-dimensional layers in that direction orthogonal.

An *n-cube orthogonal design*, $D = [d_{ijk\ldots}]$, of propriety $(d_1,d_2,\ldots,d_n)$, side d and type $(s_1,s_2,\ldots,s_t)^n$ on the commuting variables $x_1,x_2,\ldots,x_t$ has entries from the set $\{0,\pm x_1,\ldots,\pm x_t\}$ where $\pm x_i$ occurs s_i times in each row and column of each 2-dimensional layer and in which each e_j-dimensional layer, $d_i - 1 \leq e_j \leq n-1$, in the i^{th} direction is orthogonal.

Shlichta [3] found n-dimensional Hadamard matrices of size $(2^t)^n$ and propriety $(2,2,\ldots,2)$. In [2] the concept of higher dimensional m-suitable matrices was introduced to show that if t is the side of 4 Williamson matrices there is a 3-dimensional Hadamard matrix of size $(4t)^3$ and propriety $(2,2,2)$.

2. n-DIMENSIONAL ORTHOGONAL DESIGNS OF TYPE $(1,1)^n$ AND SIDE 2

Theorem. *There exists an* n-*dimensional orthogonal design of type* $(1,1)^n$, *side* 2 *and propriety* $(2,2,\ldots,2)$.

Proof. Let a and b be commuting variables and $[h_{ijk\ldots}]$ be the orthogonal design. Define $w = i+j+k+\ldots+v$, the weight of the subscripts which can only assume the values 0 and 1 for side 2.

Now define

$$h_{ijk\ldots v} = \begin{cases} (-1)^{\frac{1}{2}w+1}a & w \equiv 0 \pmod 2, \\ (-1)^{\frac{1}{2}w-1}b & w \equiv 1 \pmod 2. \end{cases}$$

In order to check the orthogonality we consider

$$h_{00x}h_{01x} + h_{10x}h_{11x} \tag{*}$$

and

$$h_{00x}h_{10x} + h_{01x}h_{11x} \tag{**}$$

where x is a constant vector of n-2 subscripts. For convenience we put the two varying constants first but of course we are really checking then in each of n(n-1) positions. Suppose v = sum of the subscripts in x. Then we have four cases:

(1) $v \equiv 0 \pmod 4$ then (*) and (**) both become

$$-ab + ba = 0;$$

(2) $v \equiv 1 \pmod 4$ then (*) and (**) both become

$$ba + a(-b) = 0;$$

(3) $v \equiv 2 \pmod 4$ then (*) and (**) both become

$$a(-b) + (-b)(-a) = 0;$$

(4) $v \equiv 3 \pmod 4$ then (*) and (**) both become

$$(-b)(-a) + (-a)b = 0 .$$

Hence each face of this orthogonal n-cube is a 2-dimensional orthogonal design and so we have a proper n-dimensional orthogonal design of type $(1,1)^n$.

Corollary. *There exist* n-*dimensional orthogonal designs of types* $(2^t,2^t)^n$, *side* 2^{t+1} *and propriety* $(2,2,\ldots,2)$.

Proof. Take the Kronecker product of Shlichta's proper n-dimensional Hadamard matrices of order $(2^t)^n$ with the orthogonal design established in the theorem.

This is illustrated in the Figure.

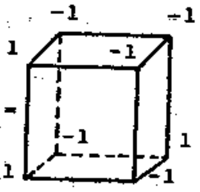
REFERENCES

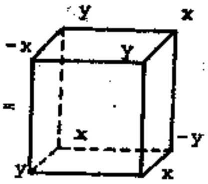
[1] A.V. Geramita and Jennifer Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices* (Marcel Dekker, New York, 1979).

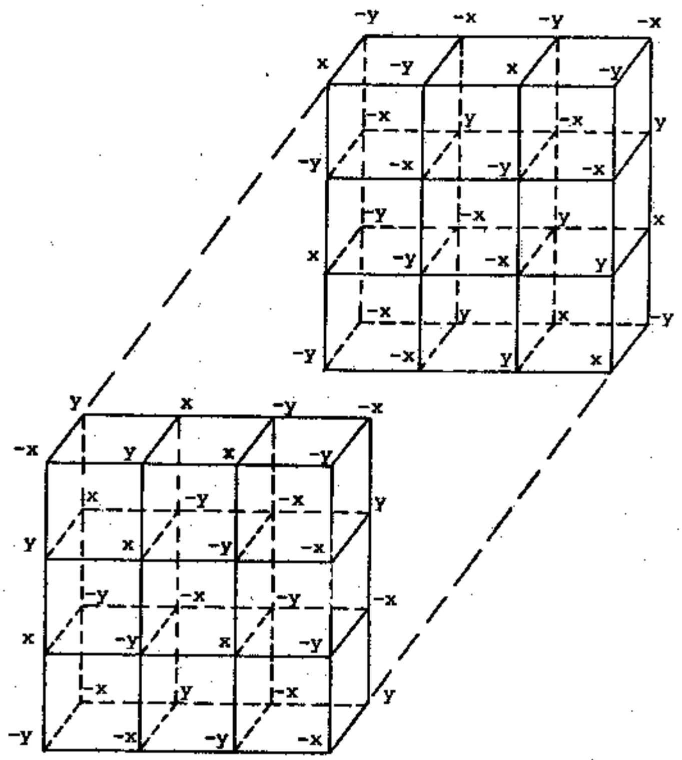
[2] Joseph Hammer and Jennifer Seberry, Higher dimensional orthogonal designs and applications, *IEEE Trans. Inform. Theory*, (to appear).

[3] P.J. Shlichta, Higher dimensional Hadamard matrices, *IEEE Trans. Inform. Theory*, (to appear).

Department of Mathematics
University of Sydney
Sydney

$A_1 =$ 

$A_2 =$ 



$A_1 \otimes A_2$.