ALL DBIBDs WITH BLOCK SIZE FOUR EXIST

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ABSTRACT. A directed balanced incomplete block design with parameters (v,b,r,k,λ^*), is a balanced incomplete block design with parameters $(v,b,r,k,2\lambda^*)$, in which the blocks are regarded as ordered k-tuples and in which each ordered pair of elements occurs in λ^* blocks. By generalizing results of Hanani, we show that the necessary conditions for the existence of these designs, when $k = 4$, are sufficient.

A directed balanced incomplete block design (DBIBD), with parameters (v,b,r,k,λ^*), is a balanced incomplete block design (BIBD), with parameters $(v,b,r,k,2\lambda^*)$, in which the blocks are regarded as ordered k-tuples and in which each ordered pair of elements occurs in λ^* blocks. Thus given the block (a,b,c,d) we will say the six ordered pairs $(a,b),(a,c),(a,d),(b,c)(b,d)$, and (c,d) occur in it. These designs are, in fact, designs on a directed graph and as such have been studied by several authors (see Harary, Wallis and Heinrich [2] and the references therein).

When $k = 3$, Seberry and Skillicorn [3] have shown that the necessary conditions for the existence of a DBIBD are sufficient. In this paper, by generalizing results of Hanani [1], we show that for $k = 4$ the necessary conditions are sufficient.

A directed group divisible design (DGD) is a group divisible design (GD) in which each ordered pair formed from elements of different groups occurs in the same number of blocks (for the definition of a group divisible design see Hanani [1]).

The existence of a $\text{BIBD}(v,b,r,k,\lambda)$ implies the existence of a $\text{DBIBD}(v,2b,2r,k,\lambda)$. The directed design is obtained by writing each block of the BIBD twice - once in some order and once in the reverse order. The same is, of course, true for GD designs.

As in Hanani [1], $B(k,\lambda)$ is the set of all v such that a $\text{BIBD}(v,b,r,k,\lambda)$ exists, $DB(k,\lambda^*)$ is the set of all v such that a $\text{DBIBD}(v,b,r,k,\lambda^*)$ exists, $GD(k,\lambda,m)$ is the set of all v such that a

UTILITAS MATHEMATICA Vol. 18 (1980), pp. 27-34.

$GD[k,\lambda,m;v]$ exists and $DGD(k,\lambda^*,m)$ is the set of all v such that a $DGD[k,\lambda^*,m;v]$ exists.

As a consequence of the remarks above and the results in Hanani [1] we have the following.

LEMMA 1 (cf. [1, Lemma 4.10]).

$$\{12,15\} \subset DGD(4,1^*,3) .$$

LEMMA 2 (cf. [1, Lemma 5.11]). *If* $v \equiv 1$ *or* 4 (mod 12), *then*

$$v \subset DB(4,1^*) .$$

LEMMA 3 (cf. [1, Lemma 5.13]). *If* $v \equiv 0$ *or* 1 (mod 4), *then*

$$v \in DB(4,3^*) .$$

LEMMA 4 (cf. [1, Lemma 2.26]). *If* $n \in GD(S,1,R)$, $mR+1 \subset DB(k,\lambda^*)$ *and* $mS \subset DGD(k,\lambda^*,m)$, *then*

$$mn+1 \in DB(k,\lambda^*) .$$

Proof. Let the points of the design be $\{1,2,\ldots,m\} \times \{1,2,\ldots,n\} \cup \{\infty\} = (X \times Y) \cup \{\infty\}$. Using the elements of Y, construct a $GD[S,1,R;n]$. For each group G of this design, $|G| \in R$ and so we may construct a $DBIBD(m|G|+1,b,r,k,\lambda^*)$ on $(X \times G) \cup \{\infty\}$. For each block B of the GD, construct a $DGD[k,\lambda^*,m;m|B|]$ on $X \times B$.

We now prove several lemmas which are used to construct a few initial designs from which the existence of all others may be deduced.

LEMMA 5 (cf. [1, Lemma 2.11].

$$DGD(k,\lambda^*,k-1) + 1 \subset DB(k,\lambda^*) .$$

Proof. Adjoin a fixed additional point to each group of the DGD design. Then the additional blocks are the groups with fixed additional point written in some order, and the reverse of that order, each repeated λ^* times.

LEMMA 6 (cf. [1, Lemma 2.16]). *If* $n \in DB(K,\lambda^*)$ *and* $mK \subset GD(k,\lambda,m)$, *then*

$$mn \in DGD(k,\lambda\lambda^*,m) .$$

Proof. Let the n groups of the $DGD(k,\lambda\lambda^*,m)$ be $G_1,G_2,\ldots,G_n$ and use the set of symbols $\{G_1,G_2,\ldots,G_n\}$ to construct a $DBIBD(n,b,r,k,\lambda^*)$. For each block of this design use the groups in that block to construct a $GD(k,\lambda,m)$ where, if G_i appears before G_j in the block, then elements from G_i appear before elements from G_j in the $GD(k,\lambda,m)$.

COROLLARY 6.1. *If* $n \in DB(K,\lambda^*)$ *and* $(k-1)K+1 \subset B(k,1)$ *then* $(k-1)n + 1 \subset DB(k,\lambda^*)$.

Proof. Use Lemmas 5 and 6, and Lemma 2.12 of Hanani [1].

LEMMA 7 (cf. [1, Lemma 4.1]. *Let* q *be a prime power, where* $q = 2f+1, f$ *odd. Then* $q \in DB(4,3^*)$.

Proof. Let $C_0 = \{1,x^f\}$, where x is a generator of $GF(q)^*$ (the multiplicative group of $GF(q)$) and let $C_i = x^iC_0 = \{x^i,x^{f+i}\}$, $0 \le i \le f-1$. We will consider these sets as ordered pairs, writing D_i for the pair $\{x^{f+i},x^i\}$. It is then straightforward to check that the required blocks are $C_0 \cup D_1, C_1 \cup D_2, \ldots, C_{f-1} \cup D_0$ (regarding these as ordered 4-tuples).

LEMMA 8 (cf. [1, Lemma 5.12]). *If* $v \equiv 1 \pmod 3$ *then*

$$v \in DB(4,1^*) .$$

Proof. Let $v = 3n+1$, where n is a positive integer. By Hanani [1, Lemma 5.9], $n \in GD(\{4,5\},1,M_4')$, where $M_4' = \{1,2,\ldots,15,26,27\}$. By Lemmas 1 and 4 it suffices to show that $3M_4'+1 \subset DB(4,1^*)$. For $n \in M_4'$, $n \equiv 0$ or $1 \pmod 4$, the result follows from Lemma 2; for the remaining values of v a solution is given in Table I.

LEMMA 9 (cf. [1, Lemma 5.14]). *For every integer* $v \geq 4$,

$$v \in DB(4,3^*) .$$

Proof. By Hanani [1, Lemma 5.10]), it suffices to show that $v \in DB(4,3^*)$ for all $v \in \{4,5,\ldots,12,14,15,18,19,23,27\}$. If $v \equiv 1 \pmod 3$ then Lemma 8 gives the result; if $v \equiv 0$ or $1 \pmod 4$, then Lemma 3 gives the result; for the remaining values of v a solution is given in Table II.

THEOREM. *Let* λ^* *and* $v \geq 4$ *be given positive integers. A necessary and sufficient condition for the existence of a DBIBD*$(v,b,r,4,\lambda^*)$ *is that*

$$\lambda^*(v-1) \equiv 0 \pmod 3 \quad \textit{and} \quad \lambda^* v(v-1) \equiv 0 \pmod 6 .$$

Proof. That these conditions are necessary follows from the usual counting arguments for BIBDs. We need only consider values of λ^* which are factors of 3, as if $\lambda_1^* | \lambda_2^*$ then $DB(k,\lambda_1^*) \subset DB(k,\lambda_2^*)$. Thus we have the following cases:

$\lambda^* = 1$	$v \equiv 1 \pmod 3$
$\lambda^* = 3$	all v.

In Lemmas 8 and 9 we have established the existence of the required designs.

Table I.

v	DBIBD(v,b,r,4,1*)
7	Initial block (6,0,3,5) developed mod 7 (in [3]).
10	Form the residual of the (16,6,1*) constructed from the initial block (a,b,c,d,ab,cd).
19	Initial blocks to be developed mod 19: (0,3,12,1); (13,1,5,0); (4,9,6,0).
22	Use Corollary 6.1 and 7 ∈ DB(4,1*) from above.
31	Initial blocks to be developed mod 31: (0,1,8,11); (14,11,0,2); (7,13,5,0); (0,15,5,9); (1,17,0,13).
34	Use Corollary 6.1 with n = 11, K = 5, λ* = 1, and k = 4. (The initial block for the DBIBD(11,11,5,5,1*) is (3,5,1,4,9).)
43[a]	Initial blocks to be developed mod 43: (1,0,6,36); (26,0,33,27); (35,0,13,38); (31,0,41,14); (3,0,22,18); (23,9,0,11); (39,19,0,28).
46[b]	We show 45 ∈ DGD(4,1*,3) and apply Lemma 5. Initial blocks to be developed mod (3;3,5): ((0;0,0), (0;1,0), (1;2,2), (1;2,3)); ((1;2,4),(0;1,0), (0;0,0), (1;2,1)); ((0;2,1), (2;1,3), (1;2,0), (0;1,0)); ((0;2,4), (0;1,0), (2;1,2), (1;2,0)); ((0;2,0), (0;1,2), (2;2,1), (1;1,0)); ((1;1,0), (0;2,0), (2;2,4), (0;1,3)); ((0;0,1), (0;0,4), (0;1,3), (0;1,2)).
79[a]	Initial blocks to be developed mod 79: (1,0,23,55); (18,0,42,19); (54,0,57,47); (8,0,45,26); (3,69,0,7); (6,0,59,14); (4,0,13,62); (29,0,15,35); (9,49,0,21); (2,0,46,31); (12,0,39,28); (36,0,5,38); (27,68,0,63).

82[a,b] We show $81 \in DGD(4,1^*,3)$ and apply Lemma 5. Initial blocks to be developed mod (3;27):

$((0;1),(0;2),(1;x),(1;2x))$,

$((1;2x^2+x+1),(1;x^2+2x+2),(0;2x^2+2x+2),(0;x^2+x+1))$,

$((1;x^2+1),(1;2x^2+2),(0;x^2+2x+2),(0;2x^2+x+1))$,

$((0;x^2+2),(0;2x^2+1),(1;2),(1;1))$,

$((0;2x),(0;x),(1;x^2),(1;2x^2))$,

$((1;x^2+x+1),(1;2x^2+2x+2),(0;x^2+2x+1),(0;2x^2+x+2))$,

$((1;2x+2),(1;x+1),(0;2x^2+2),(0;x^2+1))$,

$((0;x^2+x+2),(0;2x^2+2x+1),(1;2x^2+1),(1:x^2+2))$,

$((0;2x^2),(0;x^2),(1;x+2),(1;2x+1))$,

$((1;2x^2+x+2),(0;x^2+2x),(1;x^2+2x+1),(0;2x^2+x))$,

$((1;2x^2+2x),(1;x^2+x),(0;x+1),(0;2x+2))$,

$((1;2x^2+2x+1),(0;x^2+x),(1;x^2+x+2),(0;2x^2+2x))$,

$((0;2x+1),(0;x+2),(1;2x^2+x),(1;x^2+2x))$.

a) Found by William H. Wilson using a backtrack algorithm written in Pascal.

b) We use the notation of Hanani [1] except that the numbers are elements of the field instead of exponents of elements of the field.

Table II.

v	DBIBD(v,b,r,4,3*)
6	Blocks: (1,2,3,4); (5,3,2,1); (1,2,3,6); (1,2,4,5); (6,4,2,1); (6,5,2,1); (5,4,3,1); (6,4,3,1); (1,3,5,6); (1,4,5,6); (3,2,4,5); (6,2,3,5); (5,4,2,6); (6,5,4,3); (3,4,2,6).
11	Use Lemma 7.
14	Initial blocks to be developed mod 13 (∞ is a fixed element): (0,1,3,9); (9,3,1,0); (0,1,3,9); (9,3,1,0); (1,0,3,9); (9,3,1,∞); (∞,5,2,6).
15	Initial blocks to be developed mod (3,5): ((1,1), (1,4), (2,2), (2,3)); ((2,3), (2,2), (1,4), (1,1)); ((1,2), (1,3), (2,1), (2,4)); ((2,4), (2,1), (1,3), (1,2)); ((0,0), (0,1), (1,0), (2,0)); ((2,0), (1,0), (0,0), (0,2)); ((2,3), (1,1), (1,4), (2,2)).
18	Initial blocks to be developed mod 17 (∞ is a fixed element): (1,4,13,16); (16,13,4,1); (3,5,12,14); (14,12,5,3); (2,8,9,15); (15,9,8,2); (6,11,10,7); (∞,16,15,11); (9,10,13,∞).
23	Use Lemma 7.
27	Use Lemma 7.

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Received February 20, 1980.