Example of construct in Seberry ~~#19 in your text~~

Then

$$\begin{matrix} E & A \\ B & D \end{matrix}$$

is the required $GH\left[p^r(p^r-1), C_{p^r}\right]$.

Before we justify this algorithm we look at two examples of its use.

<u>Example of construction of $GH(20, Z_5)$.</u>

The multiplication table

$$\begin{array}{c|ccccc}
 & e & x & x^2 & x^3 & x^4 \\ \hline
e & e & x & x^2 & x^3 & x^4 \\
x & x & x^2 & x^3 & x^4 & e \\
x^2 & x^2 & x^3 & x^4 & e & x \\
x^3 & x^3 & x^4 & e & x & x^2 \\
x^4 & x^4 & e & x & x^2 & x^3
\end{array}
\quad \text{has core} \quad C = \begin{matrix} x^2 & x^3 & x^4 & e \\ x^3 & x^4 & e & x \\ x^4 & e & x & x^2 \\ e & x & x^2 & x^4 \end{matrix}$$

The generalized Hadamard matrix of order 5

$$\begin{matrix}
e & e & e & e & e \\
e & x & x^2 & x^3 & x^4 \\
e & x^2 & x^4 & x & x^3 \\
e & x^3 & x & x^4 & x^2 \\
e & x^4 & x^3 & x^2 & x
\end{matrix}
\quad \text{has core} \quad K = \begin{matrix} x & x^2 & x^3 & x^4 \\ x^2 & x^4 & x & x^3 \\ x^3 & x & x^4 & x^2 \\ x^4 & x^3 & x^2 & x \end{matrix} .$$

5 elts

The matrices

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{bmatrix} \quad \text{and} \quad ML = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

represent ε, a, b, ab where $a^2 = b^2 = \varepsilon$. The generalized Hadamard matrix $GH(4, Z_2 \times Z_2)$ is

$$\begin{array}{cccc} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & a & b & ab. \\ \varepsilon & b & ab & a \\ \varepsilon & ab & a & b \end{array}$$

Now $D =$

$$\begin{array}{cccc} x^2K & x^3K & x^4K & eK \\ x^3K & x^4KM & eKL & xKLM \\ x^4K & eKL & xKLM & x^2KM \\ eK & xKLM & x^2KM & x^4KL \end{array}$$

So required matrix is

$e = 0$
$x^i \to i$

$$\begin{array}{cccccccc} e & e & e & e & \underset{\sim}{e} & \underset{\sim}{e} & \underset{\sim}{e} & \underset{\sim}{e} \\ e & e & e & e & \underset{\sim}{x}^2 & \underset{\sim}{x}^4 & \underset{\sim}{x} & \underset{\sim}{x}^3 \\ e & e & e & e & \underset{\sim}{x}^3 & \underset{\sim}{x} & \underset{\sim}{x}^4 & \underset{\sim}{x}^2 \\ e & e & e & e & \underset{\sim}{x}^4 & \underset{\sim}{x}^3 & \underset{\sim}{x}^2 & \underset{\sim}{x} \\ \underset{\sim}{e}' & \underset{\sim}{x}^{2'} & \underset{\sim}{x}^{3'} & \underset{\sim}{x}^4 & & & & \\ \underset{\sim}{e}' & \underset{\sim}{x}^{4'} & \underset{\sim}{x}' & \underset{\sim}{x}^{3'} & & D & & \\ \underset{\sim}{e}' & \underset{\sim}{x}' & \underset{\sim}{x}^{4'} & \underset{\sim}{x}^{2'} & & & & \\ \underset{\sim}{e}' & \underset{\sim}{x}^{3'} & \underset{\sim}{x}^{2'} & \underset{\sim}{x}' & & & & \end{array}$$

where $\underset{\sim}{x} = [x\ x\ x\ x]$ and $\underset{\sim}{x}' = \begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix}$.

Example of construction of GH(12, $Z_2 \times Z_2$).

	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

has core $C = \begin{matrix} e & ab & b \\ ab & e & a \\ b & a & e \end{matrix}$.

The generalized Hadamard matrix of order 4:

$$\begin{matrix} e & e & e & e \\ e & a & b & ab \\ e & b & ab & a \\ e & ab & a & b \end{matrix}$$

has core $K = \begin{matrix} a & b & ab \\ b & ab & a \\ ab & a & b \end{matrix}$.

4 elts.

Let I, T, T^2 of order 3 be a matrix representation of $\varepsilon, \omega, \omega^2$ where ω is a cube root of unity, then

$$W = \begin{matrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \omega & \omega^2 \\ \varepsilon & \omega^2 & \omega \end{matrix}$$

is a generalized Hadamard matrix of order 3.

Now define

$$C * W = \begin{matrix} e\varepsilon & ab\varepsilon & b\varepsilon \\ ab\varepsilon & e\omega & a\omega^2 \\ b\varepsilon & a\omega^2 & e\omega \end{matrix}$$

$\varepsilon \to K$
$e\omega^i = KT^i$

$e\omega$ KT

$abK = \begin{matrix} b & a & e \\ a & e & b \\ e & b & a \end{matrix}$

a a ab a b
e a ab b
e ab b

e ab b

$$\text{and} \qquad D = \begin{array}{ccc} eK & abK & bK \\ abK & eKT & aKT^2 \\ bK & aKT^2 & eKT \end{array}$$

$\omega^i \to T^i$

$I^0 = \begin{matrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{matrix} \quad T^1 = \begin{matrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{matrix}$

$T^2 = \begin{matrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{matrix}$

The following is the required matrix:

$$\begin{array}{ccc|ccc}
e & e & e & \underset{\sim}{e} & \underset{\sim}{e} & \underset{\sim}{e} \\
e & e & e & \underset{\sim}{b} & \underset{\sim}{ab} & \underset{\sim}{a} \\
e & e & e & \underset{\sim}{ab} & \underset{\sim}{a} & \underset{\sim}{b} \\
\underset{\sim}{e}' & \underset{\sim}{b}' & \underset{\sim}{ab}' & eK & abK & bK \\
\underset{\sim}{e}' & \underset{\sim}{ab}' & \underset{\sim}{a}' & abK & eKT & aKT^2 \\
\underset{\sim}{e}' & \underset{\sim}{a}' & \underset{\sim}{b}' & bK & aKT^2 & eKT
\end{array}$$

where $\underset{\sim}{a} = [a\ a\ a]$ and $\underset{\sim}{a}' = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$. Explicitly

$$G = \begin{array}{ccc|ccc|ccc|ccc}
e & e & e & e & e & e & e & e & e & e & e & e \\
e & e & e & b & b & b & ab & ab & ab & a & a & a \\
e & e & e & ab & ab & ab & a & a & a & b & b & b \\
\hline
e & ab & b & a & b & ab & b & a & e & ab & e & a \\
e & ab & b & ab & a & b & e & b & a & a & ab & e \\
e & ab & b & b & ab & a & a & e & b & e & a & ab \\
\hline
e & b & a & ab & e & a & b & e & ab & b & ab & a \\
e & b & a & a & ab & e & ab & b & e & a & b & ab \\
e & b & a & e & a & ab & e & ab & b & ab & a & b \\
\hline
e & a & ab & b & a & e & b & ab & a & b & e & ab \\
e & a & ab & e & b & a & a & b & ab & ab & b & e \\
e & a & ab & a & e & b & ab & a & b & e & ab & b
\end{array}$$

eK abK bK abK bK

is a $GH(12, Z_2 \times Z_2)$.