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Note

All Directed BIBDs with $k = 3$ Exist

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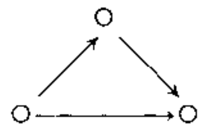
A directed BIBD with parameters (v, b, r, k, λ^*) is a BIBD with parameters $(v, b, r, k, 2\lambda^*)$ in which each ordered pair of varieties occurs together in exactly λ^* blocks. It is shown that $\lambda^* v(v-1) \equiv 0 \pmod 3$ is a necessary and sufficient condition for the existence of a directed (v, b, r, k, λ^*) BIBD with $k = 3$.

A directed BIBD (DBIBD) with parameters (v, b, r, k, λ^*) is a BIBD with parameters $(v, b, r, k, 2\lambda^*)$ in which each ordered pair of varieties occurs together in precisely λ^* blocks. A pair a, b is said to occur in a block if a is written to the left of b. Thus, the block abc contains the pairs a, b; a, c; and b, c. Such a design is a block design on the directed graph.



Some such designs arise as modifications of those generated by difference sets. If a BIBD with parameters (v, b, r, k, λ) exists, with λ even, and if it is generated by a difference set, then if the difference set can be rearranged so that each difference appears $\lambda/2$ times when the differences are taken left to right, the resulting design will have the directed property.

For example, the (7, 7, 4, 4, 2) BIBD is generated by the difference set 0 3

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5 6 (mod 7). When this block is rearranged as 6 0 3 5 (mod 7), the generated design is the DBIBD with parameters (7, 7, 4, 4, 1*).

It is also clear that the derived and residual designs of a directed design will be directed, giving another construction technique.

We now describe two results which enable recursive construction of large designs with $k = 3$. Both results were proved indirectly by Hung and Mendelsohn [2] but we provide direct proofs.

LEMMA A. *If there exists a* DBIBD *with parameters* $(v, b, r, 3, \lambda^*)$ *then there exists a* DBIBD *with parameters* $(2v + 1, 2v(2v + 1)\lambda^*/3, 2v\lambda^*, 3, \lambda^*)$.

Proof. Take the triples of the design with parameters $(v, b, r, 3, \lambda^*)$ and to them adjoin all triple of the form

$$(v + i, j, v + i + j) \qquad (i = 1,..., v + 1; j = 1,..., v),$$

where elements in the third position are reduced, when greater than $2v + 1$, by subtracting $v + 1$, so as to remain in the set $\{v + 1,..., 2v + 1\}$. For fixed j, as i varies, the resulting set of triples cover all pairs of the forms (x, j), (j, x) and $(x, x + j)$, where x and $x + j$ come from $\{v + 1,..., 2v + 1\}$ (by reduction, if necessary). These blocks should be repeated λ^* times.

For example, one extends the directed triple system on four elements (123, 214, 342, 431) by adding the blocks

516,	617,	718,	819,	915
527,	628,	729,	825,	926
538,	639,	735,	836,	937
549,	645,	746,	847,	948

LEMMA B. *If there exists a* DBIBD *with parameters* $(v, b, r, 3, \lambda^*)$ *then there exists a* DBIBD *with parameters* $(2v + 4, (2v + 4)(2v + 3)\lambda^*/3, \lambda^*(2v + 3), 3, \lambda^*)$.

Proof. Take the triples of the design with parameters (v, b, r, k, λ^*) and to them adjoin all triples of the forms

$$(v + i, j, v + i + j) \qquad (i = 1,..., v + 4; j = 1,..., v),$$

where elements in the third position are reduced, as before, so as to remain in the set $\{v + 1,..., 2v + 4\}$, and

$$(v + i, 2v + 2 + i, 2v + 1 + i) \qquad (i = 1,..., v + 4),$$

where, again, elements in the second and third position are reduced to remain in $\{v + 1,..., 2v + 4\}$.

For fixed j, in the first and third positions of triples of the first type, all elements from $\{v+1,\ldots, 2v+4\}$ appear, ensuring that all pairs of the form ab and ba, when $a \in \{1,\ldots, v\}$ and $b \in \{v+1,\ldots, 2v+1\}$, appear.

Triples of this type also account for pairs of elements, chosen from $\{v+1,\ldots, 2v+4\}$, whose elements are less than v apart (possibly allowing for wraparound).

Triples of the second type account for pairs of numbers from $\{v+1,\ldots, 2v+4\}$ which are $v+1$ and $v+2$ apart as well as those which are -1 (which corresponds to $v+3$) apart.

Thus all pairs occur and, by repeating the new blocks λ^* times, a DBIBD is obtained.

An example will illustrate the procedure. We show the extension of the (4, 4, 3, 3, 1*) DBIBD to a (12, 44, 11, 3, 1*) design. Add te blocks of the first type

5, 1, 6;	6, 1, 7;	7, 1, 8;	8, 1, 9;	9, 1, 10;	10, 1, 11;	11, 1, 12;	12, 1, 5
5, 2, 7;	6, 2, 8;	7, 2, 9;	8, 2, 10;	9, 2, 11;	10, 2, 12;	11, 2, 5;	12, 2, 6
5, 3, 8;	6, 3, 9;	7, 3, 10;	8, 3, 11;	9, 3, 12;	10, 3, 5;	11, 3, 6;	12, 3, 7
5, 4, 9;	6, 4, 10;	7, 4, 11;	8, 4, 12;	9, 4, 5;	10, 4, 6;	11, 4, 7;	12, 5, 8

where each row consists of blocks with fixed j, and each column, blocks with fixed i. The blocks of the second type are:

5, 11, 10	9, 7, 6
6, 12, 11	10, 8, 7
7, 5, 12	11, 9, 8
8, 6, 5	12, 10, 9

a difference set.

2. Sufficiency of the Necessary Conditions

A DBIBD with parameters (v, b, r, k, λ^*) is (by disregarding the order) a BIBD with parameters $(v, b, r, k, 2\lambda^*)$ and the necessary conditions for existence of such a desing give, for $k = 3$, the equations

$$2\lambda^*(v-1) = 2r,$$
$$3b = vr.$$

Thus we may write the parameters of the DBIBD as $(v, v\lambda^*(v-1)/3, \lambda^*(v-1), 3, \lambda^*)$. Since the number of blocks must be integral, there are three possibilities: $3 \mid \lambda^*$, $3 \mid v$ or $3 \mid v-1$ or, more succinctly, $\lambda^* v(v-1) \equiv 0 \pmod 3$.

Hung and Mendelsohn [2] have shown that all directed designs with parameters $(v, b, r, 3, 1^*)$ exist. Hence, when $3 \nmid \lambda^*$, all DBIBDs with parameters $(v, v\lambda^*(v-1)/3, (v-1)\lambda^*, 3, \lambda^*)$ can be obtained by taking λ^* copies of the DBIBD with parameters $(v, v(v-1)/3, v-1, 3, 1^*)$.

It remains to consider te case where $3 \mid \lambda^*$. Suppose, further, that $\lambda^* = 6t$. By the remark in the previous paragraph all DBIBDs with parameters $(v, 2v(v-1)/3, 2(v-1), 3, 2^*)$ exist and so, by taking $\lambda^*/2$ copies, we obtain all DBIBDs with parameters $(v, \lambda^* v(v-1)/3, \lambda^*(v-1), 3, \lambda^*)$.

If $\lambda^* = 3t$ (that is, t is odd) we can obtain the required result by taking $\lambda^*/3$ copies of the directed BIBD with parameters $(v, v(v-1), 3(v-1), 3, 3^*)$, if they exist.

It remains to show that all such directed designs do, in fact, exist. If $v \equiv 0$ or $1 \pmod 3$ then we can easily obtain the design by taking three copies of the DBIBD with parameters $(v, v(v-1)/3, v-1, 3, 1^*)$. The case $v \equiv 2 \pmod 3$ remains to be investigated.

First, we give some directed BIBDs with parameters $(v, v(v-1), 3(v-1), 3, 3^*)$

$v = 3$	$(3, 6, 6, 3, 3^*)$ design 123 123 123 321 321 321.
$v = 4$	$(4, 19, 9, 3, 3^*)$ design 123 214 342 431 (each block repeated three times).
$v = 5$	$(5, 20, 12, 3, 3^*)$ design 123 124 125 134 135 145 234 235 245 345 321 421 521 431 531 541 432 532 542 543.
$v = 6$	$(6, 30, 15, 3, 3^*)$ design 215 320 423 341 524 035 450 102 513 014 (each block repeated three times).
$v = 7$	$(7, 42, 18, 3, 3^*)$ Difference set (4, 2, 1)(1, 2, 4) mod 7 with each initial block taken three times.
$v = 8$	$(8, 56, 21, 3, 3^*)$ Difference set (0, 4, 1) (0, 4, 2) (0, 4, 3) (0, 2, 1) (1, 2, 0) (0, 5, 3) (3, 5, 0) mod 8.

The existence of these DBIBDs for $v = 3$, 4, 6, and 7 was previously established but we give them as starting examples for the recursive procedure we now describe.

We use induction to show that, if $v \equiv 2 \pmod 3$, then all DBIBDs with parameters $(v, v(v-1), 3(v-1), 3, 3^*)$ are known. The required designs are given above for $v = 5$ and 8. Suppose that all DBIBDs with the required parameters are known for $v \leqslant 6u + 8$. Then, in particular, the DBIBD with

$v = 3u + 5$ is known. We may then use Lemma A to establish the existence of a DBIBD with $v = 6u + 11$ and Lemma B to establish the existence of a DBIBD with $v = 6u + 14$. Thus all designs with $v \equiv 2 \pmod 3$ and the required parameters exist, completing the result.

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