AN INFINITE FAMILY OF SKEW-WEIGHING MATRICES

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Abstract

We show that orthogonal designs of type (1,k) exist for all $k = 0,1,\dots,2^t.15-1$, in order $2^t.15$, $t \geq 4$ a positive integer. Hence there exist skew-symmetric weighing matrices $W(2^t.15,k)$ for all $k = 0,1,\dots,2^t.15-1$.

1. Introduction.

All definitions and references are to the book [1] of Geramita and Seberry. We, at times, use OD as an abreviation for orthogonal design.

Our first result supposes that all ODs of type (1,a,b) exist in order 4t, this is certainly not true for t odd, but for t sufficiently divisible by 2, and probably just divisible by 2 is sufficient, this is almost certainly always true but is not yet proved.

LEMMA 1. *Suppose in order* 4t *all* ODs *of type* (1,a,b) *exist then replacing the variables by the circulant matrices (or back-circulant where necessary) with first rows*

(i)	$xy\bar{y}$, $\bar{y}yy, yyy$	*(v)*	$x00, 0y\bar{y}$, yyy
(ii)	$xy\bar{y}$, $\bar{y}yy, 0yy$	*(vi)*	$x00, 0y\bar{y}$, $0yy$
(iii)	$x00$, $\bar{y}yy, yyy$	*(vii)*	$xy\bar{y}, 0yy$, $y00$
(iv)	$x00$, $\bar{y}yy, 0yy$		

(where $\bar{y}$ *means* $-y$*)*

we obtain orthogonal designs in order 12t *of types*

(i)	$(1,12b-1)$, $b \leq t$	*(ii)*	$(1,5b-1)$, $b \leq 2t-1$;
(iii)	$(1,12b)$, $b \leq t-1$;	*(iv)*	$(1,5b)$, $b \leq 2t-1$;
(v)	$(1,9b)$, $b \leq t-1$;	*(vi)*	$(1,4b)$, $b \leq 2t-1$;
(vii)	$(1,4+b)$, $b \leq 4t-2$;		

respectively.

2. Results.

By Lemma 8.39 of [1], there exists an orthogonal design of type (1,k) in order 60 for $k \in \{x: x \neq 7,15,23,28,31,34,39,42,47,48,53,54,55,56,57,\ 0 \leq x \leq 59\}$.

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We now use Proposition 5.1 of [1] to see that orthogonal designs of type $(1,k)$ exist in order 120 for $k \in \{x: 0 \le x \le 119 \; x \ne 15,31,47,56,57,62,63,68,69,78,79,84,85,94\text{-}97,106\text{-}115\}$. Now there exists an $A = W(60,j)$ for $0 \le j \le 60$. So

$$B = \begin{bmatrix} xI & yA \\ yA^T & -xI \end{bmatrix}$$

gives an OD of type $(1,k)$ in order 120 for every k, $0 \le k \le 60$.

Lemma 1 can now be used to obtain ODs of type $(1,k)$ in order 120 for $k \in \{68,85,94,95,96,107\}$ by using the designs of the types specified in order 40 with the part of the lemma indicated: (1,17,17) and (vi); (1,17,17) and (iv), (1,18,19) and (ii); (1,19,19) and (iv); (1,24,8) and (iii); (1,26,9) and (i).

Replacing the variables of the (1,10,14,15) design in 40 (see [p 380]) by A,bI,C,B gives (1,84) in 120; (1,4,13,13) by aI,bI,B,C gives (1,69); (1,8,11,11) by aI,bI,B,C gives (1,63); (1,16,15) (from (1,11,5,15)) by A,B,C gives (1,79); where A is the back circulant matrix with first row $ab\bar{b}$ and B,C are the circulant matrices with first rows $0bb$ and $\bar{b}bb$.

From Table H.2 [1, p.403] we see there is an orthogonal design of type (1,2,2,36) in order 60 and hence design of type (1,2,4,72) in order 120 exists giving a (1,78).

Let E and F be the two circulant matrices (see [p.334]) from which a $W(30,29)$ can be constructed. Let G and H be the back-circulant and circulant matrices with first rows

$$a \; 0_6 \; b \; - b \; 0_6 \quad \text{and} \quad 0_7 bb 0_6$$

respectively where 0_t means the $1 \times t$ vector of zeros. Then replacing the variables of a (1,1,2,2) design in order 8 by G,H,E,F gives a (1,62) design in order 120.

Let C be the back-circulant matrix with first row $abb\bar{b}\bar{b}$, and B be the circulant matrix with first row $0\bar{b}\bar{b}bb$. Replace the variables of the (1,1,1,2,9,10) design in order 24 by $C,B+bI,b(J-I),B-bI,B,b(J-2I)$ this gives a (1,109) design in 120; of the (1,1,1,3,9,9) design by $aI,bI,b(J-I),bJ, B+bI,b-bI$ to get a (1,110) design in 120; of the (1,1,2,3,8,9) design by $C,B+bI,B-bI,b(J-2I),B,b(J-2I)$ to get a (1,111) design; of the (1,1,9,12) design by aI,b(J-I),b(J-2I),B to get a (1,97) design 120.

Summarizing we have:

SUMMARY 2. *All orthogonal designs of type* (1,k) *exist in order* 120 *except possibly for* k ∈ {106,108,112-115}.

We now use proposition 5.1 of [1], to see that orthogonal designs of type (1,k) exist in order 240 for all orders except possibly k ∈ {106,108,112-115,212,213,216,217,224-231}. Since all orthogonal designs of type (1,k) exist in orders $128 = 2^5$ and $112 = 2^4.7$ we take the direct sum to see the designs (1,k), k ∈ {106,108} exist. The (1,56), and (1,57) in 120 give the (1,112), (1,113), (1,114) and (1,115) in 240.

Since there are amicable orthogonal designs of types ((1,19);(1,19)) in order 20 [1, Theorem 5.52] we can use Corollary 5.100 of Geramita and Seberry to show there is a (1,1,1,1,19,19,19,19) design in order 80. Let X,V be the circulant matrices with first rows $\bar{x}xx$, $\bar{v}vv$ and T,Y,Z be the back-circulant matrices with first rows $01\bar{1}$, $yx\bar{x}$ and $zw\bar{w}$. Then replacing the variables by (i), aI,bI,cI,dI,xJ,X,X,Y;
(ii) Z,w(J-I),cI,dI,xJ,X,X,X; (iii) Z,w(J-I),v(J-I),V,xJ,X,X,X;
(iv) V,v(J-I),cI,dI,xJ,X,X,Y: (v) aI,bI,y(J-I),yT,xJ,X,X,xT
are orthogonal designs of types (1,1,1,1,19,209), (1,1,1,4,228), (1,4,5,228), (1,1,5,19,209) and (1,1,4,209) in order 240.

Similarly there are amicable orthogonal designs of types ((1,11);(1,11)) in 12 so there is a (1,1,1,1,11,11,11,11) design in 48. Let B,N be the circulant matrices with first rows $0b\bar{b}\bar{b}b$ and 00dd0. Let M be the back circulant matrix with first row $x0\bar{d}d0$. Then replacing the variables of the designs by (i) aI,cI,dI,eI,b(J-2I), b(J-2I),B+bI,B-bI; (ii) M,N,fI,eI,b(J-2I),b(J-2I),B+bI,B-bI gives (1,1,1,1,220) and (1,1,1,4,220) designs in 240. Replacing the first four variables by the matrices given on p. 379 of [1] allows us to also establish the existence of the designs (1,1,9,220), (1,11,220), (1,4,9,220), (1,1,10,220), (1,14,220), (1,17,220), (1,19,220). Replacing the first four variables by the four symmetric matrices on p. 379 and the other variables by b(J-2I),b(J-2I),B-bI,A, where A is the circulant matrix with first row $dbb\bar{b}\bar{b}$ gives the designs (1,1,11,9,209), (1,11,11,209), and (1,1,1,1,11,209).

Summarizing

SUMMARY 3. *Orthogonal designs of types*

(1,1,1,1,220)	(1,1,9,11,209)	(1,1,1,1,11,209)
(1,1,1,4,220)	(1,11,11,209)	
(1,1,9,220)	(1,1,1,1,19,209)	
(1,11,220)	(1,1,1,4,228)	
(1,4,9,220)	(1,4,5,228)	
(1,1,10,220)	(1,1,5,19,209)	
(1,14,220)	(1,1,4,209)	
(1,17,220)	(1,19,220)	

exist in order 240. *In particular designs of type* (1,k) *exist in order* 240 *for* $k \in \{209\text{-}215, 218\text{-}226, 228\text{-}234, 237, 239\}$.

Let B be the symmetric W(6,5), C = B+I and D = B-I. Now we observe from [1; p. 349-351] that an orthogonal design of type (1,1,9,9) exists in order 20 and hence by Lemmas 4.8 and 4.11 there exists an orthogonal design of type (1,1,2,18,18) in order 40.

Replacing the variables of the following designs in 40 by the matrices indicated we obtain orthogonal designs of types (1,k) for $k \in \{216, 217, 227\}$ in order 240:

(1,18,18) by xI,yC,yD; (1,1,18,18) by xI,yI,yC,yD;
(1,1,2,18,18) by xI,yI,yB,yC,yD.

Summarizing we have:

THEOREM 4. *All orthogonal designs of type* (1,k), $0 \le k \le 239$ *exist in order* 240. *Hence all orthogonal designs of type* (1,k), $0 \le k \le 2^t.15-1$ *exist in order* $2^t.15$, $t \le 4$ *an integer. Further skew-symmetric weighing matrices* $W(2^t.15,k)$ *exist for* $0 \le k \le 2^t.15-1$, $t \ge 4$.

Using this result:

In fact we can say a little more. The designs (1,1,8,8) and (1,1,9,9) exist in every order 4n, $n \ge 5$. So the designs (1,1,2,16,16) and (1,1,2,18,18) exist in every order 8n, $n \ge 5$. By various choices for replacing the variables of these designs by I,A,A+I,A-I where A is a W(2m,2m-1) (and for m odd, A can always be chosen to be symmetric with zero diagonal) we obtain, in order 16mn, $n \ge 5$, designs of types (1,1,2,16(2m-1),16(2m-1)),(1,1,2,64m),(1,2,2m-1,16(2m-1), 16(2m-1)),(1,2,2m-1,64m),(1,1,4m-2,16(2m-1),16(2m-1)),(1,1,4m-2,64m), (1,2m-1,4m-2,16(2m-1),16(2m-1)),(1,2m-1,4m-2,64m), (1,1,2,18(2m-1),

18(2m-1)),(1,1,2,72m),(1,2,2m-1,18(2m-1),18(2m-1)),(1,2,2m-1,72m),
(1,1,4m-2,18(2m-1),18(2m-1)),(1,1,4m-2,72m),(1,2m-1,4m-2,18(2m-1),
18(2m-1)),(1,2m-1,4m-2,72m).

For $m = 3$ we have (1,1,2,80,80),(1,1,2,192),(1,2,5,80,80),
(1,2,5,192),(1,1,10,80,80),(1,1,10,192),(1,5,10,80,80),(1,5,10,192),
(1,1,2,90,90),(1,1,2,216),(1,2,5,90,90), (1,2,5,216), (1,1,10,90,90),
(1,1,10,216), (1,5,10,90,90), (1,5,10,216), in every order $48n$, $n \ge 5$.

So we have:

LEMMA 5. *There exists an orthogonal design of type* (1,k) $k \in$ {80-83,85,87,90-93,95,97,100,101,105,160-163,165,167,170,171,175, 180-183,185,187,190-195,197,199,202,203,207,216-219,221,223,226,227,231} *in every order* $48n$, $n \ge 5$.

We note that orthogonal designs of types (1,k), $k \in$ {1-71}, exist in order 72, 96 and 120 (by Summary 2). Hence orthogonal designs (1,k) for $k \in$ {1-71} exist in every order $24n$, $n \ge 3$ and orthogonal designs of types (1,j), $j \in$ {1-143} exist in every order $48n$, $n \ge 3$.

There are amicable orthogonal designs I+X, Y, Z in order 12 with I+X,Y of types ((1,11);(12)) and I+X,Z of types ((1,11);(11)): These may be used with the design of type (1,2,4,8) which exists in every order $4n$, $n \ge 3$ to obtain orthogonal designs of types (1,11,24,48,96),(1,11,22,44,88),(1,2,48,96) in every order $48n$, $n \ge 3$.

There are amicable orthogonal designs of types ((1);(5)) in order 6: these may be used with the OD of type (1,1,2,4,8,16) in order $8n$, $n \ge 3$ to obtain designs of types (1,5,2,20,40,80),(1,1,10,20,40,80) and (1,5,10,20,40,80) in $48n$, $n \ge 3$.

Hence we have obtained orthogonal designs of type (1,k) for $k \in$ {1-147,150,151,154,155,165,168,179} in $48n$, $n \ge 3$.

Orthogonal designs of types (1,k), $k \in$ {72-95}, exist in orders 96,120 and 144. We now look briefly at orders 84 and 168. From Lemma 8.46 of Geramita and Seberry we see a $W(84,j)$ exists for $j \in$ {72-76,78,80-84} and hence (1,j) exists in 168 for these j. (1,77) exists in 80 and 88 and hence in order 168. Let W be the back circulant matrix of order 7 with first row -110100 (where - represents -1). Now there is an orthogonal design of type (1,1,1,1,1,19) in order 24 and replacing its variables variously by xI_7, I_7 and W we obtain orthogonal designs of types (1,j) for $j \in$ {79,85,86,88,89,92} in order 168. Designs of types (1,18),(1,3,18) and (1,19) exist in 28, hence (1,1,18,18),(1,3,18,18) and (1,1,19,19) exist in 56 and replacing the variables by the circulant matrices $xI_3, yI_3, \bar{z}zz, 0zz$ we obtain designs of types (1,1,90),(1,3,90) and (1,1,95) in order 168. Thus we have orthogonal designs of types (1,j), j ε {72-86,88-93,95,96} in order 168. Hence orthogonal designs of type (1,j) exist in every order 24n, n ≥ 4 and orthogonal designs of types (1,j), j ε {144-173,176-187,190,191} exist in every order 48n, n ≥ 4.

From Table H.2 a design of type (1,43) exists in order 4n, n ≥ 11. Hence a design of type (1,1,2,172) exists in order 16n, n ≥ 11 and thus 48n, n ≥ 4.

So we have orthogonal designs of types (1,k), k ε {1-187,190,191} in 48n, n ≥ 4.

Further we observe that designs of types (1,j), j ε {20,21,22,24, 25,26,27}, (1,i,18), i ε {1,2,3,6,8,9}, (1,4,20) and (1,2,25) exist in order 28. Thus there are designs of types (1,1,j,j),(1,i,18,18), (1,4,20,20) and (1,2,25,25) in order 56 and proceeding as before we obtain designs of types (1,1,5j),(1,i,90),(1,4,100) and (1,2,125) in order 168. Let J be the circulant matrix with first row all ones, I_7 be the identity matrix and R, S the back circulant matrices with first rows 011-1-- and 111-1-- respectively (all or order 7). The following designs exist in order 24: (i) (1,1,3,15), (ii) (1,3,5,15), (iii) (1,1,1,9,12), (iv) (1,2,3,14), (v) (1,1,1,2,5,14), (vi) (1,2,2,3,16), (vii) (1,1,3,5,14), (viii) (1,1,1,4,17), (ix) (1,1,1,3,18), (x) (1,1,1,2,8,11).

We replace the variables as follows: aI, bI, c(J-I), cR and aI, bI, c(J-I), cS in (i) and (ii); aI, cJ, c(J-I), dI, cR and aI, cJ, c(J-I), dI, cS in (iii); aI, cJ, dI, cR and aI, cJ, dI, cS in (iv); aI, bI, dI, cJ, eI, cR or cS in (v); aI, c(J-I), c(J-2I), bI, cR or cS in (vi); aI, c(J-I), c(J-2I), bI, cR in (vii); aI, bI, c(J-I), c(J-2I), cR or cS in (viii); aI, bI, c(J-2I), c(J-I), cR or cS in (ix); aI, bI, c(J-I), c(J-2I), dI, cR in (x); to obtain designs of types (1,1,108), (1,1,123), (1,5,108), (1,5,123), (1,9,85), (1,9,97), (1,3,98), (1,3,112), (1,1,1,5,98), (1,1,1,5,112), (1,3,122), (1,3,138), (1,5,111), (1,1,136), (1,1,153), (1,1,133), (1,1,151), (1,1,8,86) in order 168.

Let Q, P be the incidence matrices of the quadratic residues and the quadratic non-residues respectively in order 17. Now replace the variables of the $(1,1,1,1,2,2)$ design in order 8 by aI, $b(Q-I)$, $b(P-I)$, $cI+b(Q-P)$, $cI-b(Q-P)$ to obtain a design of type $(1,4,98)$ in order 168.

These designs just constructed give us:

LEMMA 6. *There exist orthogonal designs in order* 168 *of type* $(1,k)$ *for* $k \in \{1-106,108-128,130,131,133-138,141,151-154\}$.

All orthogonal designs of types $(1,k)$ for $k \in \{1,\ldots,n-1\}$ exist in orders $n = 96$, 112, 144, and 192. Now orthogonal designs of types $(1,i)$, $i \in \{49,50,51\}$ exist in order 52. Hence designs of types $(1,j)$, $j \in \{98-103\}$ exist in order 104 and also in order 112 and thus in order 216. The results in order 120 obtained before with the result for 96 show that designs of types $(1,j)$, $j \in \{1-95\}$ exist in 216. Since the designs $(1,i)$, $i \in \{1-95,98-103\}$ exist in 120, 144, 168, 192 and 216 they exist in $24n$, $n \geq 5$. Hence designs of types $(1,k)$, $k \in \{1-191,196-207\}$ exist in $48n$, $n \geq 5$.

From Table H.4 orthogonal designs of types $(1,1,58)$ exist in $4n$, $n \geq 15$ and $(1,1,50)$ in $4n$, $n \geq 13$. Hence designs of types $(1,1,2,4,232)$ exist in $16n$, $n \geq 15$ and $(1,1,2,4,200)$ exist in $16n$, $n \geq 13$ that is in orders $48n$, $n \geq 5$. We use the designs of type $(13,13)$ which exist in orders $2n$, $n \geq 14$ from Table H.3 with Lemma 4.118 of Geramita and Seberry to obtain designs of types $(1,1,52,52)$ in every order $4n$, $n \geq 29$ and hence designs of types $(1,1,2,104,104)$ in every order $8n$, $n \geq 29$, that is $48n$, $n \geq 5$.

These results are summarized as follows:

LEMMA 7. *There exist orthogonal designs of types* $(1,k)$ *in orders* $48n$ *with*

(a) $n \geq 3$, $k \in \{1-147,150,151,154,155,167,168,179\}$,

(b) $n \geq 4$, $k \in \{1-187,190,191\}$,

(c) $n \geq 5$, $k \in \{1-211,216-219,221,223,226,227,231-239\}$.

REFERENCE

[1] Anthony V. Geramita and Jennifer Seberry, *Orthogonal Designs: quadratic forms and Hadamard matrices*, Marcel Dekker, New York, 1979.

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