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SOME REMARKS ON AMICABLE ORTHOGONAL DESIGNS

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We present some new results on amicable orthogonal designs. We obtain amicable Hadamard matrices of order $2^4 \cdot 211$ and a skew Hadamard matrix of order $2^4 \cdot 295$ which were previously not known.

1. Introduction

For all definitions we refer the reader to the book of Geramita and Seberry [1].

Our aim in writing this note has been to try to fill gaps in the tables of [2]. A greater knowledge of amicable orthogonal designs in small orders would have been useful. In fact, little is known for orders >8.

2. A note on amicable orthogonal designs

Theorem 1. *There are amicable orthogonal designs of types*

(a) $((1, 1, 2, 4, \ldots, 2^{t-1}); (2^t))$,

(b) $((1, 2, 4, \ldots, 2^{t-1}); (1, 2, 4, \ldots, 2^{t-1}))$,

(c) $((1, 1, 2, 4, \ldots, 2^{t-2}, 2^{t-1}-1); (2^{t-1}-1, 2^{t-1}))$,

(d) $((1, 1, 2, 4, \ldots, 2^{t-2}); (1, 2, 4, \ldots, 2^{t-2}, 2^{t-1}))$

in order 2^t.

Proof. In Geramita and Seberry [1, Corollary 5.128], it is established that for order 2^t, there exist matrices P, Q, H such that P and Q are skew-symmetric orthogonal designs of type $(1, 2, 4, \ldots, 2^{t-1})$, and H is a symmetric orthogonal design of type (2^t), and such that $PQ^T = QP^T$, $PH^T = HP^T$, and $QH^T = HQ^T$.

Then the required matrices are

(a) P and H,

(b) P and Q,

(c) $\begin{bmatrix} P+xI & Q \\ Q & -P+xI \end{bmatrix}, \begin{bmatrix} H & Q \\ -Q & H \end{bmatrix}$ (using the matrices from order 2^{t-1}),

(d) $\begin{bmatrix} P+xI & 0 \\ 0 & -P+xI \end{bmatrix}, \begin{bmatrix} H & Q \\ -Q & H \end{bmatrix}$.

Corollary 2. *All amicable orthogonal designs of types* $((1, a); (b))$, *where* $1 \leq a \leq 2^{t-1}$ *and* $1 \leq b \leq 2^t$, *exist in order* 2^t. *All amicable orthogonal designs of types* $((1, e); (b))$, *where* e *is even, and* $((1, c); (d))$ *where* $d \in \{2^{t-2}, 2^{t-1}-1, 2^{t-1}, 2^{t-1}+2^{t-2}, 2^t-1, 2^t\}$ *and* $c \in \{1, 2, \ldots, 2^t-2\}$, *exist in order* 2^t. *In addition, design of type* $((1, 2^t-1); (f))$, *where* $f \in \{2^{t-2}, 2^{t-1}, 2^{t-1}+2^{t-2}, 2^t\}$ *exist in order* 2^t.

3. Two constructions for amicable Hadamard matrices

Theorem 3. *Suppose there exist amicable Hadamard matrices of order b of the form*

$$X = \begin{bmatrix} x & 1 \cdots 1 \\ - & \\ \vdots & xI+P \\ - & \end{bmatrix}, \qquad Y = \begin{bmatrix} 1 & 1 \cdots 1 \\ 1 & \\ \vdots & D \\ 1 & \end{bmatrix}.$$

Further, suppose there are amicable orthogonal designs of types $((1, a, ab-a-1); (a, ab-a))$ *in order* ab. *Then there exist amicable Hadamard matrices of order* $ab(b-1)$.

Proof. Since $XY^T = YX^T$ we have $PJ = 0$, $DJ = -J$, $D^T = D$, $P^T = -P$ and $PD^T = DP^T$. We replace the variables of the amicable orthogonal designs of types $((1, a, ab-a-1); (a, ab-a))$ in order ab by $xI+P$, J, D, J and D respectively, then since

$$DD^T = bI - J \quad \text{and} \quad PP^T = (b-1)I - J$$

and the matrices pairwise satisfy $AB^T = BA^T$ we have amicable orthogonal designs of types $((1, ab(b-1)-1); (ab(b-1)))$ in order $ab(b-1)$. We use [1, Theorem 5.43] now to obtain the required structure of the amicable Hadamard matrices.

We use the fact that amicable orthogonal designs of types $((1, p); (1, p))$ exist in order $p+1$, $p \equiv 3 \pmod 4$ a prime power to obtain the next result.

Corollary 4. *Let* $p \equiv 3 \pmod 4$ *be a prime power. If there exist amicable orthogonal designs of types* $((1, a, ap-1); (a, ap))$ *in order* $a(p+1)$ *there exist amicable Hadamard matrices of order* $ap(p+1)$.

Example. If there exist amicable orthogonal designs of types $((1, a, 3a-1); (a, 3a))$ in order $4a$ there exist amicable Hadamard matrices of order $12a$.

Comment. At this stage more knowledge of the existence of amicable orthogonal designs would be invaluable.

We obtain one more new construction by observing that if there are amicable orthogonal designs of types $((a,\ b);\ (a+b))$ in order $a+b$, then there exists amicable orthogonal designs of types $((a,\ b,\ a+b);\ (a+b,\ a+b))$ in order $2(a+b)$. This is now used with [2, Theorem 5.85] to see

Theorem 5. *Let* $p \equiv 3 \pmod 4$ *be a prime power. Let* $q \equiv 5 \pmod 8$ *be a prime power with proper representation* $q = s^2 + 4(2p+1)^2$. *Then there exist amicable Hadamard matrices of order* $2(p+1)(q+1)$.

Proof. Since $p \equiv 3 \pmod 4$ is a prime power, there are amicable orthogonal designs of types $((1, p);\ (1, p))$ in order $p+1$ [1, Theorem 5.52] and types $((1,\ p,\ p+1);\ (p+1,\ p+1))$ in order $2(p+1)$. Now using $|t| = 2p+1$ in [1, Theorem 5.85] we have the result.

Example. With $q = 421 = 15^2 + 4 \cdot 49^2$ we see there are amicable Hadamard matrices of order $2^4 \cdot 211$ which is new. This construction also gives amicable Hadamard matrices of orders $2^4 \cdot 279$ and $2^4 \cdot 411$.

4. A result on skew Hadamard matrices

Using the fact that amicable Hadamard matrices exist for orders which are powers of two and [3, Theorem 4.20, p. 337] we have

Lemma 6. *Let n be the order of amicable Hadamard matrices. Suppose there exists a skew Hadamard matrix of order* $(2^t - 1)n$. *Then there exists a skew Hadamard matrix of order* $2^t n(2^t n - 1)(2^t - 1)$. *In particular, if there are amicable Hadamard matrices of order n there is a skew Hadamard matrix of order* $2n(2n-1)$.

Example. $m = 4$ and $n = 20$ in the corollary gives a skew Hadamard matrix of order $2^4 \cdot 295$. $n = 36$ and $n = 44$ in the last part of the corollary give skew Hadamard matrices of orders $2^3 \cdot 639$ and $2^3 \cdot 957$. This improves a result of [2].

References

[1] A.V. Geramita and J. Seberry, Orthogonal Designs: Quadratic Forms and Hadamard matrices (Marcel Dekker, New York, 1979).

[2] J. Seberry, On skew Hadmard matrices, Ars Combinatoria 6 (1978) 255–275.

[3] J. Seberry Wallis, Hadamard matrices, in: Combinatorics: Room Squares, Sum Free Sets, Hadamard Matrices, W. D. Wallis, A. Penfold Street and J. Seberry Wallis, Lecture Notes in Mathematics, 292 (Springer-Verlag, Berlin, 1972).