COMPLEX WEIGHING MATRICES AND ORTHOGONAL DESIGNS.

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Abstract

Galois fields $GF(q^2)$ are used to obtain a new infinite family of complex weighing matrices $CW(q+1,q)$, $q \equiv 1 \pmod 8$, and type

$$P = \begin{bmatrix} R & -S \\ S^* & -R^* \end{bmatrix}$$

where R and S are symmetric complex circulants.

These matrices are used to construct orthogonal designs. Some unsolved cases of Geramita and Geramita are also settled.

1. *Introduction.*

An *orthogonal design* of order n and type $(s_1,\ldots,s_u)$ on the commuting variables $x_1,\ldots,x_u$ is an nxn matrix X with entries chosen from the set $\{0,\pm x_1,\ldots,\pm x_u\}$ such that

$$XX^T = (\sum_{i=1}^{u} s_i x_i^2) I_n.$$

Such generically orthogonal matrices have played a significant role in the construction of Hadamard matrices (see eg. the book [5]) and they have already been extensively used in the study of *weighing matrices* $W(n,k) = W$ which have entries $0,1,-1$ and satisfy

$$WW^T = kI_n.$$

These have been studied extensively [1,2,5,8,9,12,13] and one of us (Seberry) has conjectured [13] that for $n \equiv 0 \pmod 4$ there is a weighing matrix of order n and weight k for every $k \leq n$.

Later the notion of a complex Hadamard matrix, i.e., an nxn matrix C whose entries are chosen from $\{\pm 1,\pm i\}$ and satisfying $CC^* = nI_n$ ($*$ = complex conjugate) were studied in order to obtain new Hadamard

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matrices (see [11, p. 347-353]).

Recently the study of *complex orthogonal designs (cod)* of order n and type $(s_1,\ldots,s_u)$ (s_i positive integers) which is an nxn matrix X with entries chosen from $\{0,i_1x_1,\ldots,i_ux_u\}$, where i_j is a fourth root of 1 and $x_1,\ldots,x_u$ are real variables, satisfying

$$XX^* = (\sum_{j=1}^{u} s_j x_j^2)I_n$$

was commenced in [4].

We note that if $\{i_1,\ldots,i_u\} = \{\pm 1\}$ we have an orthogonal design as previously defined. If $\sum_{j=1}^{u} s_j = n$ and $\{i_1,\ldots,i_u\} = \{\pm 1,\pm i \mid i^2 = -1\}$ we have a complex Hadamard matrix but if $\{i_1,\ldots,i_u\} = \{\pm 1\}$ we have a Hadamard matrix.

If $\sum_{j=1}^{u} s_i = k$, $k \leq n$ we have a *complex weighing matrix* $CW(n,k)$ or weighing matrix according as $\{i_1,\ldots,i_u\} = \{\pm 1,\pm i\}$ or $\{\pm 1\}$. Complex weighing matrices have been studied with great interest lately [2,10] because of their connection with combinatorial designs. The notation $CW(n,k,z_4)$ is sometimes used to denote the elements are from the cyclic group z_4 and distinguish it from $CW(n,k,f_2xf_2)$.

2. A Preliminary Lemma.

Let $GF(q^2)$ denote a finite field of q^2 elements where $q = p^r$ is an odd prime power. Let τ denote a generator of the cyclic group of non-zero elements of $GF(q^2)$. Put $\gamma = \tau^{(q+1)/2}$ and $g = \gamma^2$. Then g is a generator of the cyclic group of non-zero elements of a finite field $GF(q)$ of order q. It follows that $\gamma^q = -\gamma$. We also have $GF(q^2) = \{a\gamma+b; a,b\varepsilon GF(q)\}$.

For arbitrary $\xi\varepsilon GF(q^2)$ define

$$tr(\xi) = \xi + \xi^q \tag{1}$$

so that $tr(\xi)\varepsilon GF(q)$. It follows at once from this definition that

$$tr(\tau^\kappa) = \tau^{(q+1)\kappa} tr(\tau^{-\kappa}), \tag{2}$$

for any integer κ. For $\xi \in GF(q^2)$, $\xi \neq 0$, let $\mathrm{ind}(\xi)$ be the least non-negative integer t such that $\tau^t = \xi$. Let β denote a primitive eighth root of unity. Then

$$\chi(\xi) = \begin{cases} \beta^{\mathrm{ind}(\xi)} & (\xi \neq 0) \\ 0 & (\xi = 0), \end{cases} \tag{3}$$

defines an eighth power character χ of $GF(q^2)$. For $a \in GF(q)$, $a \neq 0$, put $q^j = a$. Then (3) implies that $\chi(a) = \beta^{(q+1)j}$.

Suppose henceforth that $q \equiv 1 \pmod 8$ so that $\chi(a) = i^j$. This means that the restriction of χ to $GF(q)$ reduces the character $\chi(a)$ to a fourth power character over $GF(q)$. Thus $\chi(a) = 1, -1, i$ or $-i$ according as $j \equiv 0, 2, 1$ or $3 \pmod 4$. Accordingly we deduce from (2) that

$$\chi\mathrm{tr}(\tau^\kappa)\bar{\chi}\mathrm{tr}(\tau^{-\kappa}) = i^\kappa \qquad (\mathrm{tr}(\tau^\kappa) \neq 0), \tag{4}$$

where $\bar{\chi}(\xi)$ is the complex conjugate of $\chi(\xi)$.

It will later be convenient to introduce $\theta = \tau^{q-1}$, an element of $GF(q^2)$ of multiplicative order $q+1$. Since $\mathrm{tr}(\tau^\kappa) = \tau^\kappa(\theta^\kappa + 1)$ we have

$$\chi\mathrm{tr}(\tau^\kappa) = \beta^\kappa \chi(\theta^\kappa + 1). \tag{5}$$

The proof of Theorem 1 in the next section is based upon the following lemma.

LEMMA 1. *If r is a non-negative integer, then*

$$\sum_{\kappa=0}^{q} \chi\mathrm{tr}(\tau^\kappa)\bar{\chi}\mathrm{tr}(\tau^{\kappa+r}) = \begin{cases} (-i)^j q & (q+1 \mid r), \\ 0 & (q+1 \nmid r), \end{cases} \tag{6}$$

where, in the first case, $r = j(q+1)$.

Proof. For fixed $\eta \in GF(q^2)$ put $\eta = c\gamma + d$, $c, d \in GF(q)$. Then $\eta \in GF(q)$ if $c = 0$ and $\eta \notin GF(q)$ if $c \neq 0$. We first show that

$$\sum_{\xi} \chi\mathrm{tr}(\xi)\bar{\chi}\mathrm{tr}(\eta\xi) = \begin{cases} \bar{\chi}(d)q(q-1) & (c = 0), \\ 0 & (c \neq 0), \end{cases} \tag{7}$$

where the summation extends over all $\xi \in GF(q^2)$. Put $\xi = a\gamma+b, a,b \in GF(q)$. By (1) we have $tr(\xi) = 2b$ and $tr(\eta\xi) = 2(gac+bd)$. Therefore

$$\sum_{\xi} \chi tr(\xi)\overline{\chi}tr(\eta\xi) = \sum_{b}\chi(b)\sum\chi(gac+bd),$$

and (7) follows at once.

For $\eta \neq 0$ we may put $\eta = \tau^r$, $0 \leq r \leq q^2-2$, so that $c = 0$ if $q+1 \mid r$ and $c \neq 0$ if $q+1 \nmid r$. If $c = 0$ put $r = j(q+1)$ then we get $\chi(d) = i^j$. The sum in (7) now becomes

$$\sum_{\kappa=0}^{q^2-2} \chi tr(\tau^\kappa)\overline{\chi}tr(\tau^{\kappa+r}) = \sum_{h=0}^{q-2}\sum_{\kappa=h(q+1)}^{h(q+1)+q} \chi tr(\tau^\kappa)\overline{\chi}tr(\tau^{\kappa+r}).$$

In view of (7), the double sum on the right has the value 0 if $q+1 \nmid r$. Since $\chi tr(\tau^{\kappa+q+1}) = i\chi tr(\tau^\kappa)$, the value of the inner sum is the same for each h. In particular, for $h = 0$ we obtain the result stated in the lemma.

3. The main theorem.

The idea of associating with a circulant matrix A of order n the polynomial

$$\psi(\zeta) = a_0 + a_1\zeta + \ldots + a_{n-1}\zeta^{n-1},$$

where the coefficients $a_0, a_1, \ldots, a_{n-1}$ comprise the top row of A and ζ is an nth root of unity has been exploited by Williamson [14]. One may also associate with $\psi(\zeta)$ a finite Parseval relation. If the coefficients of $\psi(\zeta)$ are complex numbers, this relation is given for fixed r by

$$\sum_{i=0}^{n-1} a_i\bar{a}_{i+r} = \frac{1}{n}\sum_{j=0}^{n-1} |\psi(\zeta^j)|^2\zeta^{jr}, \tag{8}$$

where $\bar{a}_{i+r}$ is the conjugate of a_{i+r} and $\zeta = \exp(2\pi i/n)$.

We shall employ the finite Parseval relation to prove the following theorem.

THEOREM 1. *Let* q *be a prime power* $\equiv 1 \pmod 8$ *and put* $n = (q+1)/2$. *Let* τ *be a primitive element of* $GF(q^2)$. *Put* $\tau^k = a\gamma+b, a,b\in GF(q)$, *and define*

$$a_k = \chi(a), \qquad b_k = \chi(b). \tag{9}$$

Then the sums

$$f(\zeta) = \sum_{i=0}^{n-1} a_{8i}\zeta^i, \qquad g(\zeta) = \sum_{i=0}^{n-1} b_{8i}\zeta^i \tag{10}$$

satisfy the identity

$$|f(\zeta)|^2 + |g(\zeta)|^2 = q \tag{11}$$

for each nth *root of unity including* $\zeta = 1$.

Note that (with the exception of $a_0 = 0$) the coefficients a_{8i}, b_{8i} are $\pm 1, \pm i$. When q is a prime $\equiv 1 \pmod 8$ and $\zeta = 1$, the identity (11) reduces to the classical result that every prime $\equiv 1 \pmod 8$ is representable in the form c^2+2d^2 with integral values of c and d. This is true for the following reason. In the proof of the corollary of Theorem 1 we show that $f(1)$ reduces $\pm d(1+i)$. Moreover, $g(1)$ is actually real and reduces to c with $c \equiv 1 \pmod 4$. Consequently $|f(1)|^2+|g(1)|^2 = c^2 = 2d^2 = q$.

Proof of Theorem 1. Since τ is a primitive element of $GF(q^2)$ the integer $k = (q+1)/2 = n$ is the only value of k in the interval $0 \le k \le q$ for which $tr(\tau^k) = 0$. Since $\tau^n = \gamma$, the numbers a_k, b_k in (9) satisfy the periodicity relation

$$\begin{aligned} a_{k+2jn} &= i^j a_k, & b_{k+2jn} &= i^j b_k, \\ a_{k+(2j+1)n} &= i^j b_k, & b_{k+(2j+1)n} &= i^{j+1} a_k. \end{aligned} \tag{12}$$

From (4) we deduce the second of the two relations

$$a_k = i^k a_{8n-k}, \qquad b_k = i^k b_{8n-k}.$$

The first relation follows from the second upon replacing k by $k+n$ and then applying the last relation in (12). For $k = 8r$ we get in particular

$$a_{8r} = a_{8(n-r)}, \qquad b_{8r} = b_{8(n-r)} \qquad (r = 0,1,\ldots,n-1) \tag{13}$$

Let us now apply the finite Parseval relation (8) to the two sums in (10). We obtain

$$\sum_{i=0}^{n-1} (a_{8i}\bar{a}_{8i+8r}+b_{8i}\bar{b}_{8i+8r}) = \frac{1}{n}\sum_{j=0}^{n-1} (|f(\zeta^j)|^2+|g(\zeta^j)|^2)\zeta^{jr}, \tag{14}$$

where $\zeta = \exp(2\pi i/n)$.

Let $8i \equiv s \pmod{q+1}$, s even, $0 \leq s < q+1$, and $8i+n \equiv t \pmod{q+1}$, t odd, $1 \leq t < q+1$. As i runs over the integers from 0 to $n-1$, the numbers s run over the even integers from 0 to $q-1$ in some order, and the numbers t run over the odd integers from 1 to q in some order. Making use of the third relation in (12) we find that the sum in the left member of (14) reduces to

$$\begin{aligned} &\sum_{i=0}^{n-1} b_{8i}\bar{b}_{8i+8r} + \sum_{i=0}^{n-1} b_{8i+n}\bar{b}_{8i+n+8r} \\ &= \sum_{j=0}^{n-1} b_{2j}\bar{b}_{2j+8r} + \sum_{j=0}^{n-1} b_{2j+1}\bar{b}_{2j+1+8r} = \sum_{k=0}^{q} b_k\bar{b}_{k+8r} \end{aligned} \tag{15}$$

We next employ Lemma 1. We denote the sum in (6) by $F(r)$. The sum in the last member of (15) is equal to $F(8r)$. Combining (14) and (15) we get

$$F(8r) = \frac{1}{n}\sum_{j=0}^{n-1} (|f(\zeta^j)|^2+|g(\zeta^j)|^2)\zeta^{jr}. \tag{16}$$

The inverted form of (16) is

$$|f(\zeta^j)|^2 + |g(\zeta^j)|^2 = \sum_{r=0}^{n-1} F(8r)\zeta^{-rj} \qquad (j = 0,1,\ldots,n-1) \tag{17}$$

The assumption $q \equiv 1 \pmod 8$ implies that $r = 0$ is the only value of r in the summation range $0 \leq r \leq n-1$ for which $8r$ is divisible by $q+1$. Therefore Lemma 1 yields

$$F(8r) = \sum_{k=0}^{q} b_k \overline{b}_{k+8r} = \begin{cases} q & (r = 0) \\ 0 & (1 \leq r \leq n-1). \end{cases} \tag{18}$$

It follows from (18) that the right member of (17) reduces to g. This completes the proof of Theorem 1.

When $\zeta = 1$ the sums $f(\zeta)$ and $g(\zeta)$ in Theorem 1 may be evaluated separately.

COROLLARY 1. *In the case* $\zeta = 1$ *the identity* (11) *reduces to the identity* $q = c^2+2d^2$ *where* c *and* d *are integers.*

Proof. There does not appear to be any straightforward way to derive the corollary from the theorem. It is conveneient to put $\theta = \tau^{q-1}$ as in (5). We begin by expressing the sum

$$L(\chi) = \sum_{k=0}^{q} \chi(\theta^k+1)$$

in terms of $f(1)$ and $g(1)$. We have

$$L(\chi) = \sum_{j=0}^{n-1} \chi(\theta^{2j}+1) + \sum_{j=0}^{n-1} \chi(\theta^{2j+1}+1)$$

$$= \sum_{k=0}^{n-1} \chi(\theta^{8k}+1) + \sum_{k=0}^{n-1} \chi(\theta^{8k+n}+1).$$

From (5) we obtain $\chi(\theta^{8k}+1) = \chi(2)b_{8k}$ and $\chi(\theta^{8k+n}+1) = \chi(2)\beta^{-n+2}a_{8k}$. Since 2 is the square of an element in $GF(q)$ the value of $\chi(2)$ is ± 1. The value of $\chi(\theta^{8k}+1)$ is an even power of β, and the value of $\chi(\theta^{8k+n}+1)$ is an odd power of β. Since $n \equiv 1$ or $5 \pmod 8$ according as $q \equiv 1$ or $q \pmod{16}$ we deduce

$$L(\chi) = \chi(2)\{g(1) \pm \beta f(1)\}, \tag{19}$$

where the ambiguous sign is plus if $q \equiv 1 \pmod{16}$ and minus if $q \equiv 9 \pmod{16}$.

We next transform $L(\chi)$ into the Eisenstein sum

$$E(\chi) = \sum_{b} \chi(1+b\gamma),$$

where b runs over the elements of $GF(q)$.

We follow the method of P.A. Leonard and K.S. Williams [7; p. 306]. For $b \in GF(q)$ the elements $(1-b\gamma)/(1+b\gamma)$ are distinct and different from -1, as $((1-b\gamma)/(1+b\gamma))^q = (1+b\gamma)/(1-b\gamma)$, each of them satisfies $y^{q+1} = 1$, and so these q elements of $GF(q^2)$ are simply θ^k, $1 \leq k \leq q+1$, $k \neq (q+1)/2$. Therefore

$$\{\theta^k+1 \mid 1 \leq k \leq q+1,\ k \neq \tfrac{q+1}{2}\} = \{\frac{2}{1+b\gamma} \mid b \in GF(q)\},$$

so that

$$\sum_{k=1}^{q+1} \chi(\theta^k+1) = \sum_{k=1}^{q+1}{}' \chi(\theta^k+1) = \sum_{b} \chi(\frac{2}{1+b\gamma}) = \chi(2)\sum_{b} \overline{\chi}(1+b\gamma)$$

where the dash (') over the second sum indicates that the value $k = (q+1)/2$ is excluded. We have thus established

$$L(\chi) = \chi(2)E(\overline{\chi}), \tag{20}$$

where $\overline{\chi}$ is the character conjugate to χ.

Comparing (19) and (20) we obtain

$$E(\overline{\chi}) = g(1) \pm \beta f(1). \tag{21}$$

By employing the theory of Jacobi sums Bruce C. Berndt and Ronald J. Evans [3; Chapter 4] have evaluated $E(\chi)$ for q an odd prime power. The following lemma is part of the Theorem 4.1.

LEMMA 2. *Let* $q \equiv 1 \pmod 8$, *and let* χ *be a character of* $GF(q^2)$ *of order* 8. *Then*

$$E(\chi) = c + d\sqrt{-2} \tag{22}$$

where c *and* d *are integers such that* $c^2+2d^2 = q$ *and* $c \equiv 1 \pmod 4$.

Finally we compare (21) and (22). Since $\overline{\chi}$ is a character on $GF(q^2)$ of order 8 we have $E(\chi) = E(\overline{\chi})$. Furthermore, $\beta+\beta^3 = \beta(1+i) = \sqrt{-2}$. Since the terms of $f(1)$ and $g(1)$ are even powers of β it follows that $g(1) = c$ and $f(1) = \pm d(1+i)$. This completes the proof of the corollary.

4. Applications.

Theorem 1 yields the following analogue of a valuable result of Goethals and Seidel [6 or 11, Theorem 3.18].

THEOREM 2. *Let* $q \equiv 1 \pmod 8$ *be a prime power. Then there exists a square matrix* P *of order* $q+1$ *with diagonal elements* 0 *and all other elements* $\pm 1, \pm i$ *such that*

$$PP^* = qI_{q+1} \quad \textit{and} \quad P = \begin{pmatrix} R & S \\ S^* & -R^* \end{pmatrix} \tag{23}$$

where R *and* S *are symmetric circulants. This is a* $CW(q+1,q,z_4)$.

Proof. The matrices R and S in Theorem 2 are obtained from the polynomials $f(\zeta)$ and $g(\zeta)$ in Theorem 1. The elements in the top row of the circulant R are the consecutive coefficients $a_0,a_8,\ldots,a_{8(n-1)}$ of $f(\zeta)$, and the elements in the top row of the circulant S are the consecutive coefficients $b_0,b_8,\ldots,b_{8(n-1)}$ of $g(\zeta)$. The symmetry of R and S is expressed in (13). Since $a_0 = 0$ the diagonal elements in P are all 0. For $n = (q+1)/2$ the matrix equation

$$RR^* + SS^* = qI_n \tag{24}$$

is equivalent to the orthogonality relation

$$\sum_{i=0}^{n-1} (a_{8i}\bar{a}_{8i+8r} + b_{8i}\bar{b}_{8i+8r}) = \begin{cases} q & (r = 0), \\ 0 & (1 \le r \le n-1), \end{cases}$$

which follows from (14), (15) and (18). Consequently, (23) may be established by a straightforward verification.

Examples. In the following two examples the top rows of the circulants R and S are given.

$q = 9$: $R = (0,i,1,1,i)$, $S = (1,-i,i,i,-i)$

$f(1) = 2 + 2i$, $g(1) = 1$

$|f(1)|^2 + |g(1)|^2 = 1^2 + 2.2^2 = 9.$

$q = 17$: $R = (0,i,1,-i,i,i,-i,1,i)$

$S = (1,i,-i,-1,-1,-1,-1,-i,i)$

$f(1) = 2 + 2i, \qquad g(1) = -3$

$|f(1)|^2 + |g(1)|^2 = (-3)^2 + 2.2^2 = 17.$

The following Theorem of Williamson [14, 11 p. 382] as adopted by Geramita, Geramita and Wallis is useful.

LEMMA 3. *Suppose there exist four symmetric circulant matrices* A,B,C,D *of order* n *which satisfy*

$$AA^T + BB^T + CC^T + DD^T = fI_n$$

where f *is a quadratic form. Then*

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix} \tag{25}$$

satisfies

$$HH^T = fI_{4n}.$$

Hence, if A,B,C,D, *have entries from the set of commuting variables* $\{0,\pm x_1,\pm x_2,\ldots\}$, H *is an orthogonal design.*

We shall now employ Theorem 2 to produce the following family of orthogonal designs and weighing matrices to give insight into the internal structure. It should be noted that there is another proof which follows easily from a theorem of Delsante, Goethals and Seidel.

THEOREM 3. *Let* $q \equiv 1 \pmod 8$ *be a prime power. Then there exists an orthogonal design of type* (q,q) *in order* $2(q+1)$ *as well as a* $W(2(q+1),q)$ *and a* $W(2(q+1),2q)$.

Proof. Let R and S be the symmetric circulants of Theorem 2. Put $R = U+iV$ and $S = X+iY$. Then we have

$$RR^* + SS^* = (U+iV)(U-iV) + (X+iY)(X-iY)$$
$$= U^2 + V^2 + X^2 + Y^2 + i(VU-UV+YX-XY)$$

so that

$$U^2 + V^2 + X^2 + Y^2 = UU^T + VV^T + XX^T + YY^T \quad qI_n$$

$$VU - UV + YX - XY = 0$$

in view of (24).

Now suppose x and y are commuting variables and write $A = xU+yV$, $B = yU-xV$, $C = xX+yY$, $D = yX-xY$. Now

$$A^2 + B^2 + C^2 + D^2 = (x^2+y^2)\ (U^2+V^2+X^2+Y^2)$$
$$= (qx^2+qy^2)I_n.$$

Hence, substituting A,B,C,D in (25) gives the required orthogonal design. Setting $x = y = 1$ gives the $W(2(q+1),2q)$ and $x = 1$, $y = 0$ gives the $W(2(q+1),q)$.

Example. In the following two examples (writing $\bar{x} = -x$ and $\bar{y} = -y$) the top rows of the circulants A,B,C,D are given.

$q = 9$: $A = \{0yxxy\}$, $B = \{0x\bar{y}\bar{y}x\}$,

$C = \{x\bar{y}yy\bar{y}\}$, $D = \{yx\bar{x}\bar{x}x\}$.

$q = 17$: $A = \{0yx\bar{y}yyy\bar{y}xy\}$, $B = \{0x\bar{y}x\bar{x}x\bar{x}\bar{y}x\}$

$C = \{xy\bar{y}x\bar{x}x\bar{x}yy\}$, $D = \{\bar{y}x\bar{x}yyyy\bar{x}x\}$.

Also using lemma 4.54 of Geramita and Seberry [5] we can establish.

THEOREM 4. *Let* $q \equiv 1 \pmod 8$ *be a prime power. Then there exists an orthogonal design of type* $(4,2q,2q)$ *in order* $4(q+1)$.

Proof. Let x,y,z be commuting variables and U,V,X,Y be the circulant symmetric matrices of the last proof. Now define

$A = zI + xU + yV$, $B = -zI + xU + yV$, $C = zI + yU - xV$,
$D = -zI + yU - xV$, $E = xX + yY$, $F = xX + yY$,
$G = yX - xY$, $H = yX - xY$,

which may be used in the lemma quoted above to obtain the result.

We now derive another result using Theorem 2.

Suppose $z = X+iY$ is a cod of order m and type $(z_1,z_2,\ldots,z_r)$ then $XX^T+YY^T = fI_m$, where f is a quadratic form, and $XY^T = YX^T$.

THEOREM 5. *Suppose there exists an orthogoanl design of type* (a,a) *in order* n *and a cod of order* m *and type* $(z_1,z_2,\ldots,z_r)$. *Then there exists an orthogoanl design of type* $(az_1,az_2,\ldots,az_r)$ *in order* mn.

Proof. Write $z = X+iY$ for the cod. Replace the first and second variables of the orthogonal design xA_1+yA_2 by X and Y respectively to obtain $W = A_1XX + A_2XY$ then

$$\begin{aligned} WW^T &= A_1A_1^TXXX^T + A_1A_2^TXXY^T + A_2A_1^TXYX^T + A_2A_2^TXYY^T \\ &= aI \times (XX^T+YY^T) + (A_1A_2^T+A_2A_1^T) \times XY^T \\ &= afI_{nm}. \end{aligned}$$

Hence W is the required orthogonal design.

Corollary 4.116 of Geramita and Seberry establishes that an orthogonal design of type (r,r) exists in order $2n$, $n \geq 2^a10^b26^c6^d14^e$ for each r of the form $2^a10^b26^c5^d13^e$ where a,b,c,d,e are non-negative integers.

COROLLARY 2. *Let* r *and* n *be as just described. Suppose there exists a complex orthogonal design of type* $(z_1,z_2,\ldots,z_r)$ *in order* m. *Then there exists an orthogonal design of type* $(rz_1,rz_2,\ldots,rz_r)$ *in order* $2mn$.

This corollary is a generalization of Construction 19 of [4] which gives this result for $r = 2$.

Using Theorem 2 with Theorem 5 we have:

COROLLARY 3. *Let* $q \equiv 1 \pmod 8$ *be a prime power. Let* r *and* n *be as described above. Then there exists an orthogonal design of type* (rq) *in every order* $\geq n(q+1)$. *Equivalently, there exists a* $W(m(q+1),rq)$ *for every* $m \geq n$.

5. *New Cods.*

We note in passing that if a,b,c are commuting variables and $i^2 = -1$ then

(i) a c $-a$ and a ib a

(ii) a ic a a b −a and a ic a −a −b a

are sequences with zero-autocorrelation which can be used as first rows of circulant matrices of order $n \geq 3$ and $n \geq 5$ respectively by adding a suitable number of zeros to the end of each sequence and which can be used in Construction 8 of [4] to obtain:

LEMMA 3. *There exist cods of type* (1,1,4) *in all orders* 2n, $n \geq 3$ *and of types* (2,2,8), (2,10), (4,8) *and* (12) *in all orders* 2n, $n \geq 5$.

This gives some unsolved cases of Geramita and Geramita [4].

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