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SOME REMARKS ON GENERALISED HADAMARD MATRICES AND THEOREMS OF RAJKUNDLIA ON SBIBDS

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Constructions are given for generalised Hadamard matrices and weighing matrices with entries from abelian groups.

These are then used to construct families of SBIBDs giving alternate proofs to those of Rajkundlia.

1. DEFINITION

A *generalised Hadamard matrix* $GH(n,G)$ is an $n \times n$ matrix with elements from the abelian group G of order $|G|$ such that if $\underset{\sim}{a} = (a_1,\ldots,a_n)$ and $\underset{\sim}{b} = (b_1,\ldots,b_n)$ are any two rows of $GH(n,G)$ then the elements $a_i b_i^{-1}$, $i = 1,\ldots,n$ give $n/|G|$ copies of G. These matrices were considered by Butson [4,5], by Shrikhande [18] in connection with combinatorial designs, by Delsarte and Goethals [6,7] in connection with codes and Drake [8] in connection with λ- geometries.

A *generalised weighing matrix* $GW(n,k,G)$ is an $n \times n$ matrix with elements from the abelian group G of order $|G|$ and zero, there are k non-zero elements per row and column and if $\underset{\sim}{a} = (a_1,\ldots,a_n)$ and $\underset{\sim}{b} = (b_1,\ldots,b_n)$ are any two rows of $GW(n,k,G)$ then the elements $a_i b_i^{-1}$, $i = 1,\ldots,n$ give λ_{ab} copies of G. If λ_{ab} is a constant for all a and b we have a *balanced weighing matrix*.

Weighing matrices, the special case with G the cyclic group of order 2 have been studied extensively [10,11,13,19,22]. Their name comes from Yates [25] who gave an application in the accuracy of measurements. Balanced weighing matrices have been studied in connection with combinatorial designs by Mullin and Stanton [14,15,16,21] and Berman [2]. Complex weighing matrices have been studied by Berman [3] and Geramita and Geramita [9].

To illustrate that Berman's generalised weighing matrices and ours are not the same we consider

$$A = \begin{bmatrix} 0 & 1 & 1 & i \\ 1 & 0 & i & 1 \\ i^2 & i & 0 & 1 \\ i & i^2 & 1 & 0 \end{bmatrix}$$

which satisfies $AA^* = 3I$ and is a $W(4,3,Z_4)$ when $i^2 = -1$ but is not a generalised weighing matrix by our definition as the product of rows 1 and 2 is $\{i,i^3\}$ and we need $\{1,i,i^2,i^3\}$.

Notation. Throughout this paper we use Z_q for the cyclic group on q symbols and C_{p^r} for the elementary abelian group $Z_p \times Z_p \times \ldots \times Z_p$.

For our purposes an $SBIBD(v,k,\lambda)$ is a matrix with entries 0 and 1 of order v with k ones per row and column and inner product between rows of λ.

David Glynn [12] has found the only $GW(v,k,G)$ known to the author where G is not an abelian group. Consider the multiplication table for S_3

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	5	6	4
3	3	1	2	6	4	5
4	4	6	5	1	3	2
5	5	4	6	2	1	3
6	6	5	4	3	2	1

1 ↔ e
2 ↔ (123)(456)
3 ↔ (132)(465)
4 ↔ (14)(26)(35)
5 ↔ (15)(24)(36)
6 ↔ (16)(25)(34)

Then the circulant matrix with first row

$$[0\ 5\ 1\ 4\ 0\ 1\ 1\ 6\ 5\ 6\ 0\ 4\ 0]$$

is a generalised weighing matrix $GW(13,9,S_3)$.

2. A FAMILY OF GENERALISED WEIGHING MATRICES

We first give a more direct construction for a result implicit in the work of Rajkundlia. We note that our matrix implies the one of Berman but has an additional property and is obtained quite differently.

Let γ be a primitive element of $GF(p^r)$. Let $q \mid p^r - 1$ and let α be a generator of Z_q, the cyclic group. Write $g_1 = 0, g_2, \ldots, g_{p^r}$ for the elements of $GF(p^r)$ and define $M = (m_{ij})$ of order $p^r + 1$ as follows:

$$\begin{aligned} m_{ii} &= 0 \\ m_{ij} &= \alpha^k \qquad \text{where } g_j - g_i = \gamma^k \\ m_{0j} &= m_{j0} = 1. \end{aligned}$$

Example. Let γ be a primitive element of $GF(2^2)$ and ω be a primitive element of $GF(3)$ where $q = 3$. Write $g_1 = 0$, $g_2 = 1$, $g_3 = \gamma$, $g_4 = \gamma + 1$ for the elements of $GF(2^2)$ using $\gamma^2 = \gamma + 1$. Now

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{bmatrix}$$

We note that M is a $GW(5,4,Z_3)$.

Example. Let $\gamma = 3$ be a primitive element of $GF(7)$ and ω be a primitive element of $GF(3)$ where $q = 3$. Write $g_i = i - 1$, $i = 1, \ldots, 7$ for the elements of $GF(7)$. Now

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega^2 & \omega & \omega & \omega^2 & 1 \\ 1 & 1 & 0 & 1 & \omega^2 & \omega & \omega & \omega^2 \\ 1 & \omega^2 & 1 & 0 & 1 & \omega^2 & \omega & \omega \\ 1 & \omega & \omega^2 & 1 & 0 & 1 & \omega^2 & \omega \\ 1 & \omega & \omega & \omega^2 & 1 & 0 & 1 & \omega^2 \\ 1 & \omega^2 & \omega & \omega & \omega^2 & 1 & 0 & 1 \\ 1 & 1 & \omega^2 & \omega & \omega & \omega^2 & 1 & 0 \end{bmatrix}$$

We note M is a $GW(8,7,Z_3)$.

Theorem 1. *Suppose p^r is a prime power and $q \mid p^r - 1$. Then there exists a balanced* $GW(p^r+1,p^r,Z_q)$.

Proof. Construct M of order $p^r + 1$ as above. We show M is the required $GW(p^r+1,p^r,Z_q)$. First M has the elements 0,1 ($(p^r-1)/q+1$ times), and $\alpha,\alpha^2,\ldots,\alpha^{q-1}$ (each $(p^r-1)q$ times) in each row (column) but the first. So we have the group property with respect to the first row.

We now consider the other rows. We consider $q = p^r - 1$. Suppose $g_j - g_i = \gamma^b$, $g_j - g_k = \gamma^s$ then $m_{ij} = \alpha^b$, $m_{kj} = \alpha^s$. We wish to show that $m_{ij}m_{kj}^{-1} = \alpha^{b-s}$ cannot arise in any other way. We proceed by reductio ad absurdum. Suppose there exists other entries so $g_m - g_i = \alpha^a$ and $g_m - g_k = \gamma^r$, where $m_{mi} = \gamma^a$ and $m_{mk} = \alpha^r$. That is, $m_{mi}m_{mk}^{-1} = \alpha^{a-r}$ where $a - r = b - s$. Then $g_k - g_i = \gamma^a - \gamma^r = \gamma^r(\gamma^{a-r}-1)$ and $g_k - g_i = \gamma^b - \gamma^s = \gamma^s(\gamma^{b-s}-1)$. So $s = r$ and $a = b$. But this means there were no other entries. Hence each of the $p^r - 2$ elements $m_{ij}m_{kj}^{-1}$, $i \neq j$, $k \neq j$, $j = 1,\ldots,q$ is different. It is not possible for $m_{ij} = m_{kj}$ so the $p^r - 2$ elements are $\alpha,\ldots,\alpha^{q-1}$. The 1 comes from $m_{i0}m_{k0}^{-1}$.

We saw that when $q = p^r - 1$ the $p^r - 2$ elements $m_{ij}m_{kj}^{-1}$, $i \neq j$, $k \neq j$, $j = 1,\ldots,q$ where $\alpha,\ldots,\alpha^{q-1}$. Hence if $q_1 \mid p^r - 1$ so $\alpha^{q_1} = 1$ these $p^r - 2$ elements will be $\alpha,\ldots,\alpha^{q_1-1}$ ($(p^r-1)/q_1$ times) and 1 ($(p^r-1)/q_1 - 1$ times). The additional 1 comes from $m_{i0}m_{k0}^{-1}$.

So we have a generalised $GW(p^r+1,p^r,Z_q)$. The matrix is balanced as the underlying SBIBD is (p^r+1,p^r,p^r-1).

Remark. This construction was first given for $q = 4$ in [19, p.297].

3. SOME GENERALISED HADAMARD MATRICES $GH(p^r,C_{p^r})$ and $GH(p^r(p^r-1),C_{p^r})$

The $GH(p^r,Z_p\times\ldots\times Z_p)$ was first noted by Drake [8] but we give it here for illustrative purposes.

Let x be a primitive element of $GF(p^r)$. We form

$$X = \left(x^{j-i+1(\text{mod } p^r-1)}\right) .$$

Now the generalised Hadamard matrix on the elementary abelian group in additive form is formed by reducing the elements of X modulo a primitive polynomial and adding a zeroth row and column which is the additive identity. This matrix can now be written multiplicatively to obtain $GH(p^r,Z_p\times Z_p\times\ldots\times Z_p)$.

For example, let x be a primitive element of $GF(3^2)$. We form

$$X = \begin{array}{cccccc} x & x^2 & x^3 & \ldots & x^8 \\ x^8 & x & x & \ldots & x^7 \\ \vdots & & & & \\ x^2 & x^3 & x^4 & \ldots & x \end{array}$$

then the generalised Hadamard matrix using $x^2 = x + 1$ is

$$\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & x+1 & 2x+1 & 2 & 2x & 2x+2 & x+2 & 1 \\ 0 & 1 & x & x+1 & 2x+1 & 2 & 2x & 2x+2 & x+2 \\ 0 & x+2 & 1 & x & x+1 & 2x+1 & 2 & 2x & 2x+2 \\ \vdots & & & & & & & & \\ 0 & x+1 & 2x+1 & 2 & 2x & 2x+2 & x+2 & 1 & x \end{array}$$

or in multiplicative form

$$\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & ab & a^2b & b^2 & a^2 & a^2b^2 & ab^2 & b \\ 1 & b & a & ab & a^2b & b^2 & a^2 & a^2b^2 & ab^2 \\ 1 & ab^2 & b & a & ab & a^2b & b^2 & a^2 & a^2b^2 \\ \vdots & & & & & & & & \\ 1 & ab & a^2b & b^2 & a^2 & a^2b^2 & ab^2 & b & a \end{array}$$

The corresponding matrices, if x (=3) is a primitive element of GF(5), are

$$\begin{array}{cccc} & & & \\ x & x^2 & x^3 & x^4 \\ x^4 & x & x^2 & x^3 \\ x^3 & x^4 & x & x^2 \\ x^2 & x^3 & x^4 & x \end{array} \qquad \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 2 & 1 \\ 0 & 1 & 3 & 4 & 2 \\ 0 & 2 & 1 & 3 & 4 \\ 0 & 4 & 2 & 1 & 3 \end{array} \qquad \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & a^3 & a^4 & a^2 & a \\ 1 & a & a^3 & a^4 & a^2 \\ 1 & a^2 & a & a^3 & a^4 \\ 1 & a^4 & a^2 & a & a^3 \end{array}$$

For reference purposes we note the following theorem. A direct proof of (ii), inspired by Rajkundlia, will appear elsewhere.

<u>Theorem 2.</u> *(i) Suppose* p^r *is a prime power. Then there is a* $GH(p^r, C_{p^r})$ *where* C_{p^r} *is the elementary abelian group.*

(ii) Suppose p^r *and* $p^r - 1$ *are both prime powers. Then there is a* $GH(p^r(p^r-1), C_{p^r})$ *where* C_{p^r} *is the elementary abelian group.*

Example of construction of $GH(12, Z_2 \times Z_2)$

	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

has core $C = \begin{array}{ccc} e & ab & b \\ ab & e & a \\ b & a & e \end{array}$

The generalised Hadamard matrix of order 4:

$$\begin{matrix} e & e & e & e \\ e & a & b & ab \\ e & b & ab & a \\ e & ab & a & b \end{matrix} \quad \text{has core } K = \begin{matrix} a & b & ab \\ b & ab & a \\ ab & a & b \end{matrix}$$

Let I, T, T^2 of order 3 be a matrix representation of ε, w, w^2 where w is a cube root of unity, then

$$W = \begin{matrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & w & w^2 \\ \varepsilon & w^2 & w \end{matrix}$$

is a generalised Hadamard matrix of order 3.

Now define

$$C{*}W = \begin{matrix} e\varepsilon & ab\varepsilon & b\varepsilon \\ ab\varepsilon & ew & aw^2 \\ b\varepsilon & aw^2 & ew \end{matrix}$$

and

$$D = \begin{matrix} eK & abK & bK \\ abK & eKT & aKT^2 \\ bK & aKT^2 & eKT \end{matrix}$$

and the following is the required matrix:

$$\begin{matrix} e & e & e & \underset{\sim}{e} & \underset{\sim}{e} & \underset{\sim}{e} \\ e & e & e & \underset{\sim}{b} & \underset{\sim}{ab} & \underset{\sim}{a} \\ e & e & e & \underset{\sim}{ab} & \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{e}' & \underset{\sim}{b}' & \underset{\sim}{ab}' & eK & abK & bK \\ \underset{\sim}{e}' & \underset{\sim}{ab}' & \underset{\sim}{a}' & abK & eKT & aKT^2 \\ \underset{\sim}{e}' & \underset{\sim}{a}' & \underset{\sim}{b}' & bK & aKT^2 & eKT \end{matrix}$$

where $\underset{\sim}{a} = [a \ a \ a]$ and $\underset{\sim}{a}' = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$. Explicitly

$$G = \begin{matrix}
e & e & e & e & e & e & e & e & e & e & e & e \\
e & e & e & b & b & b & ab & ab & ab & a & a & a \\
e & e & e & ab & ab & ab & a & a & a & b & b & b \\
e & ab & b & a & b & ab & b & a & e & ab & e & a \\
e & ab & b & ab & a & b & e & b & a & a & ab & e \\
e & ab & b & b & ab & a & a & e & b & e & a & ab \\
e & b & a & ab & e & a & b & e & ab & b & ab & a \\
e & b & a & a & ab & e & ab & b & e & a & b & ab \\
e & b & a & e & a & ab & e & ab & b & ab & a & b \\
e & a & ab & b & a & e & b & ab & a & b & e & ab \\
e & a & ab & e & b & a & a & b & ab & ab & b & e \\
e & a & ab & a & e & b & ab & a & b & e & ab & b
\end{matrix}$$

is a $GH(12, Z_2 \times Z_2)$.

4. USING $GW(v,k,G)$ TO CONSTRUCT SBIBD

Write P for the matrix with 1 where David Glynn's $GW(13,9,S_3)$ has zeros and 0 where the GW is non-zero and $e = (1,1,1,1,1,1)$. Then, as Glynn observed,

$$DG = \begin{bmatrix} P^T & I_{13} \times e \\ I_{13} \times e^T & \begin{matrix} GW(13,9,S_3) \text{ with the} \\ \text{group elements replaced} \\ \text{by their permutation} \\ \text{matrix representation} \end{matrix} \end{bmatrix}$$

is the incidence matrix of the Hughes plane of order 9.

In general, we can say

Lemma 3. *Suppose there exists a* $GW(p^2+p+1, p^2, G)$, $|G| = p(p-1)$. *Then forming* DG *similarly to the above we have the incidence matrix of a tangentially transitive projective plane of order* p^2.

Remark. If G is an "interesting group" then the related projective plane will also be "interesting".

We now give some other constructions using generalised weighing matrices.

Theorem 4. *Suppose there is a generalised balanced weighing matrix* $W = GW(v,k,Z_d)$ *with entries,* θ^i, *which are* d^{th} *roots of unity. Suppose the underlying SBIBD has parameters* (v,k,λ). *Then if* $d(v-k) = k-1$ *there exists a BIBD*

$$(vd^2, vd(d+1), k(d+1), kd, k)$$

and an SBIBD

$$(vd(d+1)+1, vd+1, k).$$

Proof. Each entry θ^i, of the $GW(v,k,Z_d)$, is first replaced by $\theta^i GH(d,Z_d)$ where $GH(d,Z_d)$ is the generalised Hadamard matrix. Now W_B of order vd^2 with kd ones per row and column is formed by replacing each element, θ^i, by its permutation matrix representation A_i of order d. W_B has inner products $0,k,\lambda$.

W_A and W_C are formed by replacing 0 by 0_d, the $d \times d$ zero matrix, and θ^i by $e \times A_i$ and $e^T \times A_i$ respectively in W, with e the $d \times 1$ matrix of ones. Now W_A has inner products $k, 0, \lambda/d$, is of size $vd^2 \times vd$, and has k ones per row and kd ones per column. $\begin{bmatrix} W_A & W_B \end{bmatrix}$ is the required BIBD.

The matrix W_D is now obtained by replacing each zero element of W by J_d the $d \times d$ matrix of ones and each non-zero element by 0_d. Then, with f the $1 \times vd$ matrix of ones,

$$\begin{bmatrix} 1 & f & 0 \\ f^T & W_D & W_C \\ 0 & W_A & W_B \end{bmatrix}$$

is the required SBIBD.

Example. Berman has shown that there is a circulant matrix $W = W\left((2^{t+1}-1)/3, 2^{t-}\right)$, $t \geq 3$ odd, with entries the cube roots of unity $1, \omega, \omega^2$. Since 3 is prime, W is a balanced $GW\left((2^{t+1}-1)/3, 2^{t-1}, Z_3\right)$. We replace each element ω^i by

$$\omega^i \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

and 0 by 0_3. We form W_B by replacing ω^i by $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^i$ and 0 by 0_3.

W_A and W_C are obtained by replacing 0 by 0_3 and ω^i by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^i \quad \text{and } [1\ 1\ 1] \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^i$$

respectively.

Since W is orthogonal, the inner product of any two rows is a multiple λ of $1 + \omega + \omega^2$. Further, since replacing $1, \omega, \omega^2$ by 1 gives the incidence matrix of a $\left((2^{t+1}-1)/3, 2^{t-1}, 3.2^{t-3}\right)$ difference set, we see $\lambda = 2^{t-3}$. Now W_B is of order $3(2^{t+1}-1)$ has 3.2^{t-1} ones per row and column and has inner products of rows $0, 2^{t-1}$ or 3.2^{t-3}; W_A is of size $3(2^{t+1}-1) \times (2^{t+1}-1)$, has 2^{t-1} ones per row and 2^{t-1}, 0 or 2^{t-3}; W_C is of size $(2^{t+1}-1) \times 3(2^{t+1}-1)$, has 3.2^{t-1} ones per row and 2^{t-1} ones per column; further, it has inner products 0 or 3.2^{t-3}.

$\begin{bmatrix} W_A & W_B \end{bmatrix}$ is a BIBD $\left(3(2^{t+1}-1), 2^{t+3}-4, 2^{t+1}, 3.2^{t-1}, 2^{t-1}\right)$.

We form W_D by replacing the zeros of W by J_3 and all other elements by 0_3. Since W_D is based on a $\left((2^{t+1}-3)/3, (2^{t-1}-1)/3, (2^{t-3}-1)/3\right)$ difference set, it has $2^{t-1}-1$ ones per row and column and inner products $2^{t-3}-1$ and $2^{t-1}-1$. So with e the $1 \times (2^{t+1}-1)$ matrix of ones we have

$$\begin{bmatrix} 1 & e & 0 \\ e^T & W_D & W_C \\ 0 & W_A & W_B \end{bmatrix}$$

is the incidence matrix of a $(2^{t+3}-3, 2^{t+1}, 2^{t-1})$ SBIBD.

So we have a new proof of a case of a theorem of Radjundlia.

Corollary 5. *Let* $t > 3$ *be odd. Then there exists an SBIBD with parameters* $(2^{t+3}-3, 2^{t+1}, 2^{t-1})$.

Example. Berman exhibits a $W(16,21)$ with entries which are cube roots of unity. Since $d = 3$, $v = 21$, $k = 16$ satisfies $3(21-16) = 16-1$, the theorem tells us there is an SBIBD (253,64,16).

Corollary 6. *Suppose there is a* $GW(p+1,p,Z_{p-1})$. *Then there exists an SBIBD with parameters* $(p(p^2-1)+1, p^2, p)$. *In particular, an SBIBD* $(p(p^2-1)+1, p^2, p)$ *exists whenever* p *is a prime power.*

This family of SBIBDs has recently been found by Becker and Piper [1] and in more general form by Rajkundlia.

Theorem 7. *Suppose there is a balanced generalised weighing matrix* $GW(v,k,Z_d)$. *Suppose the underlying SBIBD has parameters* (v,k,λ). *Then if* $v-1 = (v-k)(d-1)$ *there exists an SBIBD*

$$(dv, k+d(v-k), d(v-k)).$$

Proof. Replace each non-zero element by its $d \times d$ permutation matrix representation and each zero element by the $d \times d$ matrix of ones.

Berman found circulant $W((2^{t+1}-1)/3, 2^{t-1})$, $t \geq 3$ odd, with entries which are cube roots of unity. Since 3 is a prime, this matrix is a balanced $GW((2^{t+1}-1)/3, 2^{t-1}, Z_3)$. This satisfies the conditions of the theorem and so we have the family of SBIBDs $(2^{t+1}-1, 2^t-1, 2^{t-1}-1)$ which is, of course, well-known.

Corollary 8. *Suppose there exists a* $GW(p^2+1, p^2, Z_{p+1})$. *Then there exists an SBIBD* $\left(\frac{p^4-1}{p-1}, \frac{p^3-1}{p-1}, \frac{p^2-1}{p-1}\right)$.

This gives the well-known family of SBIBDs $\left(\frac{p^4-1}{p-1}, \frac{p^3-1}{p-1}, \frac{p^2-1}{p-1}\right)$ when p is a prime power for in this case we know the $GW(p^2+1, p^2, Z_{p+1})$ exists from Theorem 1.

5. USING GENERALISED HADAMARD MATRICES

We now give an alternate construction for the SBIBD of Corollary 6.

Theorem 9. *Suppose there exists a generalised Hadamard matrix* $GH(qp^i(p-1), C_p)$ *where* C_p *is an abelian group. Further, suppose an SBIBD* $(p(qp^i-1), qp^i, qp^{i-1})$ *exists with incidence matrix containing* $M_1 = J_{qp^i-p+1}$. *Then there exists an SBIBD* $(p(qp^{i+1}-1)+1, qp^{i+1}, qp^i)$.

Proof. Let e_t be the $1 \times t$ matrix of ones and J_t the $t \times t$ matrix of ones. Let 0_a, 0_b and 0_c be zero matrices of sizes $x \times y$, $y \times x$ and $y \times y$ respectively, where $x = p(qp^i-p+1)$ and $y = p^2-2p+1$. Let $A_1, \ldots, A_p$ be the $p \times p$ permutation matrix representation of C_p. Write GH(A) for the (0,1) matrix obtained by replacing each element of C_p by its appropriate matrix representation. Then GH(A) is a symmetrical group divisible design with parameters

$$\left(qp^{i+1}(p-1), qp^{i+1}(p-1), qp^{i}(p-1), qp^{i}(p-1), 0, qp^{i-1}(p-1), qp^{i}(p-1), p\right).$$

We write the incidence matrix of the SBIBD $\left(p(qp^{i}-1)+1, qp^{i}, qp^{i-1}\right)$ as

$$\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} = \begin{bmatrix} M_1 & X \\ M_3 & \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \\ & Y & \end{bmatrix} ,$$

where M_1 is $(qp^{i}-p+1) \times (qp^{i}-p+1)$, M_2 is $(qp^{i}-p+1) \times qp^{i}(p-1)$, M_3 is $qp^{i}(p-1) \times (qp^{i}-p+1)$ and M_4 is $qp^{i}(p-1) \times qp^{i}(p-1)$. Now form

$$M = \left[\begin{array}{cc|c} M_1 \times J_p & 0_a & \\ & & e_p \times X \\ 0_b & 0_c & \\ \hline & & \\ \multicolumn{2}{c|}{e_p^T \times Y} & GH(A) \end{array}\right]$$

which is the incidence matrix of the required SBIBD.

We note in passing that

$$[e_p^T \times Y \quad GH(A)]$$

is a pairwise balanced design $\left(qp^{i+1}(p-1); qp^{i+1}, qp^{i}(p-1); qp^{i}\right)$.

In particular, we note that if $q = 1$ and p and $p-1$ are both prime powers, the $GH\left(p^{i}(p-1), C_p\right)$ exists for all positive i, as does the SBIBD $(p^2-p+1, p, 1)$. So an SBIBD $\left[p(p^2-1)+1, p^2, p\right]$ of the right form exists by the theorem. Hence, by induction we have Rajkundlia's theorem as a corollary.

<u>Corollary 9</u>. *Suppose* p *and* p - 1 *are prime powers. Then there exists an SBIBD* $\left(p(p^{i+1}-1)+1, p^{i+1}, p^{i}\right)$ *for all positive* i.

Example.

$$\left[\begin{array}{ccccccc|cccccc}
1 & 1 & 1 & & & & & e & e & & & & \\
1 & 1 & 1 & & & & & & & e & e & & \\
1 & 1 & 1 & & & & & & & & & e & e \\
 & & & & & & & e & & e & & & \\
 & & & & & & & e & & & e & & e \\
 & & & & & & & & e & e & & & e \\
 & & & & & & & & e & & e & e & \\
\hline
f & & & f & f & & & I & I & I & I & I & I \\
f & & & & & f & f & I & I & T & T^2 & T^2 & T \\
 & f & & f & & f & & I & T & I & T & T^2 & T^2 \\
 & f & & & f & & f & I & T^2 & T & I & T & T^2 \\
 & & f & f & & & f & I & T^2 & T^2 & T & I & T \\
 & & f & & f & f & & I & T & T^2 & T^2 & T & I
\end{array}\right]$$

where $e = [1\ 1\ 1]$ and $f = \begin{bmatrix}1\\1\\1\end{bmatrix}$ is a (25,9,3).

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