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Ordered Partitions and Codes Generated by Circulant Matrices

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We consider the set of ordered partitions of n into m parts acted upon by the cyclic permutation $(12...m)$. The resulting family of orbits $\bar{\mathscr{P}}(n, m)$ is shown to have cardinality $\bar{p}(n, m) = (1/n) \sum_{d|m} \phi(d) \binom{n/d}{m/d}$, where ϕ is Euler's ϕ-function. $\bar{\mathscr{P}}(n, m)$ is shown to be set-isomorphic to the family of orbits $\bar{\mathscr{C}}(n, m)$ of the set of all m-subsets of an n-set acted upon by the cyclic permutation $(12...n)$. This isomorphism yields an efficient method for determining the complete weight enumerator of any code generated by a circulant matrix.

1. INTRODUCTION

An ordered partition (or composition, cf. [2] or m-composition, cf. [1]) of n into m parts is an ordered m-tuple $\alpha = (k_1, k_2, ..., k_m)$, where the k_i are positive integers and $k_1 + k_2 + \cdots + k_m = n$. In this paper we consider the set $\mathscr{P}(n, m)$ of all ordered partitions of n into m parts acted upon by the cyclic permutation

$$\theta = (12 \cdots m).$$

The action of group G generated by θ is defined by

$$\theta\alpha = (k_{1\theta}, k_{2\theta}, ..., k_{m\theta})$$

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and we write $\bar{\mathcal{P}}(n, m)$ for the set of orbits of G under this action. The cardinalities of $\mathcal{P}(n, m)$ and $\bar{\mathcal{P}}(n, m)$ will be denoted by $p(n, m)$ and $\bar{p}(n, m)$, respectively. Writing $\bar{p}_d(n, m)$ for the number of orbits in $\bar{\mathcal{P}}(n, m)$ having exactly d-elements, we derive in Section 3 the identities

$$\bar{p}_m(n, m) = \frac{1}{n} \sum_{d|m} \mu(d) \binom{n/d}{m/d} \tag{1.1}$$

and

$$\bar{p}(n, m) = \frac{1}{n} \sum_{d|m} \phi(d) \binom{n/d}{m/d}, \tag{1.2}$$

where μ is the Möbius function, ϕ is Euler's ϕ-function, and $\binom{n/d}{m/d}$ is defined to be zero unless d is a divisor of both n and m.

The initial reason for our interest in the set $\bar{\mathcal{P}}(n, m)$ is due to the fundamental relationship between $\bar{\mathcal{P}}(n, m)$ and the set of all m-subsets of a given n-set. Write S for the set of integers $\{1, 2, \ldots, n\}$ and $\mathcal{C}(n, m)$ for the set of all m-subsets of S. Let H be the cyclic group generated by the permutation

$$\psi = (12 \cdots n).$$

For $l = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, any element of $\mathcal{C}(n, m)$, we define the action of H on $\mathcal{C}(n, m)$ by

$$\psi l = \{\alpha_1\psi, \alpha_2\psi, \ldots, \alpha_m\psi\}, \tag{1.3}$$

i.e.,

$$\alpha_i\psi = \alpha_i \dotplus 1 \qquad \text{(modulo } n\text{)}.$$

The set $\bar{\mathcal{C}}(n, m)$ of orbits of H is shown in Section 2 to be set-isomorphic to $\bar{\mathcal{P}}(n, m)$, and the properties of the isomorphism are studied in some detail.

The isomorphism between $\bar{\mathcal{C}}(n, m)$ and $\bar{\mathcal{P}}(n, m)$ yields an efficient method for determining the complete weight enumerator of any code generated by the row vectors of a circulant matrix or a matrix of the form $[IW]$, where I is the $n \times n$ identity matrix and W is an $n \times n$ circulant matrix. This application is discussed in Section 4.

2. The Relationship between Ordered Partitions and m-Sets

The purpose of this section is to establish the fundamental relationship between the two sets $\bar{\mathcal{P}}(n, m)$ and $\bar{\mathcal{C}}(n, m)$. We will denote the cardinalities of $\mathcal{C}(n, m)$ and $\bar{\mathcal{C}}(n, m)$ by $c(n, m)$ and $\bar{c}(n, m)$, respectively. The number of orbits in $\bar{\mathcal{C}}(n, m)$ with d elements will be denoted by $\bar{c}_d(n, m)$.

Each m-subset of S has a natural ordering. Let $l = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. Associated with l we have the ordered partition of n into m parts

$$\alpha(l) = (d_1, d_2, \ldots, d_m) \tag{2.1}$$

defined by

$$d_i = \alpha_{i+1} - \alpha_i \qquad \text{for} \quad i = 1, \ldots, m-1,$$
$$d_m = n - \alpha_m + \alpha_1 .$$

Also, with each ordered partition $\alpha = (k_1, k_2, \ldots, k_m)$ we associate the m-set

$$l(\alpha) = \{1, 1 + k_1, \ldots, 1 + k_1 + k_2 + \cdots + k_{m-1}\}. \tag{2.2}$$

We prove next that (2.1) and (2.2) yield a bijection between the sets $\mathscr{P}(n, m)$ and $\mathscr{C}(n, m)$.

LEMMA 2.1. *The ordered partitions associated with a class in $\mathscr{C}(n, m)$ are contained in a class in $\mathscr{P}(n, m)$.*

Proof. Let $l = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_m \leqslant n$, and let $\alpha(l) = (d_1, d_2, \ldots, d_m)$ be defined by (2.1). Then

$$\psi^k l = \{\alpha_1 + k, \alpha_2 + k, \ldots, \alpha_m + k\},$$

where the elements are reduced modulo n. In natural order

$$\psi^k l = \{\alpha_t + k, \alpha_{t+1} + k, \ldots, \alpha_m + k, \alpha_1 + k, \ldots, \alpha_{t-1} + k\},$$

for some integer t. Hence the ordered partition associated with $\psi^k l$ is

$$\alpha(\psi^k l) = (d_t, \ldots, d_{m-1}, \alpha_1 - \alpha_m, d_1, \ldots, d_{t-2}, n - \alpha_{t-1} - k + \alpha_t + k).$$

But

$$\alpha_1 - \alpha_m \equiv d_m \qquad (\text{mod } n)$$

and

$$n - \alpha_{t-1} - k + \alpha_t + k \equiv d_{t-1} \qquad (\text{mod } n),$$

and so

$$\alpha(\psi^k l) = \theta^{t-1}\alpha(l), \tag{2.3}$$

which proves the assertion of the lemma.

LEMMA 2.2. *The m-sets associated with a class in $\mathscr{P}(n, m)$ are contained in a class in $\mathscr{C}(n, m)$. In particular*

$$l(\theta^i\alpha) = \psi^{b_i} l(\alpha) \tag{2.4}$$

for $i = 0, 1, \ldots, m-1$, where $b_i = k_{i+1} + k_{i+2} + \cdots + k_m$.

Proof. By definition

$$\psi^{b_i} l(\alpha) = \{1 + b_i, 1 + b_i + k_1, \ldots, 1 + b_i + k_1 + \cdots + k_{m-1}\}.$$

Since

$$1 + b_i + k_1 + \cdots + k_i \equiv 1 \pmod{n}$$

we have in natural order

$$\begin{aligned}\psi^{b_i} l(\alpha) = \{&1, 1 + k_{i+1}, \ldots, 1 + k_{i+1} + \cdots + k_{m-1}, 1 + k_{i+1} + \cdots + k_m, \\ &1 + k_{i+1} + \cdots + k_m + k_1, \ldots, 1 + k_{i+1} + \cdots + k_m + k_1 + \cdots \\ &+ k_{i+1}\} \\ = l(\theta^i\alpha).&\end{aligned}$$

THEOREM 2.1. *Define f: $\mathscr{P}(n, m) \to \mathscr{C}(n, m)$ by*

$$f[\alpha] = [l(\alpha)] \tag{2.5}$$

and define

$$g\colon \mathscr{C}(n, m) \to \mathscr{P}(n, m)$$

by

$$g[\alpha] = [\alpha(l)], \tag{2.6}$$

where the representative l contains 1.

Then f and g are well defined and $f \circ g = 1$, $g \circ f = 1$.

Proof. f is well defined by Lemma 2.2 and g is well defined by Lemma 2.1; hence it suffices to prove that f and g are mutual inverses.

Let $l = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and write $[l]$ for the corresponding class in $\mathscr{C}(n, m)$. Then for $\alpha(l) = (d_1, d_2, \ldots, d_m)$ defined by (2.1) we have that

$$l(\alpha(l)) = \psi^{1-\alpha_1} l;$$

hence $[l(\alpha(l))] = [l]$ and so $f \circ g = 1$.

On the other hand, let $\alpha = (k_1, k_2, \ldots, k_m)$. Then by (2.2)

$$l(\alpha) = \{1, 1 + k_1, \ldots, 1 + k_1 + \cdots + k_{m-1}\}$$

and by (2.1)

$$\alpha(l(\alpha)) = (d_1, d_2, \ldots, d_m),$$

where

$$d_1 = 1 + k_1 - 1 = k_1, \quad d_2 = 1 + k_1 + k_2 - 1 - k_1 = k_2, \ldots, d_{m-1} = k_{m-1}$$

and

$$d_m = n - (1 + k_1 + \cdots + k_{m-1}) + 1 = k_m.$$

Hence

$$\alpha(l(\alpha)) = \alpha,$$

and so $[\alpha(l(\alpha))] = [\alpha]$, which proves that $g \circ f = 1$. This completes the proof of the theorem.

An immediate consequence of Theorem 2.1 is

$$\bar{p}(n, m) = \bar{c}(n, m). \tag{2.7}$$

The next theorem shows that the bijection f preserves, in a sense, the class size.

THEOREM 2.2. *Let f be the mapping defined by Eq. (2.5) and suppose k is a divisor of m. If $[\alpha] \in \bar{\mathcal{P}}(n, m)$ is a class containing m/k elements then the class $f[\alpha]$ contains n/k elements.*

Proof. Suppose $[\alpha]$ contains m/k elements. Then

$$\alpha = (k_1, \ldots, k_d, k_1, \ldots, k_d, \ldots, k_1, \ldots, k_d),$$

where $d = m/k$ and each d-tuple $(k_1, \ldots, k_d)$ is an ordered partition of n/k into m/k parts whose class in $\bar{\mathcal{P}}(n/k, m/k)$ contains exactly m/k elements. Write $h = n/k$. Then

$$l(\alpha) = \{1, 1 + k_1, \ldots, 1 + k_1 + \cdots + k_{d-1}, 1 + h, 1 + h + k_1, \ldots, 1 + (k - 1)h + k_1 + \cdots + k_{d-1}\}.$$

Hence $\psi^h l(\alpha) = l(\alpha)$, from which it follows that

$$f[\alpha] = [l(\alpha)] \text{ contains } h = n/k \text{ distinct elements.}$$

COROLLARY. *The following identity holds for $k \mid (m, n)$,*

$$\bar{c}_{n/k}(n, m) = \bar{p}_{m/k}(n, m).$$

To each m-subset l of S there corresponds the $(n - m)$-subset $S - l$. This correspondence defines a natural bijection between $\mathscr{C}(n, m)$ and $\mathscr{C}(n, n - m)$. Moreover since

$$S - \psi l = \psi S - \psi l = \psi(S - l)$$

the mapping

$$t: \bar{\mathscr{C}}(n, m) \to \bar{\mathscr{C}}(n, n - m), \tag{2.8}$$

defined by

$$t[l] = [S - l],$$

is well defined and is a bijection.

Incorporating the results of Theorem 2.1 we have the commutative diagram

$$\begin{array}{ccc} \bar{\mathscr{C}}(n, m) & \xrightarrow{t} & \bar{\mathscr{C}}(n, n - m) \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle g} \\ \bar{\mathscr{P}}(n, m) & \xrightarrow[g\circ t\circ f]{} & \bar{\mathscr{P}}(n, n - m) \end{array} \tag{2.9}$$

where $g \circ t \circ f$: $[\alpha] \to [\alpha(S - l(\alpha))]$.

Since f, t, and g are bijections we can conclude that $g \circ t \circ f$ is also. Suppose next that $[l]$ is a class in $\bar{\mathscr{C}}(n, m)$ with n/k elements; then if $h = n/k$ we have

$$\psi^h l = l$$

and consequently

$$S - l = S - \psi^h l = \psi^h(S - l).$$

This shows that classes with n/k elements in $\bar{\mathscr{C}}(n, m)$ are in one–one correspondence with classes having n/k elements in $\bar{\mathscr{C}}(n, n - m)$.

Hence we have the following theorem.

THEOREM 2.3. *The mapping $g \circ t \circ f$ defined in* (2.9) *is a bijection between $\bar{\mathscr{P}}(n, m)$ and $\bar{\mathscr{P}}(n, n - m)$ which maps classes containing m/k elements to classes containing $(n - m)/k$ elements.*

COROLLARY. (1) $\bar{c}(n, m) = \bar{c}(n, n - m)$,

(2) $\bar{p}(n, m) = \bar{p}(n, n - m)$,

(3) $\bar{p}_{m/k}(n, m) = \bar{p}_{(n-m)/k}(n, n - m)$.

3. The Cardinality of $\mathscr{P}(n, m)$

In this section we derive (1.1) and (1.2). Since $p(n, m)$ can be interpreted as the number of ways of inserting $m - 1$ commas into $n - 1$ places [2] we have

$$p(n, m) = \binom{n-1}{m-1} = \frac{m}{n}\binom{n}{m}. \tag{3.1}$$

Also, the cardinality of each orbit is a divisor of m. Hence we immediately have the equations

$$\frac{m}{n}\binom{n}{m} = p(n, m) = \sum_{d|m} d\bar{p}_d(n, m) \tag{3.2}$$

and

$$\bar{p}(n, m) = \sum_{d|m} \bar{p}_d(n, m). \tag{3.3}$$

Perhaps the most elegant way to obtain (1.1) is to observe that $p((n/m)k, k)$ is defined for all positive integers k, if we let $p((n/m)k, k) = 0$ whenever $(n/m)k$ is not an integer; i.e., we define $\binom{nk/m}{k} = 0$ if nk/m is not an integer. Moreover, $\bar{p}_d(n, m)$ is defined for all positive integers d, being equal to 0 whenever d is not a divisor of (n, m), the greatest common divisor of n and m. With these observations, we may invert (3.2) to obtain

$$m\bar{p}_m(n, m) = \sum_{d|m} \mu(d)\, p\left(\frac{n}{m} \cdot \frac{m}{d}, \frac{m}{d}\right). \tag{3.4}$$

Equation (1.1) is a trivial consequence of (3.1) and (3.4).

To obtain (1.2) we recall that G, the cyclic group of order m, acts on the set $\mathscr{P}(n, m)$ of all ordered partitions of n into m parts. Let $\lambda(g)$ denote the number of elements of $\mathscr{P}(n, m)$ fixed by the permutation $g \in G$. If $g = e$, the identity element, then

$$\lambda(g) = \binom{n-1}{m-1}$$

since e fixes every ordered partition. If g consists of d-cycles then g fixes only those ordered partitions which are repeated copies of ordered partitions of n/d into m/d parts. For example, (1, 3, 2, 1, 3, 2, 1, 3, 2) is fixed by (147)(258)(369) = $(123456789)^3$. But the number of permutations of G consisting of d-cycles is $\phi(d)$. Hence by Burnside's lemma

$$\bar{p}(n, m) = \frac{1}{m}\sum_{d|m} \phi(d)\binom{n/d-1}{m/d-1} = \frac{1}{n}\sum_{d|m} \phi(d)\binom{n/d}{m/d}.$$

As an example suppose that $n = 24$ and $m = 4$. Then

$$\bar{p}(24, 4) = \frac{1}{24}\left[\phi(1)\binom{24}{4} + \phi(2)\binom{12}{2} + \phi(4)\binom{6}{1}\right]$$
$$= \frac{1}{24}\left[\binom{24}{4} + \binom{12}{2} + 2\binom{6}{1}\right] = 446.$$

The following corollaries may serve as further illustrations.[1]

COROLLARY 1. *If n and m are relatively prime then*

$$\bar{p}(n, m) = \bar{p}_m(n, m) = \frac{1}{n}\binom{n}{m}.$$

COROLLARY 2. *If $(n, m) = q$ is a prime then*

$$\bar{p}(n, m) = \frac{1}{n}\binom{n}{m} + \frac{q-1}{n}\binom{n/q}{m/q}.$$

COROLLARY 3.

$$\bar{p}(n, 3) = \frac{1}{n}\binom{n}{3} \qquad \text{if} \quad (n, 3) = 1$$
$$= \frac{1}{n}\binom{n}{3} + \frac{2}{3} \qquad \text{if} \quad (n, 3) = 3,$$
$$\bar{p}(n, 4) = \frac{1}{n}\binom{n}{4} \qquad \text{if} \quad (n, 4) = 1$$
$$= \frac{1}{n}\binom{n}{4} + \frac{n}{8} - \frac{1}{4} \qquad \text{if} \quad (n, 4) = 2$$
$$= \frac{1}{n}\binom{n}{4} + \frac{n}{8} + \frac{1}{4} \qquad \text{if} \quad (n, 4) = 4.$$

4. AN APPLICATION

Let $\mathscr{C}$ be a linear code generated by the row vectors of a matrix $[IW]$, where I is $n \times n$ identity matrix and W is an $n \times n$ circulant matrix with entries in a finite field $GF(q)$. Such codes have the property that they have the same weight enumerators as their duals [4] and hence share many of the

[1] *Added in proof.* The total number of ordered partition classes of n is $\bar{p}(n) = \sum_{m=1}^{n} \bar{p}(n, m) = (1/n)\sum_{d/n} \phi(d)2^{n/d} - 1$. We are grateful to Professor G. Baron of the Technical University, Vienna, for this observation.

properties of self-dual codes. The design properties of linear codes and their subcodes of constant weight are closely related to their weight enumerators [3]. In general the problem of determining the weight enumerator (WE) of a code, or better still the complete weight enumerator (CWE), involves the determination of the WE or CWE of each of the q^n codewords. If W is circulant and W_i denotes the ith row of W then the linear combination

$$W_{i_1} + W_{i_2} + \cdots + W_{i_m}$$

has the same CWE as

$$W_{i_1+k} + W_{i_2+k} + \cdots + W_{i_m+k}$$

for any integer k, where the subscripts are reduced modulo n. Hence the codewords of $\mathscr{C}$ can be grouped into classes in which elements are "linear shifts" of one another. For given m the family of classes is in obvious correspondence with $\bar{\mathscr{C}}(n, m)$. Hence the problem of determining the CWE of $\mathscr{C}$ reduces to two problems:

(1) Finding a complete system of coset representatives of $\bar{\mathscr{C}}(n, m)$ for $m = 1,\ldots, n$.

(2) Determining the CWEs of the linear combinations corresponding to the coset representatives.

The problem of finding a complete system of coset representatives is very easy for $\bar{\mathscr{P}}(n, m)$, where such a system occurs in lexicographical order among the set of all ordered partitions of n into m parts with the first entry at most the integer part of n/m. An ordered partition in this class is a suitable representative provided that it is lexicographically less than any ordered partition in its orbit. An efficient computer algorithm exists to determine the complete system of representatives for $\bar{\mathscr{P}}(n, m)$.

We may note that in the case of binary codes Theorem 2.3 allows us to reduce the calculation time by a further factor of 2.

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