ON THE DISTRIBUTION OF THE PERMANENT OF CYCLIC (0,1) MATRICES

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ABSTRACT. Some results are obtained on the permanent of cyclic (0,1) matrices which support the conjecture that for such matrices of prime order p the number of distinct values the permanent attains is of order p.

Writing $e(n)$ for the number of distinct values the permanent of cyclic (0,1) matrices of order n can attain we found

$$e(5) = 6,\quad e(6) = 12,\quad e(7) = 9,$$
$$e(8) = 11,\quad e(9) = 21,\quad e(10) \le 44,$$
$$\text{and}\quad e(11) \le 30\,.$$

It is easy to show $e(p) \le \frac{1}{p}(2^p-2)+2$, p prime, but these answers are considerably smaller. We obtain formulae for the permanent of cyclic (0,1) matrices in several cases.

1. *Introduction.*

We consider cyclic (0,1) matrices of order n with k entries 1 in each row and column. Such matrices are of considerable interest in finite combinatorial theory. We are particularly interested in the *permanent* of such matrices. Clearly the permanent of a matrix is invariant under permutation of rows and columns. We use P_n^k to denote the permanents of those cyclic (0,1) matrices in which the first k entries of the first row are 1. These have been studied in detail by N. Metropolis, M. L. Stein, and P. R. Stein [2].

While talking with D. H. Lehmer about the distribution of the permanent of (0,1) matrices we came to believe that the permanent of the 2^n cyclic (0,1) matrices of order n actually assumed about $O(n)$ distinct values and in particular, for p prime, only about order p distinct values.

A.Schrijver [4] has given a short proof of Minc's conjecture: *If* A *is a* (0,1) *matrix of order* n *with row sums* $r_1, r_2, \ldots, r_n$ *then*

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$$\operatorname{per}(A) \le \prod_{j=1}^{n} (r_j!)^{1/r_j} .$$

In particular for $r_1 = r_2 = \dots = r_n = k$,

$$\operatorname{per}(A) \le (k!)^{n/k} .$$

In odd orders we found that for cyclic (0,1) matrices A_n^k, where n is the order and k the number of non-zero elements, max (per A_n^k) was considerably smaller than $(k!)^{n/k}$ though this became a better bound as $k \to n$.

To find the number $\bar{p}(n,k)$ of inequivalent first rows under cyclic shifts (since these clearly have the same permanent) we use the formula of Razen, Seberry, and Wehrhahn [3]

$$\text{(1)} \qquad \bar{p}(n,k) = \frac{1}{n} \sum_{d|k} \phi(d) \binom{n/d}{k/d} ,$$

which simplifies to

$$\bar{p}(n,k) = \begin{cases} \frac{1}{n}\binom{n}{k} & \text{when } \gcd(n,k) = 1 , \\ \frac{1}{n}\binom{n}{k} + \frac{q-1}{n}\binom{n/q}{k/q} & \text{when } \gcd(n,k) = \text{a prime } q. \end{cases}$$

Notation. We use per (124) to mean the permanent of the cyclic (0,1) matrix with first row (11010...0). We use per cyclic (11010...0) for exactly the same thing.

Metropolis, Stein, and Stein [2] note the following formulae (from their own work and Riordan's book):

$$P_{n+2}^3 = P_{n+1}^3 + P_n^3 - 2 , \quad n \ge 3 ;$$

$$P_{n+3}^4 = P_{n+2}^4 + P_{n+1}^4 + P_n^4 - 4 , \quad n \ge 4;$$

$$P_{n+10}^5 = 2P_{n+9}^5 + 2P_{n+8}^5 - 2P_{n+6}^5 - 8P_{n+5}^5 - 6P_{n-4}^5$$

$$- 2P_{n+3}^5 + 2P_{n+1}^5 + P_n^5 + 24 , \quad n \ge 5 ;$$

$$P^6_{n+15} = 2P^6_{n+14} + 2P^6_{n+13} + P^6_{n+12} - 4P^6_{n+10} - 18P^6_{n+9}$$
$$- 16P^6_{n+8} - 12P^6_{n+7} - 10P^6_{n+6} - 4P^6_{n+5} + 4P^6_{n+4}$$
$$+ 3P^6_{n+3} + 2P^6_{n+2} + 2P^6_{n+1} + P^6_n + 96, \quad n \geq 6 ;$$

and give an algorithm to find recurrence relations for P^k_n in other cases.

In Eades, Praeger, and Seberry [1], P^{n-1}_n (the derangement numbers) and P^{n-2}_n (the ménage numbers) are given. It is also observed that defining C^m_n to be the permanent of the matrix of order n with zero diagonal, m zeros in the above diagonal positions shown and all other elements 1, that is

$$C^m_n = \text{per}\begin{pmatrix} 0 & & & & & \\ & 0 & & & \underset{\sim}{1} & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & 00 & \\ & & & & 00 & \\ & \underset{\sim}{1} & & & \ddots & \\ & & & & & 00 \\ & & & & & 0 \end{pmatrix} \Big\} m \quad ,$$

gives, with $C^0_n = P^{n-1}_n$ and using the recurrence for derangement numbers,

$$C^1_n = C^0_n - C^0_{n-1} - C^0_{n-2} = P^{n-1}_n - P^{n-2}_{n-1} - P^{n-3}_{n-2}$$
$$= \frac{n-2}{n-1} P^{n-1}_n ;$$
$$C^m_n = C^{m-1}_n - C^{m-1}_{n-1} - C^{m-2}_{n-1} - C^{m-2}_{n-2} ;$$

and so

$$P^{n-2}_n = \text{per}\begin{pmatrix} 00 & & & \\ & 00 & \underset{\sim}{1} & \\ & & \ddots & \\ & & & 00 \\ 0 & & & 0 \end{pmatrix} = C^{n-1}_n - C^{n-2}_{n-1} \quad .$$

Also

$$C^{n-1}_n = \sum_{i=1}^{n} P^{i-2}_i \quad .$$

These matrices may be used to show per $(23\ldots x-1,x+1,\ldots n) = P_n^{n-2}+2$ when $\gcd(x-1,n) = 2$. All these results are used in the numerical results in the Appendix.

In this paper we show

(a) per $(1x) = 2^{\gcd(x-1,n)}$, n being the order of the matrix, and

(b) per $(124) = R_n$, where

$$R_n = \sum_{\substack{j=0\\2k=n+3-j}}^{k} \binom{k}{j} + 2\sum_{\substack{j=0\\2k=n-j}}^{k} \binom{k}{j} + 2\sum_{\substack{j=0\\k=n-2j-1}}^{k} \binom{k}{j} + \sum_{\substack{j=0\\k=n=2j-4}}^{k} \binom{k}{j} + 2 .$$

2. *Permanent of Cyclic (0,1) Matrix with Two Ones in Each Row.*

LEMMA 1. *Let* A *be a cyclic* $(0,1)$ *matrix of order* p, *with* 1 *in positions* 1 *and* x *of the first row, then*

$$\text{per } A = 2^k, \quad \textit{where} \quad k = \gcd(x-1,p) .$$

Proof. The permanent P_p^2 of the cyclic matrix, C_p, of order p, with first row $(110\ldots0)$ is 2.

If $x-1 \neq 1$, we rearrange the columns and rows of the matrix by first choosing the first, $(x-1)^{st}$, $2(x-1)^{st}$, ... columns; then, if $k > 1$, the second, x^{th}, $2(x-1)+1^{st}$, $3(x-1)+1^{st}$,..., third, $x+1^{st}$, $2(x-1)+2^{nd}$, ... until all the columns have been chosen. We now choose the first, $(x-1)^{st}$,..., second, x^{th}, $2(x-1)+1^{st}$,...,third, $x+1^{st}$,... rows until all the rows have been chosen.

Now if $k = 1$ the matrix we have is C_p and if $k > 1$ we have $\oplus_k C_{p/k}$ (the direct sum of k copies of $C_{p/k}$) which has permanent 2^k as required.

3. *Number of Permanents of Cyclic (0,1) matrices with Row Sum 3.*

The numerical data we have given in the tables of the Appendix shows that cyclic $(0,1)$ matrices with row sum 3 have for:

order 5 all permanents equal to per (123);
order 6 all permanents equal to per (123) or per (124) or per (135);
order 7 all permanents equal to per (123) or per (124);
order 8 all permanents equal to per (123) or per (124) or per (135);
order 9 all permanents equal to per (123) or per (124) or per (147).

Using a mapping of the set $\{d_1, d_2, d_3\}$ to $\{e_1, e_2, e_3\}$ defined by

$$e_i = d_i x + y \pmod n, \quad x,y \text{ integers modulo } n ,$$

we see every 3-typle in order

5 maps to {123},
6 maps to {123} or {124} or {135},
7 maps to {123} or {124},
8 maps to {123} or {124} or {135},
9 maps to {123} or {124} or {147}.

In another paper P. Eades, C. E. Praeger, and J. Seberry, show that these results are examples of a more general results. Hence, using (1), we have that the number of distinct values of the permanents is bounded by

$$\sum_{k=0}^{n} \bar{p}(n,k) = \frac{1}{n} \sum_{k=0}^{n} \sum_{d|k} \phi(d) \binom{n/d}{k/d} ,$$

or, for p prime, by

$$\frac{2^p - 2}{p} + 2 .$$

4. *Permanent of the Matrix with first row* (11010...0).

Define E_n and F_n to be the permanents of following (0,1) matrices:

$$E_n = \text{per}\begin{pmatrix} 101 & & & \\ 1101 & & 0 & \\ & 1101 & & \\ & & \ddots & \\ & & & 1101 \\ 0 & & & 110 \\ & & & 11 \end{pmatrix}, \quad F_n = \text{per}\begin{pmatrix} 11 & & & \\ 101 & & & \\ & 101 & 0 & \\ & 1101 & & \\ & & \ddots & \\ 0 & & & 1101 \\ & & & 11 \end{pmatrix},$$

with $E_1 = E_2 = 1, \quad E_3 = 2, \quad E_n = E_{n-1} + E_{n-3}$,

$F_1 = F_2 = 1, \quad F_3 = 2, \quad F_n = F_{n-2} + F_{n-3}$.

Then expanding the permanent R_n = per cyclic (11010...0) we have

$$R_n = E_n + 2E_{n-3} + 2F_{n-2} + F_{n-5} + 2$$

which can be used with the above relations to obtain

R_n = per cyclic (11010...0),

$R_{n+3} - R_{n+2} - R_{n+1} - R_n + R_{n-1} + R_{n-2} + R_{n-3} = 2,$

$R_5 = 13, \quad R_6 = 17, \quad R_7 = 24, \quad R_8 = 33, \quad R_9 = 45,$

$R_{10} = 65, R_{11} = 91, R_{12} = 129, R_{13} = 185.$

Solving the recurrence relations

$$E_n = E_{n-1} + E_{n-3}, \quad E_1 = E_2 = 1, \quad E_3 = 2 ,$$

and

$$F_n = F_{n-2} + F_{n-3}, \quad F_1 = F_2 = 1, \quad F_3 = 2 ,$$

we find they have generating functions

$$E(x) = (1+x+x^2)(1-x^2-x^3)^{-1} = 1 + \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} x^{j+2k-3}$$

and

$$F(x) = (1+x^2)(1-x-x^3)^{-1} = x^{-1} + \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} x^{k+2j-1} ,$$

respectively. So we have

$$E_r = \sum_{\substack{j=0 \\ 2k=r+3-j}}^{k} \binom{k}{j} \quad \text{and} \quad F_r = \sum_{\substack{j=0 \\ k=r-2j+1}}^{k} \binom{k}{j} .$$

Using this in the recurrence relation for R_n we have

$$R_r = \sum_{\substack{j=0 \\ 2k=r+3-j}}^{k} \binom{k}{j} + 2\sum_{\substack{j=0 \\ 2k=r-j}}^{k} \binom{k}{j} + 2 \sum_{\substack{j=0 \\ k=r-2j-1}}^{k} \binom{k}{j} + \sum_{\substack{j=0 \\ k=r-2j-4}}^{k} \binom{k}{j} + 2 .$$

5. Permanent of the Cyclic Matrix with Two Zeros in Each Row.

It is shown in Eades, Praeger, and Seberry that

LEMMA 2. *Let* A *be a cyclic* (0,1) *matrix of order* p *with* 0 *in positions* 1 *and* x *of the first row. Then with* k = gcd (p,x-1) = 1,2 *we find the permanent is* p_p^{p-2}, $p_p^{p-2} + 2$, *respectively.*

Example. per (12345678) = per (12345679) + 2

in order 10.

Permanent of cyclic $(0\underbrace{1\ldots1}_{k},\ 0\underbrace{1\ldots1}_{k})$.

A straightforward expansion and substitution shows

LEMMA 3. *If* M_{2k} *is the permanent of the matrix of order* 2k *of the form* $(J_k - I_k) \times J_2$ *then* M_{2k} *is given by*

$$M_{2k} = \frac{2(k-1)}{(2k-3)} \left[(2k-1)^2 M_{2k-2} + 2(2k-3)M_{2k-4} - 8(k-2)(2k-1)M_{2k-6} \right]$$

$$M_2 = 0, \qquad M_4 = 4, \qquad M_6 = 80, \qquad M_8 = 4752, \qquad M_{10} = 440192.$$

LEMMA 4. *Let* A *be a cyclic* (0,1) *matrix of order* 2k *with* 0 *in positions* 1 *and* k+1 *of the first row. Then the permanent is* M_{2k}, *where* M_{2k} *is given in the previous lemma.*

Proof. The matrix A can be put in the required form by suitable rearrangement of rows and columns.

6. *Final Remarks.*

The results quoted in this paper have been combined with a computer search undertaken at the University of Manitoba to obtain data on the distribution of the permanent for cyclic $(0,1)$ matrices. The results are given in the Appendix.

We wish to thank John Lewis for writing the programmes to search for inequivalent partitions used in the Appendix.

APPENDIX

Numerical Results

	Permanent	Number		First Rows
Order 5:	1	5		1
(6 values)	2	10		12, 13
	13	10		123, 124
	44	5		1234
	120	1		12345
Order 6:	1	6		1
(12 values)	2	6		12
	4	6		13
	8	3		14
	17	12		124
	20	6		123, 125
	36	2		135
	80	9		1234, 1245
	82	6		1235
	265	6		12345
	720	1		123456
Order 7:	1	7		1
(9 values)	2	21		12, 13, 14
	24	14		124, 126
	31	21		123, 125, 135
	144	35		1234, 1235, 1236, 1245, 1246
	579	21		12345, 12346, 12356
	1854	7		123456
	5040	1		1234567
Order 8:				
(21 values)	1	8		1
	2	16		12, 14
	4	8	$\binom{8}{2}$	13
	16	4		15
	33	32		124, 125, 126, 127
	49	16	$\binom{8}{3}$	123, 136
	81	8		135
	258	32		1235, 1237, 1246, 1257
	260	16		1236, 1245
	264	16	$\binom{8}{4}$	1234, 1247
	288	4		1256
	576	2		1357

	Permanent	Number	First Rows
Order 8:	1249	16	12346, 12347
(21 values)	1265	32 $\binom{8}{5}$	12345, 12356, 12367, 12457
(cont'd)	1313	8	12357
	4738	16	123456, 123467
	4740	8 $\binom{8}{6}$	123457
	4752	4	123567
	14833	8	1234567
	8!	1	12345678
Order 9:	1	9	1
	2	27 $\binom{9}{2}$	12, 13, 15
	8	9	14
	45	54	124, 125, 127, 128, 136, 137
	78	27 $\binom{9}{3}$	123, 126, 135
	216	3	147
	448	81	1235, 1236, 1237, 1238, 1245, 1246, 1248, 1257, 1268
	460	18 $\binom{9}{4}$	1247, 1258
	484	27	1234, 1256, 1357
	2681	18	12457, 12458
	2693	54 $\binom{9}{5}$	12346, 12348, 12356, 12357, 12368, 12378
	2702	27	12347, 12358, 12467
	2783	27	12345, 12367, 12468
	12006	54	123457, 123458, 123467, 123478, 123568, 123578
	12072	27 $\binom{9}{6}$	123456, 123468, 123567
	12096	3	124578
	43387	27 $\binom{9}{7}$	1234567, 1234568, 1234678
	43390	9	1234578
	133496	9	12345678
	9!	1	123456789
Order 10: (≤ 44 values)	1	10	1
	2	20	12, 14
	4	20 $\binom{10}{2}$	13, 15
	32	5	16
	65	80	124, 125, 126, 127, 128, 129, 136, 138
	125	20 $\binom{10}{3}$	123, 147
	169	20	135, 137
	888	≥ 20	1234, 1258
	4160	≥ 20	1267, 1368

...

at most 29 different values

...

	Permanent	Number	First rows
Order 10:	440192	5 } $\binom{10}{8}$	12346789
(≤ 44 values)	439792	20 }	12345678, 12345689
(Cont'd)	439794	20 }	12345679, 12345789
	1334961	10	123456789
	10!	1	123456789 $\overline{10}$
Order 11:	1	11	1
(≤ 30 values)	2	55	12, 13, 14, 15, 16
	91	110 } $\binom{11}{3}$	124 and 9 equivalent
	201	55 }	123 and 4 equivalent
	1632	55	1234,1259,1267,1357, 1369
		55 } $\binom{11}{4}$	1245,124$\overline{10}$,1256,1268, 1368
		110 }	1235,1237,1238,123$\overline{10}$, 1247,1258,1269,127$\overline{10}$, 1358,1359
		110 }	1236,1239,1246,1248, 1249,1257,125$\overline{10}$,126$\overline{10}$ 1279,128$\overline{10}$
	...		
	at most 18 values		
	...		
	4890741	55	123456789 and 4 equivalent
	14684570	11	123456789 $\overline{10}$
	11!	1	123456789 $\overline{10}$ $\overline{11}$

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