ON SKEW HADAMARD MATRICES

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Abstract

Recently I have proved that for every odd integer q there exists integers t and s (dependent on q) so that there is an Hadamard matrix of order $2^t q$ and a symmetric Hadamard matrix with constant diagonal order $2^s q^2$.

We conjecture that "for every odd integer q there exists an integer t (dependent on q) so that there is a skew-Hadamard matrix of order $2^t q$". This paper makes progress toward proving this conjecture. In particular we prove the result when $q \equiv 5 \pmod 8) = s^2 + 4r^2$ is a prime power and all orthogonal designs of type $(1,a,b,c,c+|r|)$, where $1+a+b+2c+|r| = 2^t$, exist in order 2^t.

1. *Introduction.*

An *orthogonal design of order* n *and type* $(u_1,u_2,\ldots,u_s)(u_i > 0)$ on the commuting variables $x_1,x_2,\ldots,x_s$ is an $n\times n$ matrix A with entries from $\{0, x_1,\ldots, x_s\}$ such that

$$AA^T = \sum_{i=1}^{s} (u_i x_i^2) I_n .$$

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In [2], where this was first defined and many examples and properties of such designs were investigated, it was shown that the numbers of variables, s, satisfies $s \le \rho(n)$, where $\rho(n)$ (Radon's function) is defined as follows:

if $n = 2^a.b$, where b is odd and, $a = 4c+d$, where $0 \le d < 4$, then

$$\rho(n) = 8c + 2^d.$$

A powerful construction for Hadamard matrices in [14] showed that the existence of orthogonal designs in powers of two was of great import. W. Wolfe and D. Shapiro showed that the cases of the problem in powers of two are crucial to the understanding of the algebraic structure involved (see [3]).

All possible designs exist in order 2, 4 and 8. The existence problem for order 16 was solved in [7] and, in [6] many designs were construced for order 32. A useful tool, *product designs*, were introduced and studied for this purpose in [6] and [8]. *Repeat designs* were introduced for the same purpose in [9] where it is shown that the following types of orthogonal designs exist (among others) in order 2^t:

(A) $(1,1,1,1,2,2,4,4,\ldots,2^{t-2},2^{t-2})$;

(B) $(1,1,2,1,2,4,8,\ldots,2^{t-3},3,6,12,\ldots,3.2^{t-3})$;

(C) $(1,1,2,4,8,\ldots,2^{t-3},2^{t-3},2^{t-2},3,3,6,\ldots,3.2^{t-4})$;

(D) $(1,1,2,4,8,\ldots,2^{t-3},3,6,9,18,\ldots,9.2^{t-5},3.2^{t-4})$;

(E) $(1,2,3,2^{t-4},3.2^{t-4},3.2^{t-3},3,3,6,6,12,12,\ldots,3.2^{t-5},3.2^{t-5})$.

We find from [6], [9], [14] that

(A1) All orthogonal designs $(1,1,a,b,c)$, $a+b+c+2 = 2^t$ exist in order 2^t.

(B1) All orthogonal designs (a,b,c), $a+b+c = 2^t$ exist in order 2^t.

(C1) All orthogonal designs (a,b,c,d), $a+b+c+d = 2^t$ exist in order 2^t, $t = 2,\dots,9$.

(D1) All orthogonal designs $(1,1,2c,2c,odd_1,odd_2)$, $2+4c+odd_1+odd_2 = 2^t$ exist in order 2^t, $t = 2,\dots,6$.

(E1) All orthogonal designs $(1,1,2c,2c,odd_1,odd_2)$, $2+4c+odd_1+odd_2 = 2^t$ exist in order 2^t for $2c = 2^a$, $1 \le a \le t-2$, and $2c = 18,34,2^{t-1}-2$, $2^{t-1}-4, 2^{t-1}-6, 2^{t-1}-8, 2^{t-1}-10, 2^{t-1}-12, 2^{t-1}-16, 2^{t-1}-18, 2^{t-1}-20$, $2^{t-1}-24, 2^{t-1}-30$.

In general we have not determined whether the orthogonal designs referred to in the theorems that follow exist but none are ruled out by the known necessary conditions.

M and N of order n are said to be *amicable orthogonal designs* of type $((m_1,\dots,m_p);(n_1,\dots,n_q))$ if M is an orthogonal design of type $(m_1,\dots,m_p)$, N is an *orthogonal design* of type $(n_1,\dots,n_q)$ and $MN^T = NM^T$. If M comprises the variables $x_1,\dots,x_p$ and N comprises the variables $y_1,\dots,y_q$ then

$$MM^T = \sum_{i=1}^{p} m_i x_i^2 I_n, \quad NN^T = \sum_{j=1}^{q} n_j y_j^2 I_n$$

and

$$ZZ^T = (m_1x_1^2 + \ldots + m_px_p^2)(n_1y_1^2 + \ldots + n_qy_q^2)I_n$$

where $Z = MN^T$. So amicability is linked with factorizing quadratic forms.

Wolfe and Shapiro (see [3]) have studied and solved the algebraic necessary conditions for amicable orthogonal designs but the sufficiency conditions are largely unresolved (see [6], [3] for partial results).

An *Hadamard matrix*, H, is an orthogonal design of order n and type (n) or alternatively a matrix with entries ± 1 satisfying $HH^T = nI_n$. H is said to be *skew-Hadamard* if $H+I$ or $H-I$ is skew-symmetric. Two Hadamard matrices $H = M+I$ and N of order n are called *amicable Hadamard matrices* if $M^T = -M$, $N^T = N$, $HN^T = NH^T$. It is shown in [3] that amicable orthogonal designs of types $((1,n-1);(n))$ in order n give amicable Hadamard matrices (they are not the same since the orthogonal designs have no symmetry or skew symmetry conditions). A *weighing matrix* $W(n,n-1)$ is an orthogonal design of order n and type $(n-1)$.

The results quoted in the abstract for Hadamard matrices and symmetric Hadamard matrices are relatively easy to find because, if h_1 and h_2 are the orders of two Hadamard matrices H_1 and H_2 then $H_1 \times H_2$ ($\times$ the Kronecker product) is an Hadamard matrix of order h_1h_2, further if H_1 and H_2 are symmetric so is $H_1 \times H_2$, and if H_1 and H_2 are symmetric with constant diagonal then so is $H_1 \times H_2$.

A similar result does not hold for skew Hadamard matrices. The Kronecker product of skew Hadamard matrices is not a skew Hadamard matrix. But if h_1 and h_2 are the orders of amicable Hadamard matrices then there are amicable Hadamard matrices of order h_1h_2, further, if g is the order of a skew Hadamard matrix there are skew Hadamard matrices of orders h_1g and h_2g. We list from [3] the orders for which amicable Hadamard matrices are known.

Summary 1.

AI	2^t	t *a non-negative integer.*
AII	$p^r + 1$	p^r (prime power) $\equiv 3 \pmod 4$.
AIII	$2(q+1)$	$2q + 1$ *is a prime power,* q(prime) $\equiv 1 \pmod 4$.
AIV	$(\lvert t\rvert+1)(q+1)$	q(prime power) $\equiv 5 \pmod 8 = s^2 + 4t^2$, $s \equiv 1 \pmod 4$, *and* $\lvert t\rvert + 1$ *is the order of amicable orthogonal designs of type* $\left((1,\lvert t\rvert);\left(\frac{1}{2}(\lvert t\rvert+1),\ \frac{1}{2}(\lvert t\rvert+1)\right)\right)$.
	$2^r(q+1)$	q(prime power) $\equiv 5 \pmod 8 = s^2 + 4(2^r-1)^2$, $s \equiv 1 \pmod 4$, r *some integer.*
AV	S	*where* S *is a product of the above orders.*

Skew-Hadamard matrices are known for the following orders (the reader should consult [15; p.451] for more details):

Summary 2.

SI	$2^t \Pi k_i$	t, r_i *all non-negative positive integers* $k_i - 1 \equiv 3 (\bmod 4)$ *a prime power.*
SII	$(p-1)^u + 1$	p *the order of a skew-Hadamard matrix,* $u > 0$ *an odd integer.*
SIII	$2(q+1)$	$q \equiv 5 \pmod 8$ *a prime power.*
SIV	$2(q+1)$	$q = p^t$ *is a prime power with* $p \equiv 5 \pmod 8$ *and* $t \equiv 2 \pmod 4$.
	$4(q+1)$	$q \equiv 9 \pmod{16}$ *a prime power.*
SV	$4m$	$m \epsilon$ {*odd integers between 3 and 25 inclusive*}.
SVI	$mn(n-1)$	n *the order of amicable orthogonal designs of types* $((1,n-1);(n))$ *and* nm *the order of an orthogonal design of type* $(1,m,mn-m-1)$; *Theorem 7.*
SVII	$(\lvert t\rvert+1)(q+1)$	$q = s^2 + 4t^2 \equiv 5 \pmod 8$ *a prime power and* $\lvert t\rvert + 1$ *the order of a skew-Hadamard matrix; from* [16]
SVIII	$4(q^2+q+1)$	q *a prime power and* $q^2+q+1 \equiv 3, 5$ or $7 \pmod 8$ *a prime or* $2(q^2+q+1)+1$

		a prime power; see [10] *or use Lemma 1.*
SIX	$2^t q$	$q = s^2 + 4r^2 \equiv 5 \pmod 8$ *a prime power and all orthogonal designs of type* $(1,a,b,c,c+\|r\|)$, *where* $1+a+b+2c+\|r\| = 2^t$, *exist in* 2^t; *Theorem 3.*
SX	hm	h *the order of a skew-Hadamard matrix,* m *the order of amicable Hadamard matrices.*

Spence [11] has found a new construction for *SIV* and Whiteman [18] a new construction for *SI* when $k_i - 1 \equiv 3 \pmod 8$. These are of considerable interest because of the structure involved and have use in the construction of orthogonal designs.

We make extensive use of the following theorems in this paper:

THEOREM (Delsarte-Goethals-Seidel [1]). *The existence of an orthogonal design of type* $(n-1)$ *in order* $n \equiv 0 \pmod 4$ *implies the existence of an orthogonal design of type* $(1,n-1)$ *in order* n *and a skew-Hadamard matrix of order* n.

THEOREM (Geramita-Verner [4]). *The existence of an orthogonal design of type* $(a_1,\dots,a_s)$ *in order* $n \equiv 0 \pmod 4$ *where* $n-1 = a_1+\dots+a_s$ *implies the existence of an orthogonal design of type* $(1,a_1,\dots,a_s)$.

2. *Preliminaries.*

Let J be the matrix of all ones and $K = J - 2I$ where I is the identity matrix. Let $I + G$ and S be the circulant symmetric matrices with entries ± 1 of order $\frac{1}{2}(q+1)$, $q \equiv 1 \pmod 4$ a prime power, found by Goethals and Seidel (see Wallis [15, p.315, Theorem 3.18]) which satisfy

$$GG^T + SS^T = qI.$$

Let Q be the circulant matrix with zero diagonal and other entries ± 1 of order q a prime defined from the quadratic residues (see Wallis [15, p.291, Lemma 1.19]) which satisfies

$$QQ^T = qI - J,\ QJ = JQ = 0,\ Q^T = (-1)^{\frac{1}{2}(q-1)}Q.$$

We use R for the back-diagonal matrix.

Szekeres difference sets which have cyclic (1,-1) incidence matrices P and V with $P + I$ skew-symmetric exist for orders $2t + 1$ if $2t + 1$ is a prime $p \equiv 3, 5$ or $7 \pmod 8$ or $4t + 3$ is a prime power [12], [13], [18]. For these

$$VV^T + PP^T = 4tI - 2J.$$

If $p \equiv 3$ or $7 \pmod 8$ both $P + I$ and $V + I$ are skew-symmetric but for $p \equiv 5 \pmod 8$ or $2p + 1$ a prime power $\equiv 3 \pmod 4$ V is symmetric.

Further, Spence [11] has shown that for orders $p \equiv 3 \pmod 4$ where $2p - 1 \equiv 5 \pmod 8$ is a prime power there are four circulant (1,-1)-matrices P_1, P_2, V_1, V_2 with $P_i + I$ skew-symmetric, Q_j symmetric $i,j = 1,2$ satisfying

$$P_1P_1^T + P_2P_2^T + V_1V_1^T + V_2V_2^T = 4pI_p.$$

This means there are orthogonal designs of type $(1,1,4p-2)$ in orders $4p$ when $p \equiv 3 \pmod 4$ and $2p - 1$ is a prime power.

We note (for the proof see Geramita and Seberry [3, Ch.4 §10]).

LEMMA 1. *Let A be the incidence matrix of a cyclic projective plane of order q. Then A is of order $q^2 + q + 1$, $W = A^2 - J$ is a circulant weighing matrix $W(q^2+q+1, q^2)$, $W * A = 0$ and*

$$(W+A)(W+A)^T + (W-A)(W-A)^T = 2(q^2+q)I + 2J.$$

Using $W+A$, $W-A$, with $P+I$, Q given above we have another proof of a result of Spence [10] and we see there are orthogonal designs of type $(1,1,4p-2)$ in order p where $p=q^2+q+1$ is a prime $\equiv 3$ or $7 \pmod 8$ and orthogonal designs of type $(1,4p-1)$ in order $4p$ where $p=q^2+q+1$ is a prime $\equiv 5 \pmod 8$ or $2p-1$ is a prime power.

We now use these results to obtain constructions for skew-Hadamard matrices.

3. *Results.*

THEOREM 2. *Let $q \equiv 5 \pmod 8$ be a prime power and $p = \frac{1}{2}(q+1)$ be a* *prime. Then there is a skew-Hadamard matrix of order* $2^t p$ *where* $t \geq [2 \log_2(p-2)]$.

Proof: Since $q \equiv 1 (\mathrm{mod}\ 4)$ we have circulant symmetric matrices G and S of order p satisfying

$$GG^T + SS^T = qI_p.$$

G has zero diagonal. Let $B = (I+Q)R$ where Q of order p is defined above and where R is the back-diagonal matrix. Then B is symmetric and satisfies

$$BB^T = (p+1)I_p - J_p, \quad BJ = JB = J, \quad AB^T = BA^T,$$

where A is any circulant matrix.

By a theorem of Sylvester the equation

$$(1) \qquad a(p+1) + b(p-3) = 2^t$$

can be solved for positive integers a and b whenever $2^t \geq p(p-4)$ or whenever $t \geq [2 \log_2(p-2)]$.

Suppose a and b satisfy (1) then since there exists an orthogonal design of type $(1,1,a,b,2^t-2-a-b)$ in order 2^t, we have on replacing the variables by the matrices G,S,J,K,B respectively, a weighing matrix $W = W(2^t p, 2^t p-1)$. Now using the theorem of Delsarte, Goethals and Seidel we see that W is equivalent to a skew-symmetric weighing matrix of the same order and hence on replacing the diagonal we have a skew-Hadamard matrix of the required order. □

For the next result we observe from [16] that if $q \equiv 5 \pmod 8$ is a prime power and $q = s^2 + 4r^2$, s odd, then with cyclotomic classes C_0, C_1, C_2, C_3 as usual, and M,N,P defined as the incidence matrices of

$$C_0 \cup C_1, \quad C_0 \cup C_2, \quad C_1 \cup C_3$$

respectively we have with $L = M+I$

$L^T = -L, \quad N^T = N, \; P^T = P, \quad LJ = 0, \quad NJ = PJ = -J.$

$$MM^T + |r|NN^T = (|r|+1)(q+1)I - (|r|+1)J$$

and

$$PP^T + NN^T = 2(q+1)I - 2J.$$

Thus $\quad LL^T + \; r\; NN^T = (|r|q + |r|+q)I - (|r|+1)J.$

THEOREM 3. *Let $q \equiv 5 \pmod 8$ be a prime power. Further suppose $q = s^2 + 4r^2$, s odd. Suppose all orthogonal designs of type $(1,a,b,c,c+|r|)$ where $2^t = 1+a+b+|r|+2c$ exist in order 2^t, $t \geq [2 \log_2 (q-2)] - 1$. Then there is a skew-Hadamard matrix of order $2^t q$.*

Proof: We choose L,N,P of order q defined using cyclotomic classes as above. Then replacing the variables of the design, D, of order 2^t and type (1,a,b,c,c+ r) by LR (R the type 2 equivalent of the back-diagonal matrix, J,K,P,N respectively we see

$$DD^T = (LL^T = aJ^2 + bK^2 + cP^2 + (c+|r|)N^2)\times I$$

$$= (2^t(q+1)-1-(q+1)a-(q-3)I\times I+(a(q+1)+b(q-3)-2^t)J\times I$$

$= (2^t q-1)I \quad \text{when} \quad 2^t = a(q+1) + b(q-3).$

By Sylvester's theorem the equation

$$(2) \qquad a(q+1) + b(q-3) = 2^t$$

can be solved for positive integers a and b when

$2^t \geq q(q-4)$ or whenever $t \geq [2 \log_2(q-2)]$.

Thus the design necessary to form a weighin matrix $W = W(2^t q, 2^t q-1)$ exists. We again used the Delsarte-Goethals-Seidel theorem to obtain a skew-symmetric weighing matrix of the same order and hence on replacing the diagonal we have a skew-Hadamard matrix of the required order. □

EXAMPLE: Let $a = 11$ and $b = 7$, then since there exists an orthogonal design of type (1,11,7,246,247) in order 2^8, there is a skew-Hadamard matrix of order $2^2.29$. We note that a skew-Hadamard matrix of order $2^2.29 = 116$ is not yet known.

THEOREM 4. *Let $q \equiv 1 \pmod 8$ be a prime power and $p = \frac{1}{2}(q+1)$ a prime or $q \equiv 3 \pmod 8$ a prime power and $p = \frac{1}{2}(q-1)$ be a prime. Suppose all orthogonal designs of type $(1,1,a,b,2^{t-1} - \frac{1}{2}(a+b), 2^{t-1} - \frac{1}{2}(a+b))$ exist in order 2^t. Then there is a skew-Hadamard matrix of order $2^t p$ for $t \geq [2 \log_2(p-2)]$.*

Proof: We proceed as before but now replace the variables by G,S,J,K,Q+I,Q-I respectively or P,V,J,K,Q+I,Q-I respectively (P,V are described in §2). Again calling on the theorems of Sylvester and Delsarte-Goethals-Seidel we have the result.

EXAMPLE. With $q = 73$ and $p = 37$ we choose $a = 1$, $b = 29$, $c = 496$ and $2^t = 2^{10}$. Since an orthogonal design of type (1,1,1,29,496,496) exists in order 2^{10} we have a skew-Hadamard matrix of order $2^{10}.37$. We note that a skew-Hadamard matrix of order $2^2.37 = 148$ is not known.

THEOREM 5. *Let* $q \equiv 1 (\mathrm{mod}\ 4)$ *be a prime power. Suppose all orthogonal designs of type* $(1,a,b,2^{t-2}-\frac{1}{4}(a+b+2),\ 2^{t-2}-\frac{1}{4}(a+b+2),\ 2^{t-1}-\frac{1}{2}(a+b))$ *exist in order* 2^t. *Then there is a skew-Hadamard matrix of order* $2^{t-1}(q+1)$, *where* $t \geq [2 \log_2 (q-3)] - 2$.

Proof: We proceed as before but now replace the variables by (G,J,K,G+I,G-I,S) of order $\frac{1}{2}(q+1)$ respectively. The result now follows as before.

EXAMPLE. Let $q = 137$. Choose $a = 17$, $b = 13$ and $2^t = 2^{11}$. Since there is an orthogonal design of type (1,17,13,504,504,1009) in order 2^{11} we have a skew-Hadamard matrix of order $2^{11}.69$. A skew-Hadamard matrix of order $2^2.69 = 276$ is not yet known.

4. *Results using Orthogonal Designs in Orders not Necessarily a Power of Two.*

In §2 we commented on the existence of orthogonal designs of types $(1,1,q-1)$ in orders $q+1 \equiv 0 (\mathrm{mod}\ 4)$ but in general little is known of these or the designs needed for the next theorem.

THEOREM 6. *Let $q \equiv 5 \pmod 8$, $q = s^2 + 4$, be a prime power. Suppose there is an orthogonal design of type $(1,1,1,q-3,q-2)$ in order $2q-2$. Then there is a skew-Hadamard matrix of order $2(q-1)q$.*

Proof: Choose L,N,P as above. Then replacing the variables of the orthogonal design by LR,J,K,P,N we have an orthogonal design of order $2q(q-1)$ and type $(2q(q-1)-1)$. Proceeding as before we have the result.

EXAMPLE: With q = 13 we obtain a skew-Hadamard matrix of order 312 since Robinson found an orthogonal design of type (1,1,1,1,1,1,9,9) in order 24.

THEOREM 7. *Suppose there is an orthogonal design of type $(1,m,mn-m-1)$ in order mn. Suppose n is the order of amicable designs of types $((1,n-1);(n))$. Then there is a skew-Hadamard matrix of order $mn(n-1)$.*

Proof: We pre and postmultiply the orthogonal designs of type $((1,n-1);(n))$ in order n until they can be written

$$xI + y\begin{bmatrix} 0 & e \\ -e^T & P \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & e \\ e^T & D \end{bmatrix}$$

where $e = (1,\ldots,1)$ is of order $1 \times n-1$. Then

$$JP = 0,\ P^T = -P,\ D^T = D,\ JD = -J$$

$$PP^T = nI - J = DD^T.$$

So replacing the variables of the design of type (1,m,nm-m-1) by P,J,D respectively and proceeding as before we have the result.

COROLLARY 8. *Suppose* $q \equiv 3 \pmod 4$ *is a prime power. Further suppose there is an orthogonal design of type* $(1,1,q-1)$ *in order* $q+1$. *Then there is a skew-Hadamard matrix of order* $q(q+1)$.

Remark: It is conjectured that an orthogonal design of type (1,1,q-1) always exists unless $q-1 \neq$ sum of two squares. So this theorem is conjectured to be

> *Suppose* $q-1 \equiv 2 \pmod 4) = a^2 + b^2$ *(a,b, integer, q a prime power) then there are skew-Hadamard matrices of orders* $q+1$ *and* $q(q+1)$.

When $q = 43$, $q-1 \neq a^2 + b^2$ so we know nothing about order 43.44.

COROLLARY 9. *Suppose there exist amicable orthogonal designs of types* $((1,n-1);(n))$ *in order* n. *Then there exist skew-Hadamard matrices of orders* $2^r n(2^r n-1)$ *for* $r > 0$.

Proof: Amicable orthogonal designs of types ((1,x-1);(x)) in order x imply amicable orthogonal designs of types ((1,1,2x-2);(2x)) in order 2x. So for m = 1 we have the conditions of the theorem satisfied and a skew Hadamard matrix of order 2x(2x-1). Proceeding by induction we have the result.

5. *Numerical Results.*

As we pointed out earlier we really need the existence of amicable Hadamard matrices of order $2^s q_i$ for each prime q_i to establish the existence of amicable and skew-Hadamard matrices for orders $2^t q_1 \ldots q_r$ where each q_i is a prime. Unfortunately little is known about amicable Hadamard matrices. In fact the only additions to Appendix J of [15] are 8.31, 4.115 and 16.31. There are also amicable Hadamard matrices of orders 16.99, 16.139 and 16.159. So we have amicable Hadamard matrices for the following orders $2^t q$, q(odd) < 250:

q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t
1	2	27	2	51	4	77	2	101	.	127	.	151	5	177	.	201	7	227	2
3	2	29	4	53	2	79	3	103	3	129	3	153	4	179	8	203	2	229	3
5	2	31	3	55	3	81	3	105	2	131	2	155	2	181	3	205	4	231	3
7	2	33	2	57	2	83	2	107	.	133	3	157	5	183	4	207	2	233	4
9	3	35	2	59	.	85	4	109	9	135	4	159	4	185	2	209	4	235	3
11	2	37	.	61	2	87	2	111	2	137	2	161	2	187	4	211	.	237	2
13	3	39	3	63	2	89	4	113	8	139	4	163	3	189	3	213	4	239	4
15	2	41	2	65	4	91	3	115	2	141	2	165	2	191	.	215	2	241	.
17	2	43	3	67	5	93	3	117	2	143	2	167	4	193	3	217	5	243	2
19	3	45	2	69	4	95	2	119	4	145	5	169	5	195	3	219	7	245	4
21	2	47	4	71	2	97	9	121	3	147	2	171	2	197	2	221	2	247	6
23	4	49	4	73	7	99	4	123	2	149	4	173	2	199	3	223	3	249	4
25	3			75	3			125	2			175	3			225	4		

Table 1. Powers of 2 for which amicable Hadamard matrices of order $2^t q$ exist.

q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t
1	2	51	2	101	10	151	5	201	3	251	6	301	3	351	2	401		451	4
3	2	53	2	103	3	153	4	203	2	253	4	303	3	353	4	403	5	453	2
5	2	55	2	105	2	155	2	205	3	255	2	305	4	355	2	405	2	455	4
7	2	57	2	107		157	3	207	2	257	4	307	2	357	2	407	2	457	
9	2	59		109	9	159	2	209	4	259	5	309	3	359	4	409	3	459	3
11	2	61	2	111	2	161	2	211	2	261	3	311		361	3	411	2	461	17
13	2	63	2	113	8	163	3	213	4	263	2	313	2	363	2	413	4	463	7
15	2	65	4	115	2	165	2	215	2	265	4	315	2	365	2	415	2	465	3
17	2	67	5	117	2	167	4	217	4	267	4	317	6	367	2	417	2	467	2
19	2	69	3	119	4	169	5	219	4	269	8	319	3	369	4	419	4	469	3
21	2	71	2	121	3	171	2	221	2	271	2	321	2	371	2	421	2	471	2
23	2	73	2	123	2	173	2	223	3	273	2	323	2	373	7	423	4	473	5
25	2	75	2	125	2	175	2	225	4	275	4	325	5	375	2	425	2	475	4
27	2	77	2	127	7	177		227	2	277	5	327	2	377	6	427	2	477	2
29	4	79	2	129	3	179	8	229	3	279	2	329	6	379	2	429	3	479	
31	2	81	3	131	2	181	3	231	2	281	2	331	3	381	2	431	2	481	3
33	2	83	2	133	3	183	2	233	4	283		333	2	383	2	433	3	483	2
35	2	85	2	135	2	185	2	235	3	285	3	335	7	385	3	435	4	485	4
37	3	87	2	137	2	187	2	237	2	287	4	337	18	387	2	437	2	487	5
39	3	89	4	139	2	189	2	239	4	289	3	339	2	389	15	439	2	489	3
41	2	91	2	141	2	191		241		291	2	341	4	391	4	441	3	491	
43	3	93	4	143	2	193	3	243	2	293	2	343	6	393	2	443	6	493	3
45	2	95	2	145	5	195	2	245	4	295	5	345	4	395	2	445	3	495	2
47	4	97	9	147	2	197	2	247	6	297	2	347		397	5	447	2	497	2
49	4	99	2	149	4	199	2	249	4	299	4	349	3	399	2	449		499	2

Table 2. The smallest value of t for which a skew-Hadamard matrix of order $2^t q$, q odd is known. A blank indicates no skew-Hadamard matrix is known for any power of 2. (The author has values tabulated to $q = 999$).

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