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A NOTE ON ASYMPTOTIC EXISTENCE RESULTS FOR ORTHOGONAL DESIGNS

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In a recent manuscript "Some asymptotic results for orthogonal designs" Peter Eades showed that for many types of orthogonal designs existence is established once the order is large enough.

This paper uses sequences with zero non-periodic and periodic auto-correlation function to establish the asymptotic existence of many orthogonal designs with four variables. Bounds are also established for orthogonal designs of type $(1, k)$ *where* $k \le 63$ *and* (ℓ) *where* $\ell \le 52$.

It is shown that any 4 *sequences with zero non-periodic auto-correlation function and* $8k - 1$ *entries* $+1$ *or* -1 *must have length at least* $2k + 1$.

Introduction

An orthogonal design of order n *and type* $(u_1, u_2, \ldots, u_s)$ $(u_i > 0)$ on the commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_s\}$ such that

$$AA^t = \sum_{i=1}^{s} \left(u_i x_i^2\right) I_n .$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$.

In [3], where this was first defined and many examples and properties of such designs were investigated, it is mentioned that

$$A^tA = \sum_{i=1}^{s} \left(u_i x_i^2\right) I_n$$

and so the alternative description of A applies equally well to the columns of A. It is also shown in [3] that $s \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

when

$$n = 2^a \cdot b \ , \quad b \text{ odd} \ , \quad a = 4c + d \ , \quad 0 \leq d < 4 \ .$$

A *weighing matrix of weight* k *and order* n, is a square $\{0, 1, -1\}$ matrix, $W = W(n, k)$, of order n satisfying

$$WW^t = kI_n \ .$$

In [3] it was shown that the existence of an orthogonal design of order n and type $(u_1, \ldots, u_s)$ is equivalent to the existence of disjoint weighing matrices $A_1, \ldots, A_s$, of order n, where A_i has weight u_i and the matrices, $\{A_i\}_{i=1}^{s}$, satisfy the matrix equation

$$XY^T + YX^T = 0$$

in pairs. In particular, the existence of an orthogonal design of order n and type $(1, k)$ is equivalent to the existence of a skew-symmetric weighing matrix of weight k and order n.

It is conjectured that:

(i) for $n \equiv 0 \pmod 4$ there is a weighing matrix of weight k and order n for every $k \leq n$;

(ii) for $n \equiv 4 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$, except possibly $k = n - 2$, where k is the sum of $\leq$ three squares of integers (equivalently, there is an orthogonal design of type $(1, k)$ in order n for every $k < n$, except possibly $k = n - 2$, which is the sum of $\leq$ three squares of integers).

D. Shapiro and W. Wolfe have found powerful algebraic non-existence theorems for orthogonal designs which supercede those of Geramita, Geramita and Wallis [3]. In addition Geramita and Verner [4] and P.J. Robinson [9] have found strong combinatorial theorems.

Let R be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to be *constructed from two circulant matrices* A and B if it is of the form

$$\begin{bmatrix} A & BR \\ BR & -A \end{bmatrix}$$

and to be of *Goethals-Seidel type* if it is of the form

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{bmatrix}$$

where A, B, C, D are circulant matrices.

Hence forth we use $\bar{x}$ for $-x$.

Let $X = \{a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn}\}$ be m sequences of commuting variables of length n .

DEFINITION. (1) The *non-periodic auto-correlation function of the family of sequences* X (denoted N_X) is given by

$$N_X(j) = \sum_{i=1}^{n-j} \left(a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j}\right) .$$

(2) The *periodic auto-correlation function of the family of sequences* X (denoted P_X) is given by

$$P_X(j) = \sum_{i=1}^{n} \left(a_{1,i}a_{1,i+j} + q_{2,1}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j}\right)$$

where we assume the second subscript is actually chosen from the complete set of residues mod n .

We can interpret the function P_X in the following way: Form the m circulant matrices which have first rows respectively,

$$[a_{11}\ a_{12} \ldots a_{1n}],\ [a_{21}\ a_{22} \ldots a_{2n}],\ \ldots,\ [a_{m1}\ a_{m2} \ldots a_{mn}] ,$$

then $P_X(j)$ is the sum of the inner products of rows 1 and $j+1$ of these matrices.

Clearly

$$P_X(j) = N_X(j) + N_X(n-j) , \quad j = 1, \ldots, n-1 ,$$

and

$$N_X(j) = 0 \quad \forall j \Rightarrow P_X(j) = 0 \quad \forall j .$$

Note: $P_X(j)$ may equal 0 for all $j = 1, \ldots, n-1$ even though the $N_X(j)$ are not.

Such sequences are used extensively in [5].

THEOREM 3 (Robinson). *All 4-tuples* (a, b, c, d) *with* $0 \le a + b + c + d \le 32$ *are the types of orthogonal designs in order* 32 .

THEOREM 4 [3]. *All 4-tuples* (a, b, c, d) *with* $0 \le a + b + c + d \le 16$ *are the types of orthogonal designs in order* 16 .

LEMMA 5 [3, 7]. *All 4-tuples* (a, b, c, d) *which are not excluded by Wolfe's necessary conditions or the Geramita-Verner theorem are*

(i) *the types of orthogonal designs in order* 12 *when* $a + b + c + d \le 12$;

(ii) *the types of orthogonal designs in order* 20 *when* $a + b + c + d \le 20$ *except possibly for* (1, 3, 6, 8), (1, 4, 4, 9) *and* (2, 2, 5, 5) *which are undecided.*

The designs may be constructed by using four circulant matrices in the Goethals-Seidel array.

Preliminary Results

LEMMA 6. *Suppose there exist circulant matrices* X_i, Y_i , $i \in \{1, 2, \ldots\} = I$ *giving orthogonal designs of type* (a_i, b_i) *in order* $2n$. *Then there exists an orthogonal design of type* (a_k, b_k, a_j, b_j) , $k, j \in I$ *in order* $4n$.

COROLLARY 7. *There exist orthogonal designs of types* (1, 1, 13, 13), (1, 4, 9, 9), (1, 2, 4, 18), (1, 8, 8, 9), (1, 9, 9, 9), (2, 4, 4, 18), (2, 8, 9, 9), (4, 4, 9, 9) *and* (5, 5, 9, 9) *in order* 52 .

Proof. First we note there exists $B = W(13, 9)$ which is circulant. So there exist (1, 9), (9, 9) and (2, 18) in 26 constructed using two circulant matrices. This fact together with the results of Table 3 give all the designs except (1, 1, 13, 13) .

Let T be the circulant matrix which is 1 above the diagonal and in the (13, 1) element and zero elsewhere. Choose i so that the positions of T^i and T^{i+1} are zero in B . Let $X = T^i + T^{i+1}$ and $Y = T^{(q^2+q)/2} - T^{(q^2+q+2)/2}$. Then

$$aB + bX , \quad bB - aX , \quad aY + cI , \quad bY + dI$$

is the (1, 1, 13, 13) design.

Table 1.

Designs with zero non-periodic auto-correlation function

(1, 1, 1, 1)	a , b , c , d
(1, 1, 1, 4)	a , b , $d\ c\ \bar{d}$, d 0 d
(1, 1, 2, 2)	a , b , c d , $c\ \bar{d}$
(1, 1, 2, 8)	$d\ a\ \bar{d}$, $d\ b\ \bar{d}$, d c d , $d\ \bar{c}\ d$
(1, 1, 4, 4)	$b\ a\ \bar{b}$, b 0 b , $d\ c\ \bar{d}$, d 0 d
(1, 2, 2, 4)	$b\ a\ \bar{b}$, b 0 b , c d , $c\ \bar{d}$
(1, 2, 3, 6)	a b c , $a\ b\ \bar{c}$, $b\ \bar{a}\ b$, $b\ d\ \bar{b}$
(2, 2, 2, 2)	a b , $a\ \bar{b}$, c d , $c\ \bar{d}$
(1, 1, 1, 9)	$d\ a\ \bar{d}$, $d\ b\ \bar{d}$, $d\ 0\ c\ 0\ \bar{d}$, d 0 d 0 d
(1, 1, 8, 8)	$c\ d\ a\ \bar{d}\ \bar{c}$, $\bar{c}\ d\ b\ \bar{d}\ c$, c d 0 d c , $c\ \bar{d}\ 0\ \bar{d}\ c$
(1, 1, 9, 9)	$b\ c\ a\ \bar{c}\ \bar{b}$, $c\ \bar{b}\ d\ b\ \bar{c}$, $c\ c\ c\ \bar{c}\ c$, $b\ b\ b\ \bar{b}\ b$
(1, 4, 4, 4)	$b\ a\ \bar{b}$, b 0 b , $c\ c\ d\ \bar{d}$, $d\ d\ \bar{c}\ c$
(2, 2, 4, 4)	a b , $a\ \bar{b}$, $c\ c\ d\ \bar{d}$, $d\ d\ \bar{c}\ c$
(2, 3, 4, 6)	$a\ d\ 0\ \bar{d}\ a$, $a\ d\ c\ d\ \bar{a}$, $b\ c\ \bar{d}$, $b\ \bar{c}\ d$
(3, 3, 3, 3)	a b c , $a\ \bar{b}\ 0\ d$, $a\ 0\ c\ \bar{d}$, b c d
(3, 3, 6, 6)	$a\ d\ b\ \bar{d}\ a$, $a\ d\ c\ d\ \bar{a}$, $c\ \bar{d}\ a\ 0\ \bar{b}$, $c\ \bar{d}\ \bar{a}\ 0\ b$
(4, 4, 4, 4)	a b c d , $a\ \bar{b}\ c\ \bar{d}$, $a\ b\ \bar{c}\ \bar{d}$, $a\ \bar{b}\ \bar{c}\ d$
(2, 12)	$a\ b\ b\ \bar{b}$, $a\ \bar{b}\ \bar{b}\ b$, b b b , $b\ \bar{b}\ b$

(1, 1, 4, 16)	$a\,0\,a\,0\,a\,0\,a$, $a\,0\,a\,b\,\bar{a}\,0\,\bar{a}$, $a\,c\,\bar{a}\,0\,\bar{a}\,c\,a$, $a\,c\,\bar{a}\,d\,a\,\bar{c}\,\bar{a}$
(1, 1, 5, 5)	a , b , $c\,c\,\bar{c}\,d\,0\,d$, $d\,d\,\bar{d}\,\bar{c}\,0\,\bar{c}$
(1, 2, 4, 8)	$c\,a\,\bar{c}$, $c\,0\,c$, $d\,b\,\bar{d}\,d\,0\,d$, $d\,b\,\bar{d}\,\bar{d}\,0\,\bar{d}$
(1, 2, 6, 12)	$a\,b\,0\,b\,a$, $a\,b\,d\,\bar{b}\,\bar{a}$, $b\,\bar{a}\,b\,b\,c\,\bar{b}$, $b\,\bar{a}\,b\,\bar{b}\,\bar{c}\,b$
(1, 4, 5, 5)	$b\,a\,\bar{b}$, $b\,0\,b$, $c\,c\,\bar{c}\,d\,0\,d$, $d\,d\,\bar{d}\,\bar{c}\,0\,\bar{c}$
(2, 2, 2, 8)	$a\,b$, $a\,\bar{b}$, $c\,d\,\bar{c}\,c\,0\,c$, $c\,d\,\bar{c}\,\bar{c}\,0\,\bar{c}$
(2, 2, 4, 16)	$a\,b\,\bar{a}\,a\,c\,a$, $a\,b\,\bar{a}\,a\,\bar{c}\,a$, $a\,b\,\bar{a}\,\bar{a}\,d\,\bar{a}$, $a\,b\,\bar{a}\,\bar{a}\,\bar{d}\,\bar{a}$
(2, 2, 5, 5)	$a\,b$, $a\,\bar{b}$, $c\,c\,\bar{c}\,d\,0\,d$, $d\,d\,\bar{d}\,\bar{c}\,0\,\bar{c}$
(2, 2, 8, 8)	$a\,b\,\bar{a}\,a\,0\,a$, $a\,b\,\bar{a}\,\bar{a}\,0\,\bar{a}$, $c\,d\,\bar{c}\,c\,0\,c$, $c\,d\,\bar{c}\,\bar{c}\,0\,\bar{c}$
(2, 2, 9, 9)	$d\,b\,\bar{d}\,c\,a\,\bar{c}$, $d\,b\,\bar{d}\,\bar{c}\,\bar{a}\,c$, $d\,c\,0\,c\,\bar{d}\,c$, $\bar{c}\,d\,0\,d\,c\,d$
(2, 2, 10, 10)	$c\,c\,\bar{c}\,d\,a\,d$, $c\,c\,\bar{c}\,d\,\bar{a}\,d$, $d\,d\,\bar{d}\,\bar{c}\,b\,\bar{c}$, $d\,d\,\bar{d}\,\bar{c}\,\bar{b}\,\bar{c}$
(2, 4, 4, 8)	$a\,b\,\bar{a}\,a\,0\,a$, $a\,b\,\bar{a}\,\bar{a}\,0\,\bar{a}$, $c\,d\,\bar{d}\,c$, $\bar{c}\,d\,d\,c$
(2, 4, 6, 12)	$a\,b\,\bar{c}\,a\,b\,c$, $a\,b\,\bar{c}\,\bar{a}\,\bar{b}\,\bar{c}$, $b\,\bar{a}\,b\,b\,d\,\bar{b}$, $b\,\bar{a}\,b\,\bar{b}\,\bar{d}\,b$
(2, 5, 5, 8)	$a\,d\,\bar{a}\,a\,0\,a$, $a\,d\,\bar{a}\,\bar{a}\,0\,\bar{a}$, $c\,c\,\bar{c}\,b\,0\,b$, $b\,b\,\bar{b}\,\bar{c}\,0\,\bar{c}$
(2, 8, 8, 8)	$a\,a\,b\,\bar{b}\,c\,d\,\bar{c}$, $a\,a\,b\,\bar{b}\,\bar{c}\,\bar{d}\,c$, $a\,\bar{a}\,b\,b\,c\,0\,c$, $a\,\bar{a}\,b\,b\,\bar{c}\,0\,\bar{c}$
(4, 4, 4, 16)	$c\,d\,\bar{c}\,c\,a\,c\,b$, $c\,d\,\bar{c}\,c\,\bar{a}\,c\,\bar{b}$, $c\,d\,\bar{c}\,\bar{c}\,a\,\bar{c}\,\bar{b}$, $c\,d\,\bar{c}\,\bar{c}\,\bar{a}\,\bar{c}\,b$
(4, 4, 5, 5)	$a\,a\,b\,\bar{b}$, $b\,b\,\bar{a}\,a$, $c\,0\,c\,d\,d\,\bar{d}$, $d\,0\,d\,\bar{c}\,\bar{c}\,c$
(4, 4, 8, 8)	$a\,a\,b\,\bar{b}\,c\,d$, $a\,a\,b\,\bar{b}\,\bar{c}\,\bar{d}$, $a\,\bar{a}\,b\,b\,c\,\bar{d}$, $a\,\bar{a}\,b\,b\,\bar{c}\,d$
(4, 4, 10, 10)	$b\,c\,a\,c\,d\,d\,\bar{d}$, $b\,\bar{c}\,a\,\bar{c}\,\bar{d}\,\bar{d}\,d$, $b\,d\,\bar{a}\,d\,\bar{c}\,\bar{c}\,c$, $b\,\bar{d}\,\bar{a}\,\bar{d}\,c\,c\,\bar{c}$
(5, 5, 5, 5)	$a\,a\,\bar{a}\,b\,0\,b$, $\bar{b}\,\bar{b}\,b\,a\,0\,a$, $\bar{c}\,\bar{c}\,c\,\bar{d}\,0\,\bar{d}$, $\bar{d}\,\bar{d}\,d\,c\,0\,c$
(6, 6, 6, 6)	$a\,a\,b\,\bar{b}\,c\,d$, $\bar{b}\,\bar{b}\,a\,\bar{a}\,d\,\bar{c}$, $\bar{c}\,\bar{c}\,\bar{d}\,d\,a\,b$, $\bar{d}\,\bar{d}\,c\,\bar{c}\,\bar{b}\,a$
(7, 7, 7, 7)	$a\,a\,\bar{a}\,b\,c\,b\,d$, $\bar{b}\,\bar{b}\,b\,a\,d\,a\,\bar{c}$, $\bar{c}\,\bar{c}\,c\,\bar{d}\,a\,\bar{d}\,b$, $\bar{d}\,\bar{d}\,d\,c\,\bar{b}\,c\,a$

LEMMA 8. *There exists a* (1, 1, 2, 4, 9) *design in* 24 *and* (1, 1, 1, 4, 20) *and* (1, 1, 9, 8, 8) *designs in* 40 .

Proof. Use

$$ab b,\ \bar{a}bb,\ 0cc,\ 0b\bar{b},\ d00,\ e00,\ \bar{b}bb,\ 0c\bar{c}$$

$$ae00\bar{e},\ 0e00e,\ c0000,\ d0000,\ bb\bar{b}\bar{b}b,\ \bar{b}b\bar{b}\bar{b}b,\ \bar{b}bbbb,\ \bar{b}bbbb$$

$$a0000,\ b0000,\ 0cc\bar{c}c,\ 0ccc\bar{c},\ 0d\bar{d}\bar{d}d,\ \bar{d}dddd,\ 0ee\bar{e}e,\ 0eee\bar{e}$$

as the first rows of circulant matrices in Lemma 5 parts (viii), (iii) and (viii) respectively of [11].

Main Results

THEOREM 9. *Sequences of commuting variables with zero non-periodic auto-correlation function exist which give the following orthogonal designs in order* 4n , $n \geq 7$ *:*

(1, 1, 1, 1)	(1, 2, 2, 4)	(2, 2, 5, 5)	(3, 3, 6, 6)
(1, 1, 1, 4)	(1, 2, 3, 6)	(2, 2, 8, 8)	(4, 4, 4, 4)
(1, 1, 1, 9)	(1, 2, 4, 8)	(2, 2, 9, 9)	(4, 4, 4, 16)
(1, 1, 2, 2)	(1, 2, 6, 12)	(2, 2, 10, 10)	(4, 4, 5, 5)
(1, 1, 2, 8)	(1, 4, 4, 4)	(2, 3, 4, 6)	(4, 4, 8, 8)
(1, 1, 4, 4)	(1, 4, 5, 5)	(2, 4, 4, 8)	(4, 4, 10, 10)
(1, 1, 4, 16)	(2, 2, 2, 2)	(2, 4, 6, 12)	(5, 5, 5, 5)
(1, 1, 5, 5)	(2, 2, 2, 8)	(2, 5, 5, 8)	(6, 6, 6, 6)
(1, 1, 8, 8)	(2, 2, 4, 4)	(2, 8, 8, 8)	(7, 7, 7, 7) .
(1, 1, 9, 9)	(2, 2, 4, 16)	(3, 3, 3, 3)	

Proof. Table 1 has the sequences which should be used as first rows of circulant matrices in the Goethals-Seidel array to obtain the result.

LEMMA 10. *The following orthogonal designs in order* 4n *can be constructed by using sequences of commuting variables with zero non-periodic auto-correlation function to form circulant matrices which are then used in the Goethals-Seidel array:*

(i) (1, 2, 3, 6), (1, 1, 4, 4), (1, 1, 2, 8) *for* $n \geq 3$ *;*

(ii) (1, 1, 8, 8), (1, 1, 9, 9), (1, 4, 4, 4) *for* $n \geq 5$ *;*

(iii) (1, 1, 4, 16), (1, 2, 6, 12) *for* $n \geq 6$ *;*

(iv) (1, 24), (23), (26), (1, 27) *for* $n \geq 7$ *;*

(v) (29), (30) *for* $n \geq 8$ *;*

(vi) (1, 1, 32), (1, 35), (31) *for* $n \geq 9$ *;*

(vii) (1, 36), (38) *for* $n \geq 10$ *;*

(viii) (1, 43), (1, 2, 2, 36), (39), (42) *for* $n \geq 11$ *;*

<u>Table 2.</u>

Designs with zero non-periodic auto-correlation function

(21)	$a\,a\,a\,a\,\bar{a}\,\bar{a}$, $a\,a\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,\bar{a}\,a\,a$, $a\,a\,0\,\bar{a}\,a$
(23)	$a\,\bar{a}\,a\,a\,a\,a\,\bar{a}$, $a\,a\,\bar{a}\,\bar{a}\,\bar{a}\,a\,\bar{a}$, $a\,\bar{a}\,\bar{a}\,0\,\bar{a}\,\bar{a}\,a$, $a\,0\,a\,0\,0\,0\,a$
(2, 26)	$a\,b\,b\,\bar{b}\,b\,\bar{b}\,\bar{b}$, $\bar{a}\,b\,b\,\bar{b}\,b\,\bar{b}\,\bar{b}$, $b\,b\,b\,\bar{b}\,b\,b\,b$, $b\,b\,\bar{b}\,\bar{b}\,\bar{b}\,b\,\bar{b}$
(1, 24)	$a\,a\,\bar{a}\,b\,a\,\bar{a}\,\bar{a}$, $a\,a\,a\,0\,\bar{a}\,\bar{a}\,a$, $a\,a\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,a\,a\,\bar{a}\,a$
(1, 27)	$a\,a\,\bar{a}\,b\,a\,\bar{a}\,\bar{a}$, $a\,a\,\bar{a}\,a\,\bar{a}\,a\,\bar{a}$, $a\,a\,\bar{a}\,\bar{a}\,\bar{a}\,\bar{a}\,a$, $a\,a\,\bar{a}\,a\,a\,a\,a$
(29)	$a\,a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,\bar{a}$, $a\,a\,a\,\bar{a}\,a\,\bar{a}\,\bar{a}\,a$, $a\,a\,a\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,a\,0\,\bar{a}\,a\,a$
(30)	$a\,a\,a\,\bar{a}\,a\,a\,\bar{a}\,\bar{a}$, $a\,a\,a\,a\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,a\,a\,\bar{a}\,\bar{a}\,a$
(31)	$a\,a\,\bar{a}\,a\,a\,a\,\bar{a}\,a\,\bar{a}$, $a\,a\,\bar{a}\,a\,0\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,a\,a\,a\,a\,\bar{a}\,\bar{a}$, $a\,a\,a\,\bar{a}\,\bar{a}\,a\,a$
(1, 1, 32)	$a\,a\,\bar{a}\,a\,b\,\bar{a}\,a\,\bar{a}\,\bar{a}$, $a\,a\,a\,\bar{a}\,c\,a\,\bar{a}\,\bar{a}\,\bar{a}$, $a\,a\,\bar{a}\,a\,0\,a\,\bar{a}\,a\,a$, $a\,a\,a\,\bar{a}\,0\,\bar{a}\,a\,a\,a$
(1, 33)	$a\,a\,a\,\bar{a}\,b\,a\,\bar{a}\,\bar{a}\,\bar{a}$, $a\,a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,\bar{a}\,a$, $a\,a\,a\,\bar{a}\,a\,\bar{a}\,a\,a$, $a\,a\,a\,\bar{a}\,a\,\bar{a}\,a\,a$
(1, 35)	$a\,a\,\bar{a}\,a\,b\,\bar{a}\,a\,\bar{a}\,\bar{a}$, $a\,a\,a\,\bar{a}\,\bar{a}\,\bar{a}\,a\,\bar{a}\,\bar{a}$, $a\,a\,\bar{a}\,a\,\bar{a}\,a\,\bar{a}\,a\,a$, $a\,a\,a\,a\,a\,\bar{a}\,\bar{a}\,a\,a$
(1, 36)	$a\,a\,\bar{a}\,a\,a\,a\,a\,\bar{a}\,a\,\bar{a}$, $a\,a\,\bar{a}\,a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,a\,\bar{a}\,b\,a\,\bar{a}\,\bar{a}\,\bar{a}$, $a\,a\,a\,\bar{a}\,0\,\bar{a}\,a\,a\,a$
(38)	$a\,a\,a\,a\,a\,\bar{a}\,a\,a\,\bar{a}\,\bar{a}$, $a\,a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,a\,\bar{a}\,\bar{a}\,a\,a\,\bar{a}\,a$, $a\,a\,a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,a$
(39)	$a\,a\,a\,\bar{a}\,a\,\bar{a}\,a\,a\,\bar{a}\,\bar{a}\,\bar{a}$, $a\,a\,a\,a\,\bar{a}\,a\,0\,a\,a\,\bar{a}\,a\,a$, $a\,a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,\bar{a}\,a$, $a\,a\,a\,\bar{a}\,a\,\bar{a}\,\bar{a}\,\bar{a}\,a$
(1, 2, 2, 36)	$a\,a\,a\,\bar{a}\,a\,d\,\bar{a}\,a\,\bar{a}\,\bar{a}\,\bar{a}$, $a\,a\,a\,\bar{a}\,a\,0\,a\,\bar{a}\,a\,a\,a$, $a\,a\,b\,\bar{a}\,\bar{a}\,a\,\bar{a}\,c\,a\,\bar{a}$, $a\,a\,b\,\bar{a}\,\bar{a}\,\bar{a}\,a\,c\,\bar{a}\,a$
(42)	$a\,a\,a\,\bar{a}\,a\,a\,a\,\bar{a}\,a\,\bar{a}\,\bar{a}$, $a\,a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,a\,\bar{a}\,\bar{a}\,a\,a\,a\,\bar{a}\,a$, $a\,a\,a\,a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,a$
(1, 43)	$a\,a\,a\,\bar{a}\,a\,b\,\bar{a}\,a\,\bar{a}\,\bar{a}\,\bar{a}$, $a\,a\,a\,a\,\bar{a}\,\bar{a}\,a\,a\,\bar{a}\,a\,\bar{a}$, $a\,a\,a\,a\,\bar{a}\,\bar{a}\,a\,\bar{a}\,a\,\bar{a}\,a$, $a\,a\,a\,\bar{a}\,a\,a\,\bar{a}\,\bar{a}\,a\,a\,a$

(1, 44) $a a \bar{a} \bar{a} a \bar{a} a \bar{a} \bar{a} \bar{a} \bar{a} a$, $a a \bar{a} \bar{a} a \bar{a} \bar{a} a a a a \bar{a}$, $a \bar{a} a a a b \bar{a} \bar{a} \bar{a} a \bar{a}$, $a \bar{a} a a a 0 a a a \bar{a} a$

(1, 1, 40) $a 0 a a a \bar{a} b a \bar{a} \bar{a} \bar{a} 0 \bar{a}$, $a 0 a \bar{a} \bar{a} a c \bar{a} a a \bar{a} 0 \bar{a}$, $a 0 a a a \bar{a} 0 \bar{a} a a a 0 a$,
$a 0 a \bar{a} \bar{a} a 0 a \bar{a} \bar{a} a 0 a$

(47) $a a a \bar{a} a a \bar{a} a a a \bar{a} \bar{a} \bar{a}$, $a a a \bar{a} a \bar{a} 0 \bar{a} a \bar{a} \bar{a} a a$, $a a a \bar{a} a a a a a \bar{a} \bar{a} a$,
$a a a \bar{a} a \bar{a} \bar{a} a \bar{a} a a$

(48) $a a a \bar{a} a a \bar{a} a a \bar{a} \bar{a} \bar{a}$, $a a a a \bar{a} a \bar{a} \bar{a} a a \bar{a} \bar{a}$, $a a a \bar{a} \bar{a} a \bar{a} a \bar{a} \bar{a} a a$, $a a a a \bar{a} a \bar{a} a \bar{a} a a a$

(50) $a a a a \bar{a} a \bar{a} \bar{a} a a \bar{a} \bar{a} \bar{a}$, $a a a a \bar{a} a \bar{a} \bar{a} a a \bar{a} \bar{a} a$, $a a a \bar{a} \bar{a} a \bar{a} a \bar{a} \bar{a} a a$,
$a a a a a \bar{a} a \bar{a} a \bar{a} a a$

(1, 51) $a a a \bar{a} a \bar{a} b a \bar{a} a \bar{a} \bar{a} \bar{a}$, $a a a \bar{a} a a \bar{a} \bar{a} a \bar{a} a \bar{a} \bar{a}$, $a a a \bar{a} a \bar{a} a a a \bar{a} \bar{a} a a$,
$a a a a a \bar{a} \bar{a} a \bar{a} \bar{a} a a a$

(1, 1, 17, 17) $a a a \bar{a} 0 a \bar{a} \bar{a} b b b \bar{b} b \bar{b} b b$, $b b b \bar{b} 0 b \bar{b} \bar{b} \bar{a} \bar{a} \bar{a} a \bar{a} a \bar{a} \bar{a}$, $a 0 c 0 \bar{a}$, $b 0 d 0 \bar{b}$

(1, 2, 66) $a a a \bar{a} b a \bar{a} \bar{a} \bar{a} a a a \bar{a} \bar{a} a \bar{a} \bar{a} a$, $a a a \bar{a} b a \bar{a} \bar{a} \bar{a} \bar{a} \bar{a} a a \bar{a} a a \bar{a}$,
$a a a \bar{a} a \bar{a} a a 0 a a \bar{a} a \bar{a} a a a$, $a a a \bar{a} a \bar{a} a a c \bar{a} \bar{a} a \bar{a} a \bar{a} \bar{a}$

(ix) (1, 44), (48) *for* $n \geq 12$;

(x) (47), (50), (1, 51) *for* $n \geq 13$;

(xi) (1, 1, 17, 17) *for* $n \geq 16$;

(xii) (1, 2, 66) *for* $n \geq 18$;

(xiii) (10, 10, 13, 13) *for* $n \geq 14$.

Proof. Use the sequences from Tables 1 and 2 for (i) - (x). For (xiii) use the appropriate sequences of Table 3.

COROLLARY 11. *There exist weighing matrices* $W(n, k)$ *for* $k = 0, 1, \ldots, n$ *when* $n = 12, 20, 28, 36, 44$ *constructed using four circulant matrices in the Goethals-Seidel array* EQUIVALENTLY *the weighing matrix conjecture is true for* $n = 12, 20, 28, 36, 44$.

Proof. Use Tables 1 and 2.

THEOREM 12. *There exist orthogonal designs of order* $4n$ *and type* $(1, k)$ *when*

(i) *for* $n \geq t$, $t = 3, 5, 7$, $k \in \{x : x \leq 4t-1,\ x = a^2+b^2+c^2\}$;

<u>Table 3.</u>

Sequences with zero non-periodic auto-correlation function

Length	Type	
1	(1, 1)	a , b
2	(2, 2)	$a\ b$, $a\ \bar{b}$
3	(1, 4)	$b\ a\ \bar{b}$, $b\ 0\ b$
4	(4, 4)	$a\ a\ b\ \bar{b}$, $b\ b\ \bar{a}\ a$
6	(2, 8)	$a\ b\ \bar{a}\ a\ 0\ a$, $a\ b\ \bar{a}\ \bar{a}\ 0\ \bar{a}$
6	(5, 5)	$a\ a\ \bar{a}\ b\ 0\ b$, $b\ b\ \bar{b}\ \bar{a}\ 0\ \bar{a}$
8	(8, 8)	$a\ a\ a\ \bar{a}\ b\ b\ \bar{b}\ b$, $\bar{b}\ \bar{b}\ \bar{b}\ b\ a\ a\ \bar{a}\ a$
10	(4, 16)	$a\ b\ \bar{a}\ a\ \bar{a}\ a\ \bar{a}\ \bar{a}\ b\ a$, $a\ b\ \bar{a}\ a\ a\ a\ a\ a\ \bar{b}\ \bar{a}$
10	(10, 10)	$a\ \bar{a}\ \bar{a}\ a\ b\ a\ b\ b\ b\ \bar{b}$, $b\ \bar{b}\ \bar{b}\ b\ \bar{a}\ b\ \bar{a}\ \bar{a}\ \bar{a}\ a$
11	(13)	$a\ a\ a\ 0\ \bar{a}\ a\ a\ 0\ \bar{a}\ a\ \bar{a}$, $a\ 0\ a\ 0\ 0\ 0\ \bar{a}\ 0\ 0\ 0\ a$
14	(13, 13)	$a\ a\ a\ b\ \bar{a}\ a\ a\ \bar{b}\ \bar{a}\ a\ \bar{a}\ b\ 0\ b$, $b\ b\ b\ \bar{a}\ \bar{b}\ b\ b\ a\ \bar{b}\ b\ \bar{b}\ \bar{a}\ 0\ \bar{a}$

Table 4.

Sequences with zero periodic auto-correlation function

$n \geq 5$	(1, 14)	$a\ 0_{\frac{1}{2}(n-5)}\ b\ b\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-5)}$, $b\ b\ \bar{b}\ b\ 0_{n-4}$, $b\ b\ b\ 0_{n-3}$, $b\ \bar{b}\ b\ 0_{n-3}$
$n \geq 7$	(1, 1, 1, 16)	$a\ 0_{\frac{1}{2}(n-7)}\ b\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-7)}$, $c\ 0_{\frac{1}{2}(n-5)}\ b\ 0\ 0\ \bar{b}\ 0_{\frac{1}{2}(n-5)}$,
		$d\ 0_{\frac{1}{2}(n-5)}\ b\ 0\ 0\ \bar{b}\ 0_{\frac{1}{2}(n-5)}$, $b\ b\ \bar{b}\ b\ b\ b\ 0_{n-7}$
	(1, 1, 13, 13)	$b\ 0_{\frac{1}{2}(n-7)}\ c\ c\ \bar{c}\ c\ \bar{c}\ \bar{c}\ 0_{\frac{1}{2}(n-7)}$, $c\ c\ d\ \bar{c}\ d\ c\ d\ 0_{n-7}$, $d\ d\ \bar{c}\ \bar{d}\ \bar{c}\ d\ \bar{c}\ 0_{n-7}$,
		$a\ 0_{\frac{1}{2}(n-7)}\ d\ d\ \bar{d}\ d\ \bar{d}\ \bar{d}\ 0_{\frac{1}{2}(n-7)}$
	(1, 1, 26)	$a\ 0_{\frac{1}{2}(n-7)}\ b\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-7)}$, $c\ 0_{\frac{1}{2}(n-7)}\ b\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-7)}$,
		$b\ b\ b\ \bar{b}\ b\ b\ b\ 0_{n-7}$, $b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ b\ \bar{b}\ 0_{n-7}$
	(1, 22)	$a\ 0_{\frac{1}{2}(n-7)}\ b\ b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-7)}$, $b\ b\ \bar{b}\ b\ \bar{b}\ b\ 0_{n-6}$, $b\ b\ \bar{b}\ b\ b\ 0_{n-5}$,
		$b\ b\ \bar{b}\ b\ b\ 0_{n-5}$
$n \geq 9$	(1, 25)	$a\ 0_{\frac{1}{2}(n-7)}\ b\ 0\ b\ \bar{b}\ 0\ \bar{b}\ 0_{\frac{1}{2}(n-7)}$, $b\ 0\ b\ b\ 0\ b\ b\ b\ \bar{b}\ 0_{n-9}$, $b\ 0\ b\ \bar{b}\ \bar{b}\ b\ \bar{b}\ \bar{b}\ b\ 0_{n-9}$,
		$b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ b\ 0_{n-6}$
	(1, 30)	$a\ 0_{\frac{1}{2}(n-9)}\ b\ b\ \bar{b}\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-9)}$, $b\ b\ b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ b\ 0_{n-8}$, $b\ b\ b\ \bar{b}\ b\ b\ b\ 0_{n-7}$,
		$b\ \bar{b}\ \bar{b}\ b\ \bar{b}\ \bar{b}\ b\ 0_{n-7}$

$n \geq 11$ (1, 4, 13) $a\ b\ \bar{a}\ 0_{n-3}$, $a\ 0\ a\ 0_{n-3}$, $c\ c\ c\ 0\ \bar{c}\ c\ c\ 0\ \bar{c}\ c\ \bar{c}\ 0_{n-11}$, $c\ 0\ c\ 0\ 0\ 0\ \bar{c}\ 0\ 0\ 0\ c\ 0_{n-11}$

(1, 1, 40) $a\ 0_{\frac{1}{2}(n-11)}\ \bar{b}\ b\ b\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ \bar{b}\ b\ 0_{\frac{1}{2}(n-11)}$, $c\ 0_{\frac{1}{2}(n-11)}\ b\ \bar{b}\ \bar{b}\ b\ \bar{b}\ b\ \bar{b}\ b\ b\ \bar{b}\ 0_{\frac{1}{2}(n-11)}$,

$b\ \bar{b}\ \bar{b}\ b\ b\ b\ b\ \bar{b}\ \bar{b}\ b\ 0_{n-10}$, $\bar{b}\ b\ b\ b\ b\ b\ b\ b\ b\ \bar{b}\ 0_{n-10}$

$n \geq 13$ (1, 1, 25) $a\ 0_{\frac{1}{2}(n-13)}\ b\ b\ b\ \bar{b}\ 0\ b\ \bar{b}\ 0\ b\ \bar{b}\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-13)}$, $c\ 0_{\frac{1}{2}(n-5)}\ b\ 0\ 0\ \bar{b}\ 0_{\frac{1}{2}(n-5)}$,

$b\ 0\ 0\ 0\ 0\ 0\ b\ 0\ 0\ b\ 0_{n-10}$, $b\ b\ b\ 0\ \bar{b}\ b\ \bar{b}\ b\ \bar{b}\ 0\ b\ b\ 0_{n-12}$

(1, 1, 50) $a\ 0_{\frac{1}{2}(n-13)}\ b\ b\ b\ \bar{b}\ b\ b\ \bar{b}\ \bar{b}\ b\ \bar{b}\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-13)}$, $c\ 0_{\frac{1}{2}(n-13)}\ b\ b\ b\ \bar{b}\ \bar{b}\ b\ \bar{b}\ b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-13)}$,

$b\ b\ b\ b\ \bar{b}\ b\ \bar{b}\ b\ \bar{b}\ b\ b\ b\ b\ 0_{n-13}$, $b\ b\ b\ \bar{b}\ \bar{b}\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ b\ b\ \bar{b}\ 0_{n-13}$.

$n \geq 15$ (1, 4, 26) $a\ b\ \bar{a}\ 0_{n-3}$, $a\ 0\ a_{n-3}$, $c\ c\ c\ \bar{c}\ \bar{c}\ c\ c\ c\ \bar{c}\ c\ \bar{c}\ \bar{c}\ 0\ \bar{c}\ 0_{n-14}$,

$c\ c\ c\ c\ \bar{c}\ c\ c\ \bar{c}\ \bar{c}\ c\ \bar{c}\ c\ 0\ c\ 0_{n-14}$

(1, 1, 29) $a\ 0_{\frac{1}{2}(n-15)}\ b\ 0\ 0\ 0\ \bar{b}\ 0\ 0\ 0\ 0\ b\ 0\ 0\ 0\ \bar{b}\ 0_{\frac{1}{2}(n-15)}$, $c\ 0_{\frac{1}{2}(n-13)}\ b\ b\ b\ 0\ \bar{b}\ b\ \bar{b}\ b\ 0\ \bar{b}\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-13)}$,

$b\ b\ 0\ 0\ b\ b\ \bar{b}\ 0\ \bar{b}\ b\ b\ 0\ 0\ b\ 0_{n-14}$, $b\ \bar{b}\ 0\ 0\ 0\ \bar{b}\ 0\ 0\ 0\ \bar{b}\ b\ 0\ \bar{b}\ 0_{n-13}$

(1, 1, 58) $a\ 0_{\frac{1}{2}(n-15)}\ b\ b\ b\ b\ \bar{b}\ \bar{b}\ b\ \bar{b}\ b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-15)}$,

$c\ 0_{\frac{1}{2}(n-15)}\ b\ \bar{b}\ \bar{b}\ \bar{b}\ \bar{b}\ b\ \bar{b}\ b\ \bar{b}\ b\ b\ b\ b\ \bar{b}\ 0_{\frac{1}{2}(n-15)}$,

$b\ b\ \bar{b}\ b\ b\ b\ \bar{b}\ b\ \bar{b}\ b\ b\ b\ \bar{b}\ b\ b\ 0_{n-15}$, $b\ b\ b\ \bar{b}\ b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ b\ b\ \bar{b}\ b\ b\ \bar{b}\ 0_{n-15}$

(1, 52) $a\ 0_{\frac{1}{2}(n-13)}\ b\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ b\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ 0_{\frac{1}{2}(n-13)}$, $b\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ \bar{b}\ \bar{b}\ b\ \bar{b}\ b\ b\ 0_{n-12}$,

$b\ b\ b\ \bar{b}\ b\ b\ b\ b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ b\ \bar{b}\ 0_{n-14}$, $b\ b\ b\ \bar{b}\ b\ b\ b\ \bar{b}\ \bar{b}\ b\ b\ b\ \bar{b}\ b\ 0_{n-14}$

(ii) *for* $n \geq 9$, $k \in \{x : x \leq 35,\ x = a^2+b^2+c^2,\ x \neq 34\}$;

(iii) *for* $n \geq 11$, $k \in \{x : x \leq 43,\ x = a^2+b^2+c^2,\ x \neq 34, 37, 42\}$;

(iv) *for* $n \geq 13$, $k \in \{x : x \leq 51,\ x = a^2+b^2+c^2,\ x \neq 34, 37, 42, 45, 46, 48, 49\}$;

(v) *for* $n \geq 15$, $k \in \{x : x \leq 59,\ x = a^2+b^2+c^2,\ x \neq 34, 37, 42, 45, 46, 48, 49, 53, 54, 56, 57\}$;

(vi) *for* $n \geq 16$, $k \in \{x : x \leq 63,\ x = a^2+b^2+c^2,\ x \neq 37, 42, 45, 46, 48, 49, 53, 54, 56, 57, 60, 61, 62, 63\}$.

In all cases when n *is odd it is possible to construct the designs using four circulant matrices in the Goethals-Seidel array.*

Proof. We use Table 1 to obtain the result for $n \geq 3$ and $k = 12, 16, 17, 18, 19$ for $n \geq 5$. Table 4 gives $(1, 14)$ for odd $n \geq 5$ and $(1, 13)$ for odd $n \geq 7$. We noted above that $(1, 13)$ exists in 20 and both $(1, 13)$ and $(1, 14)$ exist in 16 and 24 which gives the result.

For $n \geq 7$ Tables 2 and 4 give the result immediately except for $(1, 25)$. Table 4 gives $(1, 25)$ for $n \geq 9$ and its existence in 32, 40, 48, 56 follows because $(1, 1, 12, 12)$ exists in these orders. The existence of $(1, 25)$ in order 28 gives the result for $(1, 25)$.

Tables 1, 2 and 4 give the result immediately except for $(1, 29)$ and $(1, 30)$. Table 4 gives these two in odd orders ≥ 15 .

Table 2 gives the result immediately for $(1, 32)$ and $(1, 33)$ in orders $n \geq 9$. Table 4 gives $(1, 30)$ for n (odd) ≥ 9 and $(1, 29)$ for n (odd) ≥ 15 . Both $(1, 30)$ and $(1, 29)$ exist in orders 32, 40, 48 and 56 . We exhibit first rows of four circulant matrices which give $(1, 29)$ in orders 36, 44 and 52 :

36: $a\ b\ b\ \bar{b}\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}$, $0\ b\ b\ b\ \bar{b}\ b\ \bar{b}\ \bar{b}\ \bar{b}$, $b\ \bar{b}\ 0\ b\ b\ b\ b\ 0\ b$, $b\ \bar{b}\ 0\ b\ 0\ b\ 0\ \bar{b}\ b$

44: $a\ b\ \bar{b}\ b\ b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ b\ \bar{b}$, $\bar{b}\ b\ 0\ b\ b\ b\ 0\ 0\ 0\ b\ 0$, $b\ b\ b\ 0\ \bar{b}\ b\ b\ 0\ \bar{b}\ b\ \bar{b}$,
$b\ 0\ b\ 0\ 0\ 0\ \bar{b}\ 0\ 0\ 0\ b$

52: $b\ 0\ b\ 0_{10}$, $b\ a\ \bar{b}\ 0_{10}$, $0\ b\ \bar{b}\ \bar{b}\ \bar{b}\ b\ \bar{b}\ \bar{b}\ b\ \bar{b}\ \bar{b}\ \bar{b}\ b$,
$b\ \bar{b}\ b\ b\ \bar{b}\ \bar{b}\ \bar{b}\ \bar{b}\ \bar{b}\ \bar{b}\ b\ b\ \bar{b}$.

This gives the result.

For $n \geq 11$ we consider $k \in \{36, 38, 40, 41, 43\}$. Tables 2, 4 and the existence of $(1, 41)$ in order 48 gives the result.

For $n \geq 13$ we consider $k \in \{44, 50, 51\}$. Table 2 gives the result for $k = 44, 51$. Table 4 gives $(1, 50)$ for n (odd) ≥ 13 so the existence of $(1, 50)$ in 56, 64, 72, ... (which follows since $(1, 25)$ exists in $4m$, $m \geq 7$) , gives the result.

Table 4 gives (1, k) for $k \in \{52, 58, 59\}$ in orders $4n$, n (odd) ≥ 15. (1, 26) and (1, 29) exist in orders 32 and $4m$, $m \geq 9$. Hence we have (1, 52), (1, 58) and (1, 59) in orders 64, 72, 80, ... which gives the result.

Table 2 gives (1, 34) in all $4m$, $m \geq 16$.

Comments on the Tables

Table 2 was constructed using the Research School of Physical Sciences, DEC-10 System. The results obtained led by various devious methods to Tables 1 and 4.

We now discuss the analysis and method which led to the results of Table 2 but first a definition:

DEFINITION. The *weight* of m-complementary sequences of length n is their total number of non-zero entries.

We consider 4-complementary sequences $A = \{A_1, A_2, A_3, A_4\}$ where $A_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,n}\}$, $i = 1, 2, 3, 4$, of length n with entries 1, -1 or 0 ; we write $s_{ij} = N_{A_i}(j) \pmod 2$. Since the sequences A are complementary

$$N_A(j) = \sum_{i=1}^{4} N_{A_i}(j) = 0 \qquad \forall j = 1, \ldots, n-1 .$$

Hence

(1) $$\sum_{i=1}^{4} s_{ij} \equiv 0 \pmod 2 \qquad \forall j = 1, \ldots, n-1$$

and thus

(2) $$\sum_{j=1}^{n-1} \sum_{i=1}^{4} s_{ij} = \sum_{i=1}^{4} \sum_{j=1}^{n-1} s_{ij} \equiv 0 \pmod 2 .$$

We use this to prove

LEMMA 13. *Any 4-complementary sequences of weight* $8k - 1$ *have length at least* $2k + 1$.

Proof. It is sufficient to show this for sequences of one variable, so suppose A_i as above have weight $8k - 1$. Further suppose, contrary to the lemma, that the length n is $2k$. Then there is precisely one zero amongst the four sequences - assume it is in A_1. It follows that

$$s_{2j} \equiv s_{3j} \equiv s_{4j} \equiv n - j \pmod 2$$

and so using $n = 2k$,

(3) $$\sum_{i=2}^{4} \sum_{j=1}^{n-1} s_{ij} \equiv k \pmod 2 .$$

Now the single zero element of A_1 must occur in precisely $n - 1 = 2k - 1$ of the pairs $\{(a_{1,x}, a_{1,y}), x \neq y\}$, and thus

$$\sum_{j=1}^{n-1} s_{1j} \equiv \frac{n(n-1)}{2} - (2k-1)$$
$$\equiv k + 1 \pmod 2 \quad (\text{using } n = 2k)$$

which contradicts (2) when (3) is considered.

The lemma now follows, since the length of the four sequences must be at least $2k$ if their weight is $8k - 1$.

A computer search has been made for 4-complementary sequences with one variable, of various weights w. Whenever $w \not\equiv 7 \pmod 8$, the search has been successful for sequences of length $\left[\frac{w+3}{4}\right]$ before exhausting all possibilities (square brackets denoting integer part as usual); when $w = 8k + 7$ this is true of sequences of length $2k + 1$, which is the best possible considering the above lemma.

The search for the sequences by computer is simplified if the positions of zeros are considered before hand. Note that these positions completely determine s_{ij} for each i and j. If sequences of length n and weight w are required, the number of zeros is known, and in general there are not many configurations of positions of zeros which will satisfy equation (1). For example, if $w = 8k + 3$ and $n = 2k + 1$, the only zero must be the $(k+1)$th entry of one of the sequences.

In fact, no case has been found of a configuration of zeros which has resultant s_{ij} satisfying (1), but for which there are no complementary sequences.

Once the desired configuration of zeros was determined, the search was made in a straight-forward manner by taking first the non-zero elements amongst

$$T_1 = \{a_{1,1}, a_{2,1}, a_{3,1}, a_{4,1}, a_{1,n}, a_{2,n}, a_{3,n}, a_{4,n}\} ,$$

progressing through all possible combinations until one was found satisfying $X_{n-1} = 0$, thence proceeding to non-zero elements amongst

$$T_2 = \{a_{1,2}, a_{2,2}, a_{3,2}, a_{4,2}, a_{1,n-1}, a_{2,n-1}, a_{3,n-1}, a_{4,n-1}\}$$

and finding a combination satisfying $X_{n-2} = 0$, etc. (Exhaustion of all possibilities for T_i of course eliminates the current values for T_{i-1}.) In this manner, after suitable values for $T_{n/2}$ were found, or, if n is odd, for

$T_{(n+1)/2} = \{a_{1,(n+1)/2}, a_{2,(n+1)/2}, a_{3,(n+1)/2}, a_{4,(n+1)/2}\}$) the remainder of the conditions $X_j = 0$ were checked. In this way an exhaustive search, for a given configuration of zeros, was carried out.

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