ORTHOGONAL DESIGNS IN POWERS OF TWO

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ABSTRACT. Repeat designs are introduced and it is shown how they may be used to give very powerful constructions for orthogonal designs in powers of two.

These results are used to show all full four variable and all three variable designs exist in 2^t, $t \le 9$. We believe these constructions demonstrate the existence of all possible four variable designs with no zeros in every power of two but we have not been able to prove this.

1. *Introduction.*

An *orthogonal design of order* n *and type* $(u_1, u_2, \ldots, u_s)(u_i > 0)$ on the commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_s\}$ such that

$$AA^T = \sum_{i=1}^{s} (u_i x_i^2) I_n .$$

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In [1], where this was first defined and many examples and properties of such designs were investigated, it was shown that the numbers of variables, s, satisfies $s \le \rho(n)$, where $\rho(n)$ (Radon's function) is defined as follows:

if $n = 2^a \cdot b$, where b is odd and, $a = 4c + d$, where $0 \le d < 4$, then

$$\rho(n) = 8c + 2^d .$$

A powerful construction for Hadamard matrices in [9] showed that the existence of orthogonal designs in powers of two was of great import. W. Wolfe and D. Shapiro showed that the cases of the problem in powers of two are crucial to the understanding of the algebraic structure involved.

All possible designs exist in order 2, 4 and 8. The existence problem for order 16 was solved in [5] and, in [4] many designs were constructed for order 32. A useful tool, *product designs*, were introduced and studied for this purpose in [4] and [7]. Yet some orthogonal designs, for example (1, 1, 1, 1, 1, 1, 3, 3) and (1, 1, 1, 1, 2, 2, 3, 3) in order 16 could not be produced by the known results. This paper analyses and generalizes the ideas used earlier in order to construct these designs, and introduces repeat designs.

2. *Basic Constructions.*

The motivation for the constructions of this paper is the following observations:

CONSTRUCTION 1. *Suppose* $(M_1; M_2; N)$ *are product designs of type* $(u_1, u_2, \ldots; v_1, v_2, \ldots; w)$ *and order* n. *Then, with* x_1, x_2 *commuting variables,*

$$\begin{bmatrix} M_1 + x_1 N & M_2 + x_2 N \\ M_2 - x_2 N & -M_2 + x_1 N \end{bmatrix}$$

is an orthogonal design of order $2n$ *and type* $(w, w, u_1, u_2, \ldots, v_1, v_2, \ldots)$.

Also a construction of Geramita and Wallis [3] that

CONSTRUCTION 2. (Geramita and Wallis [3].) *Let* P_1, P_2, P_3 *be skew-symmetric orthogonal designs of types* $(p_{i1}, p_{i2}, \ldots)$ $i = 1, 2, 3$ *in order* n *and* H *a symmetric orthogonal design of type* $(h_1, h_2, \ldots)$ *in order* n. *Further suppose* $P_i P_j^T = P_j P_i^T$ *and* $P_k H^T = H P_k^T$. *Then*

$$\begin{bmatrix} x_1 I_n + P_1 & x_3 I_n + P_2 & x_5 I_n + P_3 & H \\ -x_3 I_n + P_2 & x_1 I_n - P_1 & H & -x_5 I_n - P_3 \\ -x_5 I_n + P_3 & -H & x_1 I_n - P_1 & x_3 I_n + P_2 \\ -H & x_5 I_n - P_3 & -x_3 I_n + P_2 & x_1 I_n + P_1 \end{bmatrix}$$

is an orthogonal design of order $4n$ *and type*
$(1, p_{11}, p_{12}, \ldots, 1, p_{21}, p_{22}, \ldots, 1, p_{31}, p_{32}, \ldots, h_1, h_2, \ldots)$.

To generalize this design we introduce the following definition.

DEFINITION 3. Let $R, P_1, P_2, \ldots, H$ be orthogonal designs of order n and types $(r_1, r_2, \ldots)$, $(p_{i1}, p_{i2}, \ldots)$, $i = 1, 2, \ldots$, $(h_1, h_2, \ldots)$ respectively. Then $(R; (P_1; P_2; \ldots); H)$ are *repeat designs* of order n and types $(r_1, r_2, \ldots; (p_{11}, p_{12}, \ldots; p_{21}, p_{22}, \ldots; \ldots); h_1, h_2, \ldots)$ if

(i) $R*P_i = 0$, $i = 1, 2, \ldots$,
(ii) $R + P_i$, $i = 1, 2, \ldots$ are orthogonal designs,
(iii) $R + P_i$ and H, $i = 1, 2, \ldots$ are amicable orthogonal designs,
(iv) $P_i P_j^t = P_j P_i^t$, $i \neq j$.

Then we have

CONSTRUCTION 4. *Let* $(L; M_1 + M_2 + \ldots + M_s; N)$ *be a product designs of order* n *and types* $(a_1, \ldots, a_p; b_{11}, \ldots, b_{1q_1}, b_{21}, \ldots, b_{2q_2}, \ldots, b_{s1}, \ldots, b_{sq_s}; c_1, \ldots, c_t)$ *where* M_i *is of type* $(b_{i1}, \ldots, b_{iq_i})$. *Further let* $(R; (P_1; P_2; \ldots; P_u); H)$ *be repeat designs of order* m *and types* $(r_1, \ldots, r_w; (p_{11}, \ldots, p_{1v_1}; p_{21}, \ldots, p_{2v_2}; \ldots; p_{u1}, \ldots, p_{uv_u}); h_1, \ldots, h_x)$. *Then*

$$L \times R + M_1 \times P_{j1} + \dots + M_k \times P_{jk} + N \times H$$

is an orthogonal design of order mn *and type one of*

(i) $(a_1r, \dots, a_pr, b_1p_{11}, \dots, b_1p_{1v_1}, \dots, b_sp_{s1}, \dots, b_sp_{sq_s}, ch_1, \dots, ch_x)$,

(ii) $(a_1r, \dots, a_pr, b_1p_{11}, \dots, b_1p_{1v_1}, \dots, b_sp_{s1}, \dots, b_sp_{sq_s}, c_1h, \dots, c_th)$,

(iii) $(ar_1, \dots, ar_w, b_1p_{11}, \dots, b_1p_{1v_1}, \dots, b_sp_{s1}, \dots, b_sp_{sq_s}, ch_1, \dots, ch_x)$,

(iv) $(ar_1, \dots, ar_w, b_1p_{11}, \dots, b_1p_{1v_1}, \dots, b_sp_{s1}, \dots, b_sp_{sq_s}, c_1h, \dots, c_th)$,

where a, c, r, h *are the sum of some or all of the* a_i, c_i, r_i, h_i *respectively and* $b_i = b_{i1} + \dots + b_{iq_i}$.

This construction is at first sight quite formidable but, as we shall see, it does in fact lead to new orthogonal designs.

We have previously mentioned product designs so we need to find some repeat designs to see if any new orthogonal designs can be obtained. First we see that they do lead to new designs:

EXAMPLE 5. There are repeat designs of order 4 and types $(1; (1; 3); 1, 3)$, $(1; (2; 3); 1, 3)$, $(1; (1; 2); 1, 1, 2)$ and $(1; (2; 1, 2); 1, 2)$. They are $(I; (T_1; T_4); T_0)$, $(I; (T_3; T_4); T_0)$, $(I; (T_1, T_3); T_5)$ and $(I; (T_2; T_6); T_7)$ where

$$T_0 = \begin{bmatrix} x & y & y & y \\ y & -x & -y & y \\ y & -y & y & -x \\ y & y & -x & -y \end{bmatrix}, \qquad T_1 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & + & 0 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & + & - \\ - & - & 0 & 0 \\ - & + & 0 & 0 \end{bmatrix}, \qquad T_3 = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & - & + \\ - & + & 0 & 0 \\ - & - & 0 & 0 \end{bmatrix},$$

$$T_4 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{bmatrix}, \qquad T_5 = \begin{bmatrix} u & v & w & w \\ v & -u & -w & w \\ w & -w & v & -u \\ w & w & -u & -v \end{bmatrix},$$

$$T_6 = \begin{bmatrix} 0 & a & b & b \\ -a & 0 & -b & b \\ -b & b & 0 & -a \\ -b & -b & a & 0 \end{bmatrix}, \qquad T_7 = \begin{bmatrix} u & 0 & w & w \\ 0 & -u & -w & w \\ w & -w & 0 & -u \\ w & w & -u & 0 \end{bmatrix}.$$

Repeat designs in order 4 and types (1; (1, 1; 1, 1); 1), (1; (1, 1; 1, 2); 2), (1; (1, 1; 2); 1, 2), (1; (1; 1, 2); 2, 2) and (1; (1, 2; 1, 2); 4) can be constructed using Lemma 12.

EXAMPLE 6. There are product designs of types (1, 1, 2, 3; 1, 3, 3; 1), (2, 2; 1, 1, 1, 1; 4) and (1, 1, 1; 1, 1, 1; 5) in order 8. Then using the repeat design of types (1; (2; 3) 1, 3) in order 4 with the matrix of weight 2 used once only we have orthogonal designs of types (1, 1, 2, 3, 2, 9, 9, 1, 3), (2, 2, 2, 3, 3, 3, 4, 12) and (1, 1, 1, 2, 3, 3, 5, 15) in order 32. Since all of these have weight 31, we use the Geramita-Verner theorem to obtain the following orthogonal designs in order 32: (1, 1, 1, 1, 2, 2, 3, 3, 9, 9), (1, 2, 2, 2, 3, 3, 3, 4, 12) and (1, 1, 1, 1, 2, 3, 3, 5, 15). These last two designs are new.

Remark. In the preceding example we have concentrated on constructing orthogonal designs with no zero. There is considerable scope to exploit these constructions to look for other orthogonal designs in order 32 and higher powers of 2.

We can collect the results from Example 5 in the following statement

STATEMENT 7. *In order* 4 *there exist repeat designs of types* (1; (r; s); h) *for* $0 \le r,\ s \le 3,\ 0 \le h \le 4$.

Noting that the repeat designs (R; (P); H) are just amicable orthogonal designs R + P and H we see that

COROLLARY 8. *There exist amicable orthogonal designs of types* ((1, r); (h)) *in order* 4 *for* $0 \le r \le 3,\ 0 \le h \le 4$.

Remark. The non-existence of amicable orthogonal designs of types $((1, 7); (5))$ in order 8, and $((1, 15); (1))$ in order 16 means there are *no* repeat designs of types $(1; (r; 7); 5)$ in order 8 and $(1; (r; 15); 1)$ in order 16 (see Robinson [4]).

The construction and replication lemmas given later allow us to say

COMMENT 9. In order 8 there, in fact, exist repeat designs $(1; (r); h)$ for all $0 \le r \le 7$ and $0 \le h \le 8$ *except* $r = 7$, $h = 5$ (which cannot exist).

In order 16 there exist repeat designs $(1; (r); h)$ for all $r = 1, 2, 3, \ldots, 15$, $h = 1, 2, \ldots, 16$ except possibly the following pairs (r, h): $(13, 1)$, $(13, 5)$, $(13, 9)$, $(15, 7)$, $(15, 9)$, $(15, 15)$ which are undecided and $(15, 1)$ which does not exist.

3. Construction and Replication of Repeat Designs.

We now show that many repeat designs can be constructed.

LEMMA 10. *Suppose* $((a); (b_1, b_2))$ *and* $((c); (d_1, d_2))$ *are the types of amicable orthogonal designs in orders* n_1 *and* n_2. *Then there is a repeat design in order* n_1n_2 *of type* $(b_1d_1; (ad_2, b_2d_1; b_2c, b_1d_2); ac)$.

Proof. Let A, $x_1B_1 + x_2B_2$ and C, $y_1D_1 + y_2D_2$ be the amicable orthogonal designs. Then

$(B_1 \times D_1;\ (xA \times D_2 + yB_2 \times D_1;\ uB_2 \times C + wB_1 \times D_2);\ A \times C)$
are the required repeat designs.

EXAMPLE 11. Let $A = C = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$, $B_1 = D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $B_2 = D_2 = \begin{bmatrix} 0 & 1 \\ - & 0 \end{bmatrix}$. Then the repeat design in order 4 and type $(1;\ (1,\ 2;\ 1,\ 2);\ 4)$ is

$$\left(I_4;\ \left(\left[\begin{array}{cc|cc} 0 & y & x & x \\ \bar{y} & 0 & x & \bar{x} \\ \hline \bar{x} & \bar{x} & 0 & y \\ \bar{x} & x & \bar{y} & 0 \end{array}\right];\ \left[\begin{array}{cc|cc} 0 & u & w & u \\ \bar{u} & 0 & \bar{u} & w \\ \hline \bar{w} & u & 0 & \bar{u} \\ \bar{u} & \bar{w} & u & 0 \end{array}\right]\right);\ z\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ \hline 1 & 1 & - & - \\ 1 & - & - & 1 \end{array}\right]\right).$$

Before we proceed to our uses of repeat designs we first note some replication results.

LEMMA 12. *Suppose there are repeat designs of type* $(r;\ (p_1,\ \ldots,\ p_i;\ q_1,\ \ldots,\ q_j);\ h_1,\ \ldots,\ h_k)$ *in order* n *where* $h_1 + h_2 + \ldots + h_k = h$ *and* $p_1 + \ldots + p_i = p$. *Further suppose* $A + B$ *and* $C + D$ *are amicable orthogonal designs of types* $((a,\ b);\ (c,\ d))$ *and order* m. *Then there are repeat designs of order* mn *and types*

(i) $(ar;\ (cp_1,\ cp_2,\ \ldots,\ br;\ aq_1,\ aq_2,\ \ldots,\ bh);\ ch)$,

(ii) $(ar;\ (ap_1,\ ap_2,\ \ldots;\ cq_1,\ cq_2,\ \ldots);\ ah_1,\ ah_2,\ \ldots,\ ch_1,\ \ldots)$,

(iii) $(ar;\ (ap_1,\ ap_2,\ \ldots,\ bh_1;\ cq_1,\ cq_2,\ \ldots);\ ch_1,\ ah_2,\ ah_3,\ \ldots)$,

(iv) $(ar;\ (bh_1,\ bh_2,\ \ldots;\ rb + pd,\ cq_1,\ cq_2,\ \ldots);\ rd + bp)$, *where* $d = b$,

(v) $(ar; (cq_1, cq_2, \ldots; cp); ah_1, ah_2, \ldots, bp)$,

(vi) $(ar; (br, dp_1, dp_2, \ldots; aq_1, aq_2, \ldots, bh); dh)$,

(vii) $(ar; (cp_1, \ldots; cq_1, \ldots); ch_1, ch_2, \ldots, dr)$,

(viii) $(ar; (cp_1, \ldots, dq_1, \ldots); ah_1, ah_2, \ldots, bp_1, bp_2, \ldots)$,

(ix) $(ar; (ap_1, \ldots; aq_1, \ldots); ch, dh)$,

(x) $(cr; (br; bh_1, bh_2, \ldots); ar)$,

(xi) $(cr; (br, bh); ar, abrh)$.

Proof. Use the following constructions:

(i) $(A \times R; (C \times P + xB \times R; yA \times Q + zB \times H); C \times H)$,

(ii) $(A \times R; (A \times P; C \times Q); xA \times V + C \times W)$,

(iii) $(A \times R; (A \times P + xB \times V; C \times Q); C \times V + yA \times W)$,

(iv) $(A \times R; (B \times H; xB \times R + yC \times Q - xD \times P); D \times R + B \times P)$,

(v) $(A \times R; (C \times Q; C \times P); xA \times H + yB \times P)$,

(vi) $(A \times R; (B \times R + wD \times P; xA \times Q + yB \times H); D \times H)$,

(vii) $(A \times R; (C \times P; C \times Q); C \times H + yD \times R)$,

(viii) $(A \times R; (C \times P + xD \times Q); A \times H + yB \times P)$,

(ix) $(A \times R; (A \times P; A \times Q); C \times H + yD \times H)$;

(x) $(C \times R; (B \times R; B \times H); A \times R)$;

(xi) Use Lemma on the result (x).

COROLLARY 13. *There are repeat designs of type* $(1; (1, 2, \ldots, 2^{t-1}; 1, 2, \ldots, 2^{t-1}); 2^t)$ *in order* 2^t.

Proof. Use part (i) of the lemma repeatedly with repeat designs of type $(1; (1, 2; 1, 2); 4)$ in order 4 and the amicable orthogonal designs of type $((1, 1); (2))$ in order 2.

4. *Construction of Orthogonal Designs.*

The use of repeat designs with product designs is so powerful a source of orthogonal designs that it is quite impossible to indicate all the designs constructed. Hence we give only those that were used to give the final result of this paper. We use Robinson's Ph.D. thesis [4] as a source for product designs.

COROLLARY 14. *The following types of orthogonal designs exist in order* 2^t:

(i) $(1, 1, 1, 1, 2, 2, 4, 4, \ldots, 2^{t-2}, 2^{t-2})$;

(ii) $(1, 1, 2, 1, 2, 4, 8, \ldots, 2^{t-3}, 3, 6, 12, \ldots, 3.2^{t-3})$;

(iii) $(1, 1, 2, 4, 8, \ldots, 2^{t-3}, 2^{t-3}, 2^{t-2}, 3, 3, 6, \ldots, 3.2^{t-4})$;

(iv) $(1, 1, 2, 4, 8, \ldots, 2^{t-3}, 3, 6, 9, 18, \ldots, 9.2^{t-5}, 3.2^{t-4})$;

(v) $(1, 2, 3, 2^{t-4}, 3.2^{t-4}, 3.2^{t-3}, 3, 3, 6, 6, 12, 12, \ldots, 3.2^{t-5}, 3.2^{t-5})$;

(vi) $(2, 1, 2, \ldots, 2^{t-s-4}, (2^{t-s-3} - 1)(\text{bin } 2^{s+1} - 1), 2^{t-s-2}(\text{bin}(2^{s+1} - 1)), 2^{s+1}, 2^{t-2}, 2^{t-s-3} - 1, 2^{t-s-3} - 1)$;

where $x \text{ bin}(2^y - 1)$ *means* x *times the binary expansion of* $2^y - 1$, i.e., $x, x, 2x, 4x, \ldots, 2^{y-1}x$.

Proof. (i) Proved by Robinson [7]. (ii) Use the product design $(1, 1, 2, 3, 6, 12, \ldots, 3.2^{t-4}; 1, 3.2^{t-3}; 1, 2, \ldots, 2^{t-4})$ with the amicable orthogonal designs of types $((1, 1); (1, 1))$. For the remainder we use product designs with repeat designs as

$2^t - 1$ *or* $2^{t-1} - 1$).

Proof. Call the product designs $(1, \text{bin } x, 2^j - x, 2^j, \ldots, 2^{k-2}; 1, 2^{k-1}; 1, 2, \ldots, 2^{k-2})$ the product designs A. We now use product designs with repeat designs as indicated: (i) use A with $(1; (1, 2, \ldots, 2^{t-1}; 2^t - 1); 2^t)$; (ii) use A with $(1, 2^s; (1, 2, \ldots, 2^{s-1}, 2^{s+1}, \ldots, 2^{t-1}; 2^t - 2^s - 1); 2^t)$; (iv) and (v) use A with $(2^t - a; ((\text{bin } a); a); 2^t)$ to get the orthogonal design $(a, a, a \text{ bin}(2^{k-1} - b - 1), b(\text{bin } a), 2^t - a, 2^{k-1}(2^t - a), 2^t, 2^{t+1}, \ldots, 2^{t+k-2})$. For (iii) we use the amicable orthogonal designs of type $((a, 2^{t+k-2} - a); (2^{t+k}))$ in order 2^{t+k} with product designs (1, 1, 1; 1, 1, 1; 1) in order 4.

So we have

COROLLARY 16. *All orthogonal designs of type* $(a, b, c, 2^t - a - b - c)$ *and of type* (a, b, c), $0 \le a + b + c \le 2^t$ *exist for* $t = 2, 3, 4, 5, 6, 7, 8, 9$.

Remark. We believe the results of this paper do, in fact, allow the construction of all full orthogonal designs (that is with no zero) with 4 variables in every power of 2 but we have not been able to prove this result.

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