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AN ALGORITHM FOR THE PERMANENT OF CIRCULANT MATRICES

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1. Introduction. The *permanent* of an $n\times n$ matrix $A=(a_{ij})$ is the matrix function

$$\text{per}\, A=\sum_{\pi\in S_n} a_{1\pi(1)}\cdots a_{n\pi(n)} \tag{1}$$

where the summation is over all permutations in the symmetric group, S_n. An $n\times n$ matrix A is a *circulant* if there are scalars $a_1,\ldots,a_n$ such that

$$A=\sum_{i=1}^{n} a_iP^{i-1} \tag{2}$$

where P is the $n\times n$ permutation matrix corresponding to the cycle $(12\cdots n)$ in S_n. In general the computation of the permanent function is quite difficult chiefly because it is not invariant under addition of a multiple of one row to another. Using the principle of "inclusion and exclusion", Ryser [6, p. 27] gave an expansion for the permanent. Also the Laplace expansion is available for the permanent [2, p. 20]. Neither of these methods are particularly efficient. In [4] Minc considered the permanents of matrices with entires either 0 or 1. Minc also studied tridiagonal circulants in [5]. Metropolis, Stein, and Stein [3] have given recurrence relations for evaluating the permanents of circulant matrices (2) where the first k scalars are 1 and the remaining ones are 0. Permanents of circulant matrices were also studied by Tinsley [7].

2. The algorithm. If we consider the scalars as indeterminates over an underlying field every term of the permanent (1) of a circulant matrix (2) is a monomial in the scalars $a_1,\ldots,a_n$. Our algorithm deletes appropriate monomials from the set of all n^n such monomials until only those appearing in the permanent remain. This is easily programmed because the monomials need only be considered one at a time and may be indexed by the n^n n-tuples chosen from $1,\ldots,n$ and ordered lexicographically. It is convenient to state the algorithm in terms of these indices.

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Algorithm. If $I=(i_1,\ldots,i_n)$ is an n-tuple with entries chosen from $1,\ldots,n$ then discard I if

(i) $\sum_{j=1}^{n} i_j \neq 0 \pmod n$,

or if

(ii) $i_{j+k} \equiv i_j - k \pmod n$ for any k and $j=1,\ldots,n-1$.

Condition (ii) excludes the occurrence of terms in the permanent of (2) with the following pattern

$$\cdots a_i \underbrace{*\cdots*}_{k-2\text{ entries}} a_{i+k+1}\cdots \tag{3}$$

where a_{i+n} is considered to be a_i if necessary. For example, if $n=4$ condition (ii) of the algorithm discards a monomial whenever one of the following patterns occurs:

$\cdots 14\ldots, \ldots 21\ldots, \ldots 32\ldots, \ldots 43\cdots$

$\cdot 1*3\cdot, \cdot 2*4\cdot, \cdot 3*1\cdot, \cdot 4*2\cdot$

$1**2, 2**3, 3**4, 4**1.$

Condition (i) leaves the following 4-tuples:

1111	1214	1313	1412	2114	2213	2312	2411
1124	1223	1322	1421	2123	2222	2321	2424
1133	1232	1331	1434	2132	2231	2334	2433
1142	1241	1344	1443	2141	2244	2343	2442
3113	3212	3311	3414	4112	4211	4314	4413
3122	3221	3324	3423	4121	4224	4323	4422
3131	3234	3333	3432	4134	4233	4332	4431
3144	3243	3342	3441	4143	4242	4341	4444

Condition (ii) eliminates all of the above 4-tuples which are underlined.

Hence, if $n=4$ the permanent of (2) will be

$$\sum_{i=1}^{4} a_i^4 + 2a_1^2a_3^2 + 2a_2^2a_4^2 + 4\sum_{i=1}^{4} a_i^2 a_{i+1} a_{i+3}.$$

Let R_n denote the set of n-tuples left by the algorithm. We remark that the n-tuples in R_n need not be formally distinct; e.g., 1313 and 3131 are both in R_4. The number of formally distinct diagonal products in the permanent of an arbitrary circulant has been determined by Brualdi and Newman [1].

3. Proofs

THEOREM. *Let A be a circulant matrix* (2) *with scalars $a_1, \ldots, a_n$. Then*

$$\operatorname{per} A = \sum a_{i_1} \cdots a_{i_n}$$

where the summation is over all $(i_1, \ldots, i_n) \in R_n$.

Proof. We are concerned with determining conditions for which $a_{i_1} \cdots a_{i_n}$ is a term of the permanent of the $n \times n$ matrix (2). Thus, a_{i_k} always denotes an element of the kth row of (2). The ith column of (2) is

$$\begin{bmatrix} a_i \\ a_{i-1} \\ \cdot \\ \cdot \\ \cdot \\ a_{i-n+1} \end{bmatrix}$$

where subscripts are taken modulo n. If the Laplace expansion along the first row is used to find per A the entry a_{i-k+1} cannot be chosen from row k to appear in any monomial beginning with a_i. In any monomial of the permanent the pattern (3) cannot appear since we may expand along any row.

Therefore any $(i_1, \ldots, i_n)$ in R_n satisfies

$$i_{j+k} \neq i_j - k \quad \text{for} \quad k = 1, \ldots, n-1.$$

Again, subscripts are taken modulo n when necessary.

Write $i_{j+k} = i_j - k + x_{jk} \pmod n$ where $x_{jk} \neq 0$, $1 \leq x_{jk} \leq n-1$, and $k \neq 0$. We would like to show that $s \neq t$ implies $x_{js} \neq x_{jt}$.

Suppose $x_{js} = x_{jt}$. Then

$$x_{js} = i_{j+s} - i_j + s = i_{j+t} - i_j + t = x_{jt}.$$

Hence

$$i_{j+s} = i_{j+t} - (s-t),$$

but unless $s = t$

$$i_{j+s} = i_{j+t+(s-t)} \neq i_{j+t} - (s-t).$$

So assuming $x_{js} = x_{jt}$ leads to a contradiction. Hence the contrapositive is true and $s \neq t$ implies $x_{js} \neq x_{jt}$.

Step (i) is included in the algorithm because it is easy to implement. In fact, (ii) implies (i) as we now show:

$$\begin{aligned} \sum_{k=0}^{n-1} i_{j+k} = i_j + \sum_{k=1}^{n-1} i_{j+k} &= \left(i_j + \sum_{k=1}^{n-1} (i_j - k + x_{jk})\right) \pmod n \\ &= \left(ni_j - \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} x_{jk}\right) \pmod n \\ &= (ni_j - \tfrac{1}{2}n(n-1) + \tfrac{1}{2}n(n-1)) \pmod n \\ &\equiv 0 \pmod n. \end{aligned}$$

We have shown why the n-tuples mentioned in (i) and (ii) must be discarded. It remains to show that no more should be excluded. Condition (ii) says there are n choices for a_{i_1}, $n-1$ choices for a_{i_2} and in general $n-k+1$ choices for a_{i_k}. That is, condition (ii) does not eliminate exactly $n!$ terms. But there are $n!$ terms in the permanent so precisely the right number of monomials has been excluded.

4. Numerical results. Dr. Joan Cooper wrote a Fortran programme for our algorithm which was implemented on an ICL 1904A at the University of Newcastle, N.S.W., Australia. The following various 7×7 circulants were computed using 2.54 seconds of core time.

First row of circulant matrix A							per A	row sum of $A=r$	per (A/r)
3	1	1	0	1	0	0	4416	6	0.0157750
1	1	1	0	0	0	0	31	3	0.0141747
1	1	0	0	0	0	0	2	2	0.0156250
1	1	1	1	1	1	1	5040	7	0.0061199
0	1	1	0	1	0	0	24	3	0.0109739
1	1	1	0	1	0	0	144	4	0.0087891
1	1	−1	0	0	0	0	1	1	1.0

We believe the algorithm is not shown to best advantage as most of the elapsed time is due to reading the 7-tuples of the example from disc.

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