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SOME ASYMPTOTIC RESULTS FOR ORTHOGONAL DESIGNS : II

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Résumé. — Une « configuration orthogonale » d'ordre n et de type $(u_1, u_2, ..., u_s)$ $(u_i > 0)$ sur les variables $x_1, x_2, ..., x_s$ qui commutent est une matrice carrée A d'ordre n dont les éléments appartiennent à l'ensemble $\{ 0, \pm x_1, ..., \pm x_s \}$ et telle que

$$AA^t = \sum_{i=1}^{s} (u_i x_i^2) I_n .$$

Dans un article récent [1] Peter Eades a montré que pour de nombreux types, il existe des configurations orthogonales d'ordre n pourvu que n soit assez grand. Dans cet article nous considérons les 4-uples (u_1, u_2, u_3, u_4) tels que $u_1 + u_2 + u_3 + u_4 \leqq 28$ et nous établissons des minorants pour l'existence de configurations orthogonales du type ci-dessus.

Abstract. — In a recent manuscript « Some asymptotic results for orthogonal designs » Peter Eades showed that for many types of orthogonal designs existence is established once the order is large enough. This paper examines 4-tuples (s_1, s_2, s_3, s_4) where $s_1 + s_2 + s_3 + s_4 \leqslant 28$ and establishes lower bounds for the existence of orthogonal designs of that type.

Introduction. — *An orthogonal design of order n and type* $(u_1, u_2, ..., u_s)$ $(u_i > 0)$ on the commuting variables $x_1, x_2, ..., x_s$ is an $n \times n$ matrix A with entries from $\{ 0, \pm x_1, ..., \pm x_s \}$ such that

$$AA^t = \sum_{i=1}^{s} (u_i x_i^2) I_n .$$

Alternatively, the rows of A are formally orthogonal and each row has precisely u_i entries of the type $\pm x_i$.

In [4], where this was first defined and many examples and properties of such designs were investigated, it is mentioned that

$$A^t A = \sum_{i=1}^{s} (u_i x_i^2) I_n$$

and so the alternative description of A applies equally well to the columns of A. It is also shown in [4] that $s \leqslant \rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

when

$n = 2^a.b , \quad b \text{ odd} , \quad a = 4c + d , \quad 0 \leqslant d < 4 .$

D. Shapiro and W. Wolfe have found powerful algebraic non-existence theorems for orthogonal designs which supercede those of Geramita, Geramita and Wallis [4]. In addition Geramita and Verner [5] and P. J. Robinson [9] have found strong combinatorial theorems. We quote those relevant to this paper.

Theorem 1 (Wolfe). — *Let* $n \equiv$ (mod 8) *and* $(a_i, a_j)_p$ *be the Hilbert Norm residue symbol. There exists an orthogonal design in order n and type*

(i) (a_1, a_2, a_3, a_4) *only if* $\prod_{i=1}^{4} a_i$ *is a square and*

$$\prod_{1 \leqslant i < j \leqslant 4} (a_i, a_j)_p = 1$$

at every prime p ;

(ii) (a_1, a_2, a_3) *only if*

$$(-1, a_1 a_2 a_3)_p \prod_{1 \leqslant i < j \leqslant 3} (a_i, a_j)_p = 1$$

at every prime p ;

(iii) (a_1, a_2) *only if* $a_1 a_2$ *is the sum of three squares.*

See Hall [8] for details of how to evaluate $(a_i, a_j)_p$.

Theorem 2 (Geramita-Verner). — *If* $n \equiv 0$ (mod 4) *then there exists an orthogonal design in order n and* *of type* $(u_1, \ldots, u_s)$ *where* $\sum_{i=1}^{s} u_i = n - 1$ *if and only if there exists an orthogonal design in order* n *and type* $(1, u_1, \ldots, u_s)$.

The following results will be used in the proof of our main theorem.

Theorem 3 (Geramita-Wallis). — *If there exists an orthogonal design of type* (s, t) *in order* n *there exists an orthogonal design of type* (s, s, t, t) *in order* $2n$.

Theorem 4 [11]. — *If there exists an orthogonal design of type* $(u_1, u_2, \ldots, u_s)$ *in order* n *there exists orthogonal designs of type*

(i) $(e_1 u_1, e_2 u_2, \ldots, e_s u_s)$ *where* $e_i = 1$ *or* 2,

(ii) $(u_1, u_1, fu_2, \ldots, fu_s)$ *where* $f = 1$ *or* 2, *in order* $2n$.

Theorem 5 (Robinson). — *All* 4-*tuples* (a, b, c, d) *with* $0 \leqslant a + b + c + d \leqslant 32$ *are the types of orthogonal designs in order* 32.

Theorem 6 [3]. — *All* 4-*tuples* (a, b, c, d) *with* $0 \leqslant a + b + c + d \leqslant 16$ *are the types of orthogonal designs in order* 16.

Lemma 7 [3, 7]. — *All* 4-*tuples* (a, b, c, d) *which are not excluded by Wolfe's necessary conditions or the Geramita-Verner theorem are*

(i) *the types of orthogonal designs in order* 12 *when* $a + b + c + d \leqslant 12$;

(ii) *the types of orthogonal designs in order* 20 *when* $a + b + c + d \leqslant 20$ *except possibly for* (1, 3, 6, 8), (1, 4, 4, 9) *and* (2, 2, 5, 5) *which are undecided.*

Theorem 8. — *Suppose* (s_1, s_2, s_3, s_4) *satisfies Wolfe's necessary conditions for the existence of orthogonal designs in order* $n \equiv 4 \pmod 8$. *Then*

(i) *if* $s_1 + s_2 + s_3 + s_4 \leqslant 12$ *there is an orthogonal design of type* (s_1, s_2, s_3, s_4) *and order* $4t$ *for all* $t \geqslant 3$;

(ii) *if* $s_1 + s_2 + s_3 + s_4 \leqslant 16$, *there is an orthogonal design of type* (s_1, s_2, s_3, s_4) *and order* $4t$ *for all* $t \geqslant 4$ *with the possible exception of* (2, 2, 5, 5) *which exists in order* $4t$, $t \geqslant 4$, $t \neq 5$;

(iii) *if* $16 < s_1 + s_2 + s_3 + s_4 \leqslant 28$ *the table gives the smallest known* N *such that* (s_1, s_2, s_3, s_4) *is the type of an orthogonal design which exists for all* $4t \geqslant N$.

Proof. — Theorem 6, lemma 7 and table 1 of [3] give (i) and (ii) immediately except for (1, 1, 4, 9) and (1, 2, 2, 9). Both these exist in 16, 20, 24 and 28 (Theorem 6, Lemma 7 [2] and [11]) so we have the results of (i) and (ii).

Now (1, 1, 1, 16) does not exist in order 20 and (1, 3, 6, 8) and (1, 4, 4, 9) are not known in order 20. All other (s_1, s_2, s_3, s_4) with

$$16 < s_1 + s_2 + s_3 + s_4 \leqslant 20$$

exist. The existence of (1, 8), (4, 5) and (5, 5) in order 12, (1, 1, 2, 4, 9) in order 24 (from [3]) and the results of [11] give all these (s_1, s_2, s_3, s_4) in 24 except (1, 3, 6, 8) and (2, 3, 6, 9). The existence of (1, 2, 3, 6) in $4n$, $n \geqslant 3$ gives (1, 2, 3, 6, 6) and hence (1, 3, 6, 8) in $8n$, $n \geqslant 3$. These results plus those in table 1 of [3] give the result immediately except for (1, 1, 1, 16), (1, 2, 8, 9), (1, 3, 6, 8), (1, 4, 4, 9), (1, 5, 5, 9), (2, 2, 4, 9) and (2, 3, 6, 9) : and of these all except the last two exist in 28 from [2]. A (1, 1, 1, 16) and a (1, 4, 4, 9) exist in

TABLE

N is the order such that the indicated design exists in every order $4t \geqslant N$

$12 < s_1 + s_2 + s_3 + s_4 \leqslant 16$	N	$16 < s_1 + s_2 + s_3 + s_4 \leqslant 20$	N	$20 < s_1 + s_2 + s_3 + s_4 \leqslant 24$	N	$24 < s_1 + s_2 + s_3 + s_4 \leqslant 28$	N
(1, 1, 4, 9)	16	(1, 1, 1, 16)	24	(1, 1, 2, 18)	48	(1, 1, 1, 25)	104
(1, 2, 2, 9)	16	(1, 1, 8, 8)	20	(1, 1, 4, 16)	24	(1, 1, 5, 20)	144
(1, 2, 4, 8)	16	(1, 1, 9, 9)	20	(1, 1, 10, 10)	40	(1, 1, 8, 18)	56
(1, 4, 4, 4)	16	(1, 2, 8, 9)	40	(1, 2, 2, 16)	48	(1, 1, 9, 16)	312
(1, 4, 5, 5)	16	(1, 3, 6, 8)	48	(1, 2, 6, 12)	24	(1, 1, 13, 13)	48
(2, 2, 2, 8)	16	(1, 4, 4, 9)	48	(1, 4, 8, 8)	32	(1, 2, 4, 18)	80
(2, 2, 5, 5)	24	(1, 5, 5, 9)	40	(1, 4, 9, 9)	72	(1, 3, 6, 18)	1 736
(2, 3, 4, 6)	16	(2, 2, 4, 9)	40	(2, 2, 2, 18)	48	(1, 4, 4, 16)	40
(4, 4, 4, 4)	16	(2, 2, 8, 8)	20	(2, 2, 4, 16)	24	(1, 4, 10, 10)	40
		(2, 3, 6, 9)	80	(2, 2, 9, 9)	24	(1, 8, 8, 9)	80
		(2, 4, 4, 8)	20	(2, 2, 10, 10)	24	(1, 9, 9, 9)	80
		(2, 5, 5, 8)	20	(2, 4, 6, 12)	24	(2, 4, 4, 18)	80
		(3, 3, 6, 6)	20	(2, 4, 8, 9)	160	(2, 8, 8, 8)	28
		(4, 4, 5, 5)	20	(3, 3, 3, 12)	48	(2, 8, 9, 9)	80
		(5, 5, 5, 5)	20	(3, 4, 6, 8)	56	(3, 6, 8, 9)	952
				(4, 4, 4, 9)	112	(4, 4, 4, 16)	28
				(4, 4, 8, 8)	24	(4, 4, 9, 9)	48
				(4, 5, 5, 9)	168	(4, 4, 10, 10)	28
				(6, 6, 6, 6)	24	(5, 5, 8, 8)	32
						(5, 5, 9, 9)	80
						(7, 7, 7, 7)	28

40 as a (1, 1, 16) and a (1, 4, 9) exist in 20 from [7]. The (1, 2, 3, 6) in $4n$, $n \geqslant 3$ (from [3]) gives (1, 2, 3, 3, 6, 6) and hence (2, 3, 6, 9) in $16n$, $n \geqslant 3$. These together with the existence of (1, 1, 1, 16) in $4n$, n (odd) $\geqslant 7$ and the results in 32 give the results of the table.

For $20 < s_1 + s_2 + s_3 + s_4 \leqslant 24$ table 2 of [3] plus the (1, 2, 8) in 12 giving a (1, 1, 4, 16) in 24 gives the result immediately for those with $N = 24$.

From [2] (1, 1, 2, 18), (1, 1, 10, 10), (1, 2, 2, 16), (2, 2, 2, 18) and (3, 3, 3, 12) exist in 28. From [11], the results in order 32 and the existence of appropriate results in order 20 we have these designs existing in 24, 32 and 40. Hence $N = 48$. Table 3 of [3] shows (1, 1, 10, 10) exists in $4n$, $n \geqslant 10$ so $N = 40$ for this 4-tuple.

Table 3 of [3] shows (1, 4, 8, 8) exists in $4n$, $n \geqslant 8$ so $N = 32$ for this 4-tuple.

Now (1, 4, 9, 9) exists in 24 [11], 32, 40 (since (1, 4, 9) exists in 20 from [7]), and 52 since there exists a circulant $W(13, 9)$ and a (1, 4) made of two circulant matrices in 26. This gives $N = 72$.

[11] gives a (2, 4, 8, 9) in 24, it exists in 32 and the (1, 2, 8, 9) in 20 gives it in 40 and 7×20 (this latter by replacing the first variable by a circulant $W(7, 4)$ and the other variables by the identity matrix of order 7). Hence $N = 160$.

Similarly a (4, 4, 4, 9) exists in 7×12 and a (4, 5, 5, 9) exists in 7×20. Clearly (1, 4, 4, 4, 4, 4, 4, 4) and (1, 1, 1, 1, 4, 4, 4, 4) exist in 56 by replacing some of the variables of the (1, 1, 1, 1, 1, 1, 1, 1) design in 8 by the circulant $W(7, 4)$. Hence (4, 4, 4, 9) and (4, 5, 5, 9) exist in 56. (4, 4, 9) and (4, 5, 9) exist in 20 [7] and 24 [11] so (4, 4, 4, 9) and (4, 5, 5, 9) exist in 40 and 48. They both exist in 32. Hence $N = 112$ for (4, 4, 4, 9) and $N = 168$ for (4, 5, 5, 9).

(2, 3, 4, 6) exists in $4n$, $n \geqslant 4$ (see [3]) so (2, 3, 4, 6, 6) and (3, 4, 6, 8) exists in $8n$, $n \geqslant 4$. [2] gives (3, 4, 6, 8) in 28 and so $N = 56$.

Table 2 of [3] gives the result immediately for (2, 8, 8, 8), (4, 4, 4, 16), (4, 4, 10, 10) and (7, 7, 7, 7). Table 3 of [3] gives the result for (5, 5, 8, 8), (1, 4, 4, 16) and (1, 4, 10, 10).

From above (1, 2, 4, 18), (1, 8, 8, 9), (1, 9, 9, 9), (2, 4, 4, 18), (2, 8, 9, 9), (4, 4, 9, 9) and (5, 5, 9, 9) exist in 32 and 52. From results in 20 [7], 24 [11] and 28 [2] they all exist in 40, 48 and 56. Hence $N = 80$. From table 3 of [3] (1, 1, 13, 13) exists in $4n$, $n \geqslant 14$; it is constructed above for order 52. (1, 13) exists in 24 so (1, 1, 13, 13) exists in 48 and we have $N = 48$.

(1, 1, 1, 25) exists in 28 ([2]), 32, and 48 (since (1, 1, 1, 1, 12) exists in 24). Hence $N = 104$.

(1, 1, 5, 20), (1, 1, 8, 18) and (1, 1, 9, 16) exist in 32, 40 (from above and the existence of (1, 4, 9) in 20 which gives (1, 1, 8, 18) in 40), and 48 from results in 24 from [11]. Now (1, 1, 8, 18) exists in 28 (see [2]) so $N = 56$ for this 4-tuple. The (1, 1, 5, 5) in 12 gives a (1, 1, 5, 20) in 7×12 so $N = 144$. Now (1, 1, 1, 9) exists in 12 so (1, 1, 9, 16) exists in 21×12 and $N = 132$.

(1, 3, 6, 18) exists in $16n$, $n \geqslant 2$ because (1, 2, 3, 6) in $4n$, $n \geqslant 3$ (see [3]) gives (1, 2, 3, 6, 12) in $16n$, $n \geqslant 3$ and (1, 3, 6, 18) also exists in 32. The (1, 2, 3, 6) in 12 means a (1, 3, 6, 18) exists in 12×13. Hence $N = 1\,736$.

References

[1] Peter EADES, Some asymptotic existence results for orthogonal designs, *Ars. Combinatoria* **1** (1976) 109-118.

[2] Peter EADES and Jennifer Seberry WALLIS, An infinite family of skew weighing matrices, *Combinatorial Mathematics IV : Proceedings of the Fourth Australian Conference.* Lecture Notes in Mathematics, Vol. 560, Springer-Verlag, Berlin-Heidelberg-New York (1976), p. 27-40.

[3] Peter EADES, Jennifer Seberry WALLIS, Nicholas WORMALD, A note on asymptotic existence results for orthogonal designs, *Combinatorial Mathematics V : Proceedings of the Fifth Australian Conference* (to appear).

[4] Anthony V. GERAMITA, Joan Murphy GERAMITA, Jennifer Seberry WALLIS, Orthogonal designs, *Linear and Multilinear Algebra* **3** (1975/76) 281-306.

[5] A. V. GERAMITA, J. H. VERNER, Orthogonal designs with zero diagonal, *Canad. J. Math.* **28** (1976) 215-224.

[6] Anthony V. GERAMITA, Jennifer Seberry WALLIS, Orthogonal designs III : weighing matrices, *Utilitas Math.* **6** (1974) 209-236.

[7] Anthony V. GERAMITA, Jennifer Seberry WALLIS, Orthogonal designs IV : existence questions, *J. Combinatorial Th. Ser. A* **19** (1975) 66-83.

[8] Marshall HALL, Jr., Combinatorial Theory, Blaisdell, Waltham Mass. (1967).

[9] Peter J. ROBINSON (private communication, 1976).

[10] Daniel SHAPIRO (private communication, 1976).

[11] Jennifer Seberry WALLIS, Orthogonal designs V : orders divisible by eight, *Utilitas Math.* **9** (1976) 263-281.

[12] Warren W. WOLFE, Rational quadratic forms and orthogonal designs, *J. Number Theory.*