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A NOTE ON USING SEQUENCES TO CONSTRUCT ORTHOGONAL DESIGNS

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Several constructions are given which show how to construct orthogonal designs from sequences of commuting variables with zero non-periodic auto-correlation function.

§1. GOLAY SEQUENCES AND OTHER SEQUENCES WITH ZERO AUTO-CORRELATION FUNCTION

Let $X = \{a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn}\}$ be m sequences of commuting variables of length n.

The *non-periodic auto-correlation function of the family of sequences* X (denoted N_X) is a function defined by

$$N_X(j) = \sum_{i=1}^{n-j} (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j}).$$

The *periodic auto-correlation function of the family of sequences* X (denoted P_X) is defined by

$$P_X(j) = \sum_{i=1}^{n} (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j})$$

where we assume the second subscript is actually chosen from the complete set of residues $\bmod n$.

We can interpret the function P_X in the following way: Form the m circulant matrices which have first rows respectively, $[a_{11}a_{12} \ldots a_{1n}]$, $[a_{21}a_{22} \ldots a_{2n}], \ldots, [a_{m1}a_{m2} \ldots a_{mn}]$, then $P_X(j)$ is the sum of the inner products of rows 1 and $j + 1$ of these matrices.

Clearly if X is a family of sequences as above, then

$$P_X(j) = N_X(j) + N_X(n - j), \qquad j = 1, \ldots, n - 1,$$

and

$$N_X(j) = 0 \quad \forall j \Rightarrow P_X(j) = 0 \quad \forall j.$$

Note. $P_X(j)$ may equal 0 for all $j = 1, \ldots, n - 1$ even though the $N_X(j)$ are not.

We say the *weight* of X is the number of non-zero entries in X.

Let X be as above with $N_X(j) = 0$, $j = 1, 2, \ldots, n - 1$ then we will call X *m-complementary sequences* of length n.

If $X = \{A_1, A_2, \ldots, A_m\}$ are m-complementary sequences of length n and weight $2k$ such that $Y = \{(A_1 + A_2)/2, (A_1 - A_2)/2, \ldots \ldots, (A_{2i-1} + A_{2i})/2, (A_{2i-1} - A_{2i})/2, \ldots\}$ are also m-complementary sequences (of weight k) then X will be said to be *m-complementary disjointable sequences* of length n. Y will be said to be *m-complementary disjoint sequences* of length n if all $\binom{m}{2}$ pairs of sequences are disjoint i.e. $A_i * A_j = 0$ $\forall i, j$ where $*$ is the Hadamard product.

We use $-$ for -1 and $\bar{x}$ for $-x$.

One more piece of notation is in order. If g_r denotes a sequence of integers of length r then by xg_r we mean the sequence of integers of length r obtained from g_r by multiplying each member of g_r by x. Zeros are added, if necessary, to make sequences of the same length.

If $X = \{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}\}$ are two sequences where $a_i, b_j \in \{1, -1\}$ and $N_X(j) = 0$ for $j = 1, \ldots, n-1$ then the sequences in X are called *Golay complementary sequences of length* n. E.g.

$n = 2$ 1 1 and 1 –

$n = 10$ 1 – – 1 – 1 – – – 1 and 1 – – – – – – 1 1 –

$n = 26$ 1 1 1 – – 1 1 1 – 1 – – – – – 1 – 1 1 – – 1 – – – –

– – – 1 1 – – – 1 – 1 1 – 1 – 1 – 1 1 – – 1 – – – –

Summary 1. It can be shown that Golay complementary sequences $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of length n, have the following properties:

1. $$\sum_{i=1}^{n} (x_i x_{i+j} + y_i y_{i+j}) = 0$$

for every $j \neq 0$, $j = 1, \ldots, n-1$ (where the subscripts are reduced modulo n). i.e. $P_X = 0$.

2. n is even and the sum of two squares.

3. $x_{n-i+1} = e_i x_i \Leftrightarrow y_{n-i+1} = -e_i y_i$ where $e_i = \pm 1$.

4. $$\Big[\sum_{i \in S} x_i \operatorname{Re}(\zeta^{2i+1})\Big]^2 + \Big[\sum_{i \in D} x_i \operatorname{Im}(\zeta^{2i+1})\Big]^2 + $$
$$+ \Big[\sum_{i \in S} y_i \operatorname{Im}(\zeta^{2i+1})\Big]^2 + \Big[\sum_{i \in D} y_i \operatorname{Re}(\zeta^{2i+1})\Big]^2 = \frac{1}{2} n$$

where $S = \{i\colon 0 \leqslant i < n,\ e_i = 1\}$, $D = \{i\colon 0 \leqslant i < n,\ e_i = -1\}$ and ζ is a $2n$-th root of unity.

5. exist for orders $2^a 10^b 26^c$, a, b, c non-negative integers.

6. do not exist for orders 2.9^c, c a positive integer.

7. are 2-complementary disjointable sequences.

We will use the notation A^* to mean the order of the entries in the sequence A are reversed.

Then a few simple observations are in order and for convenience we put them together as a lemma.

Lemma 1. *Let* $X = \{A_1, A_2, \ldots, A_m\}$ *be m-complementary sequences of length n. Then*

(i) $Y = \{A_1^*, A_2^*, \ldots, A_i^*, A_{i+1}, \ldots, A_m\}$ *are m-complementary sequences of length n;*

(ii) $W = \{A_1, A_2, \ldots, A_i, -A_{i+1}, \ldots, -A_m\}$ *are m-complementary sequences of length n;*

(iii) $Z = \{\{A_1, A_2\}, \{A_1, -A_2\}, \ldots$

$\ldots, \{A_{2i-1}, A_{2i}\}, \{A_{2i-1}, -A_{2i}\}, \ldots\}$

are m- (or $m+1$ *if m was odd when we let* A_{m+1} *be n zeros) complementary sequences of length* $2n$*;*

(iv) $U = \{\{A_1 / A_2\}, \{A_1 / -A_2\}, \ldots$

$\ldots, \{A_{2i-1}/A_{2i}\}, \{A_{2i-1}/A_{2i}\}, \ldots\}$,

where A_j/A_k *means* $a_{j1}a_{k1}a_{j2}a_{k2} \ldots a_{jn}a_{kn}$, *are m- (or* $m+1$ *if m was odd when we let* A_{m+1} *be n zeros) complementary sequences of length* $2n$.

By a lengthy but straightforward calculation it can be shown that

Theorem 1. *Suppose* $X = \{A_1, \ldots, A_{2_m}\}$ *are 2m-complementary sequences of length n and weight l and* $Y = \{B_1, B_2\}$ *are 2-complementary disjointable sequences of length t and weight 2k. Then there are 2m-complementary sequences of length nt and weight kl.*

The same result is true if X are 2m-complementary disjointable sequences of length n and weight 2l and Y are 2-complementary sequences of weight k.

Proof. Using an idea of R.J. Turyn we consider

$$A_{2i-1} \times (B_1 + B_2)/2 + A_{2i}^* \times (B_1 - B_2)/2$$

and

$$A_{2i-1} \times (B_1 - B_2)/2 - A_{2i}^* \times (B_1 + B_2)/2$$

for $i = 1, \ldots, m$, which are the required sequences in the first case, and

$$(A_{2i-1} + A_{2i})/2 \times B_1 + (A_{2i-1} - A_{2i})/2 \times B_2^*$$

and

$$(A_{2i-1} + A_{2i})/2 \times B_2 - (A_{2i-1} - A_{2i})/2 \times B_1^*$$

for $i = 1, \ldots, m$, which are the required sequences for the second case. $\times$ denotes the Kronecker product so if $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$

$$A \times B = \{a_1 B, a_2 B, \ldots, a_n B\} =$$
$$= \{a_1 b_1, a_2 b_2, \ldots, a_1 b_m, \ldots, a_n b_1, a_n b_2, \ldots, a_n b_m\}.$$

Corollary 1. *There are 2-complementary sequences of lengths* $2^a 6^b 10^c 14^d 26^e$ *of weights* $2^a 5^b 10^c 13^d 26^e 2$ *where* a, b, c, d, e *are non-negative integers.*

Proof. Use the sequences of Tables 1 and 2.

4-complementary sequences A, B, C, D such that $(A+B)/2, (A-B)/2, (C+D)/2, (C-D)/2$ are also 4-complementary sequences

Length	Weight	A, B, C, D
3	10	11-, 11-, 101, 101
3	12	111, 1-1, 11-, 11-
4	14	111-, 1--1, 111, 1-1
5	18	11-11, 11-1-, 111-, 111-
5	20	1-11-, 111--, 111-1, 111-1
6	10	101--1, 10111-
6	12	101--1, 10111-, 1, 1
6	14	101--1, 10111-, 11, 1-
6	16	11-1--, 11-111, 100-, 100-
6	18	11-1--, 11-111, 110-, 1-0-
6	20	111-1-, 1-1111, 1--1, 11--
6	22	11--1-, 1----1, 1-111, 1-111
6	24	11-1--, 11--11, 111-1-, 1-1111
7	26	111-111, 11---1-, 11-1--, 11-1--
7	28	1--1111, 1--1-1-, 11-111-, 1----1
9	28	1---0-11-, 1--101--1, 10110111, 101-0---
9	30	11-10111-, 11--0---1, 101101--1, 101-0-11-
9	34	111-a1---, 111--1--1, 111-1-11, 111-1-11 ; $a = \pm 1$
14	26	111--111-1--0-, 1111-11--1-101

Table 1

2-complementary disjointable sequences

Length	Type of orthogonal design	
1	(1,1)	$a\ ,b$
2	(2,2)	$ab\ ,\quad a\bar{b}$
4	(4,4)	$aab\bar{b}\ ,\quad a\bar{a}bb$
6	(2,8)	$ab\bar{a}a0a\ ,\quad ab\bar{a}\bar{a}0\bar{a}$
8	(8,8)	$aaa\bar{a}bb\bar{b}b\ ,\quad a\bar{a}aab\bar{b}\bar{b}\bar{b}$
10	(4,16)	$ab\bar{a}a\bar{a}a\bar{a}\bar{a}ba\ ,\quad ab\bar{a}aaaaa\bar{b}\bar{a}$
10	(10,10)	$a\bar{a}\bar{a}ababbb\bar{b}\ ,\quad a\bar{a}\bar{a}\bar{a}b\bar{a}b\bar{b}\bar{b}b$
14	(26)	$aaaa\bar{a}aa\bar{a}\bar{a}a\bar{a}a0a\ ,\quad aaa\bar{a}\bar{a}aaa\bar{a}a\bar{a}\bar{a}0\bar{a}$

Table 2

§2. ORTHOGONAL DESIGNS

An orthogonal design of order n *and type* $(u_1, u_2, \ldots, u_s)$ $(u_i > 0)$ on the commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_s\}$ such that

$$AA^T = \sum_{i=1}^{s} (u_i x_i^2) I_n .$$

Alternatively, the rows of A are formally orthogonal and each row has precisely u_i entries of the type $\pm x_i$.

In [1], where this was first defined and many examples and properties of such designs were investigated, it was mentioned that

$$A^T A = \sum_{i=1}^{s} (u_i x_i^2) I_n$$

and so the alternative description of A applies equally well to the columns of A. It was also shown in [1] that $s \leqslant \rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

when

$$n = 2^a \cdot b, \quad b \text{ odd}, \quad a = 4c + d, \quad 0 \leqslant d < 4.$$

A weighing matrix of weight k and *order* n, is a square $\{0, 1, -1\}$ matrix, $W = W(n, k)$, of order n satisfying

$$WW^T = kI_n.$$

Let R be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to *be constructed from two circulant matrices* A and B if it is of the form

$$\begin{bmatrix} A & BR \\ BR & -A \end{bmatrix}$$

and to be of *Goethals – Seidel type* if it is of the form

$$GS = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & -D^TR & C^TR \\ -CR & D^TR & A & -B^TR \\ -DR & -C^TR & B^TR & A \end{bmatrix}$$

where A, B, C, D are circulant matrices.

A most important type of orthogonal design is the Baumert – Hall array. First pointed out in 1965, it became clear that the solution to the existence problem for Hadamard matrices could be realized through such arrays. A *Baumert – Hall array* of order t is an orthogonal design of type (t, t, t, t) in order $4t$.

Example 1. An example for the Baumert – Hall array of order 3 is

$$
\begin{array}{ccc|ccc|ccc|ccc}
A & B & C & B & -C & D & C & D & -A & D & A & -B \\
C & A & B & -C & D & B & D & -A & C & A & -B & D \\
B & C & A & D & B & -C & -A & C & D & -B & D & A \\
\hline
-B & C & -D & A & B & C & -D & B & -A & C & -A & D \\
C & -D & -B & C & A & B & B & -A & -D & -A & D & C \\
-D & -B & C & B & C & A & -A & -D & B & D & C & -A \\
\hline
-C & -D & A & D & -B & A & A & B & C & -B & -D & C \\
-D & A & -C & -B & A & D & C & A & B & -D & C & -B \\
A & -C & -D & A & D & -B & B & C & A & C & -B & -D \\
\hline
-D & -A & B & -C & A & -D & B & D & -C & A & B & C \\
-A & B & -D & A & -D & -C & D & -C & B & C & A & B \\
B & -D & -A & -D & -C & A & -C & B & D & B & C & A
\end{array}
$$

This is an example of the following theorem which may be proved straightforwardly (in the example

$$
X_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad X_4 = 0).
$$

Theorem 2 (Cooper and Wallis). *Suppose there exist four circulant* $(0, 1, -1)$ *matrices* X_i, $i = 1, 2, 3, 4$ *which are non-zero for each of the* n^2 *entries for only one* i *and which satisfy* $\sum_{i=1}^{4} X_i X_i^t = nI$. *Then there is a Baumert – Hall array of order* n.

It is conjectured that

Conjecture 1. There exists a Baumert – Hall array of order n for every natural number n OR EQUIVALENTLY there exists an orthogonal design of type (n, n, n, n) in every order $4n$ where n is a natural number.

We now propose to use R.J. Turyn's idea of using m-complementary sequences to construct orthogonal designs.

§ 3. CONSTRUCTIONS USING COMPLEMENTARY SEQUENCES TO FORM BAUMERT – HALL ARRAYS

Lemma 2. *Consider four* $(1, -1)$ *sequences* $A = \{X, Y, Z, W\}$ *where*

$$X = \{x_1 = 1, x_2, x_3, \dots, x_m, h_m x_m, \dots$$
$$\dots, h_3 x_3, h_2 x_2, h_1 x_1 = -1\}$$
$$U = \{u_1 = 1, u_2, u_3, \dots, u_m, f_m u_m, \dots$$
$$\dots, f_3 u_3, f_2 u_2, f_1 u_1 = 1\}$$
$$Y = \{y_1, y_2, \dots, y_{m-1}, y_m, g_{m-1} y_{m-1}, \dots$$
$$\dots, g_3 y_3, g_2 y_2, g_1 y_1\}$$
$$V = \{v_1, v_2, \dots, v_{m-1}, v_m, e_{m-1} v_{m-1}, \dots$$
$$\dots, e_3 v_3, e_2 v_2, e_1 v_1\}.$$

Then $N_A = 0$ *implies that* $h_i = f_i$, $i \geq 2$, *and* $g_j = e_j$. *Also*

$$\Big[\sum_{i=1}^{m} (x_i + h_i x_i)\Big]^2 + \Big[\sum_{i=1}^{m} (u_i + f_i u_i)\Big]^2 +$$
$$+ \Big[\sum_{i=1}^{m-1} (y_i + g_i y_i) + y_m\Big]^2 + \Big[\sum_{i=1}^{m-1} (v_i + e_i v_i) + v_m\Big]^2 =$$
$$= 8m - 2.$$

Proof. We note since all variables are ± 1, $a + b \equiv ab + 1 \pmod 4$ so $x + xyz \equiv y + z \pmod 4$. Clearly $N_A(2m-1) = 0$ gives $-h_1 = f_1 = 1$, and

$$N_A(2m-2) = x_1x_2h_2 + x_2h_1x_1 + f_2u_1u_2 + u_2f_1u_1 + g_1y_1^2 + e_1v_1^2 \equiv$$
$$\equiv h_1 + h_2 + f_1 + f_2 + g_1 + e_1 \pmod 4 \equiv$$
$$\equiv h_2f_2 + g_1e_1 + 2((\mathrm{mod}\ 4) \equiv 0 \pmod 4.$$

This gives $h_2f_2 = g_1e_1$. We proceed by induction to show that $h_if_i = g_{i-1}e_{i-1}$ for all $i \leqslant m$.

Assume $h_if_i = g_{i-1}e_{i-1}$ i.e. $h_i + f_i + g_{i-1} + e_{i-1} \equiv 0 \pmod 4$, for all $i < k \leqslant m$. Now consider

$$N_A(2m-k) = (x_1x_kh_k + x_2x_{k-1}h_{k-1} + \dots + x_{k-1}x_2h_2 + x_kx_1h_1) +$$
$$+ (u_1u_kf_k + u_2u_{k-1}f_{k-1} + \dots + u_{k-1}u_2f_2 + u_ku_1f_1) +$$
$$+ (y_1y_{k-1}g_{k-1} + y_2y_{k-2}g_{k-2} + \dots + y_{k-2}y_2g_2 + y_{k-1}y_1g_1) +$$
$$+ (v_1v_{k-1}e_{k-1} + v_2v_{k-2}e_{k-2} + \dots + v_{k-2}v_2e_2 + v_{k-1}v_1e_1) \equiv$$
$$\equiv h_1 + \dots + h_k + f_1 + \dots + f_k + g_1 + \dots + g_{k-1} + e_1 + \dots + e_{k-1} \pmod 4 \equiv$$
$$\equiv h_kf_k + g_{k-1}e_{k-1} + 2 \pmod 4 \equiv 0 \pmod 4.$$

This gives the result for all $i \leqslant m$.

Suppose $k = m + j < 2m$. Then

$$
\begin{aligned}
N_A(2m-k) = {} & (x_1 x_{m-j+1} + \ldots + x_j x_m) + \\
& + (x_{j+1} h_m x_m + \ldots + x_m h_{j+1} x_{j+1}) + \\
& + (h_m x_m h_j x_j + \ldots + h_{m-j+1} x_{m-j+1} h_1 x_1) + \\
& + (u_1 u_{m-j+1} + \ldots + u_j u_m) + \\
& + (u_{j+1} f_m u_m + \ldots + u_m f_{j+1} u_{j+1}) + \\
& + (f_m u_m f_j u_j + \ldots + f_{m-j+1} u_{m-j+1} f_1 u_1) + \\
& + (y_1 y_{m-j+1} + \ldots + y_j y_m) + \\
& + (y_{j+1} g_{m-1} y_{m-1} + \ldots + y_{m-1} g_{j+1} y_{j+1}) + \\
& + (y_m g_j y_j + \ldots g_{m-j+1} y_{m-j+1} g_1 y_1) + \\
& + (v_1 v_{m-j+1} + \ldots + v_j m_j) + \\
& + (v_{j+1} e_{m-1} v_{m-1} + \ldots + y_{m-1} e_{j+1} v_{j+1}) + \\
& + (v_m e_j v_j + \ldots + e_{m-j+1} e_1 v_1) \equiv \\
& \equiv (h_1 + \ldots + h_m + h_{m-j+1} + \ldots + h_m) + \\
& + (f_1 + \ldots + f_m + f_{m-j+1} + \ldots + f_m) + \\
& + (g_1 + \ldots + g_{m-1} + g_{m-j+1} + \ldots + g_{m-1} + 1) + \\
& + (e_1 + \ldots + e_{m-1} + e_{m-j+1} + \ldots + \\
& + e_{m-1} + 1) \pmod 4 \equiv \\
& \equiv h_1 f_1 + h_{m-j+1} f_{m-j+1} \pmod 4 \equiv \\
& \equiv h_{m-j+1} f_{m-j+1} - 1 \pmod 4.
\end{aligned}
$$

Hence $h_{m-j+1} f_{m-j+1} = 1$. So in general $h_i f_i = 1$ and $e_i g_i = 1$. Now if X', Y', Z', W' are the circulant matrices of order $2m$ with first rows X, Y, Z, W respectively then

$$X'X'^t + Y'Y'^t + Z'Z'^t + W'W'^t = (8m-2)I.$$

Multiplying both sides of this equation by J, the matrix of all 1's we have the final result of the enunciation.

Corollary 2. *Consider four* $(1, -1)$ *sequences* $A = \{X, U, Y, W\}$ *where*

$$X = \{x_1 = 1, x_2, x_3, \ldots, x_m, -x_m, \ldots$$
$$\ldots, -x_3, -x_2, -x_1 = -1\}$$
$$U = \{u_1 = 1, u_2, u_3, \ldots, u_m, f_m u_m, \ldots, f_3 u_3, f_2 u_2, f_1 u_1 = 1\}$$
$$Y = \{y_1, y_2, \ldots, y_{m-1}, y_m, y_{m-1}, \ldots, y_3, y_2, y_1\}$$
$$V = \{v_1, v_2, \ldots, v_{m-1}, v_m, e_{m-1}, v_{m-1}, \ldots$$
$$\ldots, e_3 v_3, e_2 v_2, e_1 v_1\}.$$

Then $N_A = 0$ *implies that all* e_i *are* $+1$ *and all* f_i *are* -1, $i \geq 2$, $f_1 = 1$. *Here* $8m - 6$ *is the sum of two squares.*

Similarly we can prove

Corollary 3. *Four* $(1, -1)$ *sequences* $A = (X, U, Y, W)$ *where*

$$X = \{x_1 = 1, x_2, x_3, \ldots, x_m, x_{m+1}, x_m, \ldots, x_3, x_2, x_1 = 1\}$$
$$U = \{u_1 = 1, u_2, u_3, \ldots, u_m, u_{m+1}, f_m u_m, \ldots$$
$$\ldots, f_3 x_3, f_2 x_2, -1\}$$
$$Y = \{y_1, y_2, \ldots, y_m, -y_m, \ldots, -y_2, -y_1\}$$
$$V = \{v_1, v_2, \ldots, v_m, e_m v_m, \ldots, e_2 v_2, e_1 v_1\}$$

which have $N_A = 0$. *Have* $e_i = -1$ *for all* i *and* $f_i = +1$ *for all* i. *Here* $8m + 2$ *is the sum of two squares.*

Definition. Sequences such as those described in the last lemma will be called *Turyn sequences* of length l (the four sequences are of weights $l, l, l-1$ and $l-1$). $l = 2m$ in Corollary 2 and $l = 2m + 1$ in Corollary 3.

Lemma 3. *There exist Turyn sequences of lengths* 2, 4, 6, 8, 3, 5, 7, 13 *and* 15.

Proof. Consider

$l = 2$: $X = \{\{1\ -\},\{1\ 1\},\{1\},\{1\}\}$

$l = 4$: $X = \{\{1\ 1\ -\ -\},\{1\ 1\ -\ 1\},\{1\ 1\ 1\},\{1\ -\ 1\}\}$

$l = 6$: $X = \{\{1\ 1\ 1\ -\ -\ -\},\{1\ 1\ -\ 1\ -\ 1\},\{1\ 1\ -\ 1\ 1\},\{1\ 1\ -\ 1\ 1\}\}$

$l = 8$: $X = \{\{1\ 1\ -\ 1\ -\ 1\ -\ -\},\{1\ 1\ 1\ 1\ -\ -\ -\ 1\},\{1\ 1\ 1\ -\ 1\ 1\ 1\},$
$\{1\ -\ -\ 1\ -\ -\ 1\}\}$

$l = 3$: $X = \{\{1\ 1\ 1\},\{1\ 1\ -\},\{1\ -\},\{1\ -\}\}$

$l = 5$: $X = \{\{1\ 1\ -\ 1\ 1\},\{1\ 1\ 1\ 1\ -\},\{1\ 1\ -\ -\},\{1\ -\ 1\ -\}\}$

$l = 7$: $X = \{\{1\ 1\ 1\ -\ 1\ 1\ 1\},\{1\ 1\ -\ -\ -\ 1\ -\},\{1\ 1\ -\ 1\ -\ -\},$
$\{1\ 1\ -\ 1\ -\ -\}\}$

$l = 13$: $X = \{\{1\ 1\ 1\ 1\ -\ 1\ -\ 1\ -\ 1\ 1\ 1\ 1\},\{1\ 1\ 1\ -\ -\ 1\ -\ 1\ -\ -\ 1\ 1\ -\},$
$\{1\ 1\ 1\ -\ 1\ 1\ -\ -\ 1\ -\ -\ -\},\{1\ 1\ 1\ -\ -\ 1\ -\ 1\ 1\ -\ -\ -\}\}$

or

$X = \{\{1\ 1\ 1\ -\ 1\ 1\ -\ 1\ 1\ -\ 1\ 1\ 1\},\{1\ 1\ 1\ -\ -\ 1\ -\ 1\ -\ -\ 1\ 1\ -\},$
$\{1\ 1\ -\ -\ -\ 1\ -\ 1\ 1\ 1\ -\ -\},\{1\ 1\ 1\ 1\ -\ 1\ -\ 1\ -\ -\ -\ -\}\}$

$l = 15$: $X = \{\{1\ 1\ -\ 1\ 1\ 1\ -\ 1\ -\ 1\ 1\ 1\ -\ 1\ 1\},$
$\{1\ 1\ 1\ -\ 1\ 1\ -\ -\ -\ 1\ 1\ -\ 1\ 1\ -\},$
$\{1\ 1\ 1\ 1\ -\ -\ 1\ -\ 1\ 1\ -\ -\ -\ -\},$
$\{1\ -\ -\ -\ -\ 1\ -\ 1\ -\ 1\ 1\ 1\ 1\ -\}\}$

Remark. These sequences were constructed using the Research School of Physical Sciences DEC-10 System.

A complete computer search in the case of $l = 9, 10, 14$ and 16 gave no solution for any decomposition into squares. In fact we found

l	$4l - 6 = x^2 + y^2$	result
6	$18 = 3^2 + 3^2$	yes
8	$26 = 1^2 + 5^2$	yes
10	$34 = 3^2 + 5^2$	none found
12	$42 \neq x^2 + y^2$	no
14	$50 = 1^2 + 7^2$	none found
	$= 5^2 + 5^2$	none found
16	$58 = 3^2 + 7^2$	none found
18	$66 \neq x^2 + y^2$	no
20	$74 = 5^2 + 7^2$	ran out of time

In order to satisfy the conditions of Theorem 2 we are led to study sequences of a more restricted type.

Definition. Four complementary disjoint $\{0, 1, -1)$ sequences of length t and total weight t will be called *T-sequences.*

Example. Consider

$$T = \{\{1\ 0\ 0\ 0\ 0\}, \{0\ 1\ 1\ 0\ 0\}, \{0\ 0\ 0\ 1\ -\}, \{0\ 0\ 0\ 0\ 0\}\}.$$

The sequences are disjoint as the i-th entry is non-zero in one and only one of the four sequences. The total weight is 5 and $N_T = 0$.

Another example is obtained by using the Golay sequences

$$X = 1 - - 1 - 1 - - - 1 \quad \text{and} \quad Y = 1 - - - - - - 1\ 1\ -.$$

Let $\mathbf{0}$ be the vector of 10 zeros then

$$T = \left\{\left\{1, \mathbf{0}\right\}, \left\{0, \frac{1}{2}(X + Y)\right\}, \left\{0, \frac{1}{2}(X - Y)\right\}, \left\{0, \mathbf{0}\right\}\right.$$

are T sequences of length 11.

Theorem 3 (Turyn). *Suppose* $A = \{X, U, Y, V\}$ *are Turyn sequences of length* l *where* X *is skew and* Y *is symmetric for* l *even and* X *is symmetric and* Y *is skew for* l *odd. Then there are T-sequences of length* $2l - 1$ *and* $4l - 1$.

Proof. We use the notation A/B as before to denote the interleaving of two sequences $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_{m-1}\}$.

$$A/B = \{a_1, b_1, a_2, b_2, \dots, b_{m-1}, a_m\}.$$

Let $\mathbf{0}_t$ be a sequence of zeros of length t. Then

$$T_1 = \Big\{\Big\{\frac{1}{2}(X + Y), \mathbf{0}_{l-1}\Big\}, \Big\{\frac{1}{2}(X - Y), \mathbf{0}_{l-1}\Big\},$$
$$\Big\{\mathbf{0}_l, \frac{1}{2}(Y + V)\Big\}, \Big\{\mathbf{0}_l, \frac{1}{2}(Y - V)\Big\}$$

and

$$T_2 = \{\{1, \mathbf{0}_{4l-2}\}, \{0, X/Y, \mathbf{0}_{2l-1}\},$$
$$\{0, \mathbf{0}_{2l-1}, U/\mathbf{0}_{l-1}\}, \{0, \mathbf{0}_{2l-1}, \mathbf{0}_l/V\}\}$$

are the T-sequences of lengths $2l - 1$ and $4l - 1$ respectively.

Corollary 4. *There are T-sequences constructed from Turyn sequences of lengths* 3, 5, 7, 9, 11, 13, 15, 19, 23, 25, 27, 29, 31, 51, 59.

Theorem 4. *If* X *and* Y *are Golay sequences of length* r *then writing* $\mathbf{0}_r$ *for the vector of* r *zeros* $T = \Big\{\{1, \mathbf{0}_r\}, \Big\{0, \frac{1}{2}(X + Y)\Big\}, \Big\{0, \frac{1}{2}(X - Y)\Big\}, \{\mathbf{0}_{r+1}\}$ *are T-sequences of length* $r + 1$.

Corollary 5 (Turyn). *There exist T-sequences of lengths* $1 + 2^a 10^b 26^c$ *where* a, b, c *are non-negative integers.*

Corollary 6. *There exist T-sequences of lengths* $3, 5, 7, \dots, 33, 41, 51, 53, 59, 65, 81, 101$.

A desire to fill the gaps in the list in Corollary 6 leads to the following idea.

Lemma 4. *Suppose* $X = \{A, B, C, D\}$ *are* 4*-complementary sequences of length* $l, l, l-1, l-1$ *respectively and weight* k. *Then*

$$Y = \{\{A, C\}, \{A, -C\}, \{B, D\}, \{B, -D\}\}$$

are 4*-complementary sequences of length* $2l-1$ *and weight* $2k$. *Further if* $\frac{1}{2}(A+B)$ *and* $\frac{1}{2}(C+D)$ *are also* $(0, 1, -1)$ *sequences then, with* $\mathbf{0}_t$ *the sequence of* t *zeros,*

$$Z = \left\{\left\{\tfrac{1}{2}(A+B), \mathbf{0}_{l-1}\right\}, \left\{\tfrac{1}{2}(A-B), \mathbf{0}_{l-1}\right\},\right.$$
$$\left.\left\{\mathbf{0}_l, \tfrac{1}{2}(C+D)\right\}, \left\{\mathbf{0}_l, \tfrac{1}{2}(C-D)\right\}\right\}$$

are 4*-complementary sequences of length* $2l-1$ *and weight* k. *If* A, B, C, D *were* $(1, -1)$ *sequences then* Z *consists of* T*-sequences of length* $2l-1$.

In fact Turyn has found sequences satisfying these conditions.

Corollary 7. *The following four* $(1, -1)$ *sequences are of lengths* 24, 24, 23, 23

1 −1 −1 −1 1 −1 1 −1 −1 −1 −1 1 1 1 1 1 1 −1 −1 1 −1 −1 −1 1

1 −1 −1 1 −1 −1 1 −1 1 1 1 −1 −1 −1 −1 −1 1 −1 −1 −1 1 −1 −1 −1

1 1 1 −1 −1 −1 1 1 −1 −1 1 −1 −1 −1 −1 1 −1 −1 −1 −1 1 −1 1

1 1 −1 −1 1 −1 1 1 −1 1 −1 1 1 1 −1 1 −1 −1 1 −1 −1 −1 1

Hence there are T*-sequences of length* 47.

We may summarize these results in one theorem.

Theorem 5. *If there exist* T*-sequences of length* t *then*

(i) *there exists a Baumert – Hall array of order* t,

(ii) *there exists an orthogonal design of type* (t, t, t, t) *in order* $4t$.

Proof. The T-sequences are used as first rows of circulant matrices. The rest follows from Theorem 6.

§4. CONSTRUCTIONS USING COMPLEMENTARY SEQUENCES

We now give some results which are useful in constructing new sequences from old. In particular we want to use complementary sequences to construct orthogonal designs.

Remark. Since our interest is in orthogonal designs we shall not be restricted to sequences with entries only ± 1, but shall allow 0's and variables too. One very simple remark is in order. If we have a collection of sequences, X, (each having length n) such that $N_X(j) = 0$, $j = 1, \dots, n-1$, then we may augment each sequence at the beginning with k zeros and at the end with l zeros so that the resulting collection, (say $\bar{X}$), of sequences having length $k + n + l$ still has $N_{\bar{X}}(j) = 0$, $j = 1, \dots, k + n + l - 1$.

Lemma 5. *Suppose there exist 2-complementary sequences* X_i, Y_i *of length* n_i *which give an orthogonal design of type* (u_{1i}, u_{2i}) *constructed from two circulants,* $i = 1, 2$.

Then there exist 4-complementary sequences of length n *which can be used in the Goethals – Seidel array to give orthogonal designs of type* $(u_{11}, u_{12}, u_{21}, u_{22})$ *in order* $4n$ *where* $n \geq \max(n_1, n_2)$.

If in addition X_2, Y_2 *are disjointable then there exists an orthogonal design of type* $\left(u_{11}, u_{21}, \frac{1}{2} u_{12}, \frac{1}{2} u_{22}\right)$ *in order* $4n$ *where* $n \geq \max(n_1, n_2)$.

Proof. Very straightforward.

Corollary 8. *Let* r *be any number of the form* $2^a 10^b 26^c 5^d 13^e$ *and let* n *be any integer at least* $2^a 10^b 26^c 6^d 14^e$, a, b, c, d, e *non-negative integers. Then there exist orthogonal designs of order* $4n$ *and types*

(i) $(1, 1, r, r)$ *and*

(ii) $(1, 4, r, r)$.

Proof. Sequences of the weights r are by Corollary 1 disjointable and the $(1, 4)$ sequences are $\{a\, b\, \bar{a}, a\, 0\, a\}$.

Corollary 9. *There exist orthogonal designs of type*

(i) $(1, 1, 5, 5)$ *in order* $4m$, $m \geq 6$;

(ii) $(1, 1, 13, 13)$ *in order* $4m$, $m \geq 14$;

(iii) $(1, 1, 10, 10)$ *in order* $4m$, $m \geq 10$;

(iv) $(1, 1, 20, 20)$ *in order* $4m$, $m \geq 20$.

Another construction based on the existence of 2-complementary sequences which is extremely useful is

Lemma 6. *Let* $X = \{U, V\}$ *be 2-complementary sequences of length* n *giving a design constructed of two circulants of type* (a, b) *such that* $N_X(j) = 0$. *Then, with* U^* *and* V^* *their reverse sequences and* w, x, y, z *variables*

(i) $Y = \{x, y, zU, zV\}$,

(ii) $Y = \{yx - y, y0y, zU, zV\}$,

(iii) $Y = \{\{zU, 0, zU^*\}, \{zU, x, -zU^*\}, \{zV, 0, zV^*\}, \{zV, y, -zV^*\}\}$
have $N_Y(j) = 0$. *Furthermore they may be used in the Goethals – Seidel array to give orthogonal designs of type* (α) $(1, 1, a, b), (1, 4, a, b)$ *in every order* $4m$, $m \geq n$, *and* (β) $(1, 1, 4a, 4b)$ *in every order* $4m$, $m \geq$
$\geq 2n + 1$.

We see that in some cases this lemma is a considerable improvement on the previous lemma. For example

Corollary 10. *There exist orthogonal designs of types*

(i) $(1, 1, 20)$ *in every order* $4m$, $m \geq 7$;

(ii) $(1, 1, 40)$ *in every order* $4m$, $m \geq 13$;

(iii) $(1, 1, 16, 16)$ *in every order* $4m$, $m \geq 9$;

(iv) $(1, 1, 8, 8)$ *in every order* $4m$, $m \geq 5$;

(v) $(1, 1, 52, 52)$ *in every order* $4m$, $m \geq 29$.

Proof. Use

$$\{1\,0\,1, 1\,1\,-\}, \quad \{1\,0\,1\,1\,1\,-, 1\,0\,1\,-\,-\,1\},$$

$$\{a\,a\,b\,\bar{b}, a\,\bar{a}\,b\,b\}, \quad \{a\,b, \bar{a}\,b\},$$

$$\{a\,a\,a\,b\,\bar{a}\,a\,a\,\bar{b}\,\bar{a}\,a\,\bar{a}\,b\,0\,b, \mathrm{b}\,\mathrm{b}\,\mathrm{b}\,\bar{a}\,\bar{b}\,b\cdot b\,a\,\bar{b}\,b\,\bar{b}\,\bar{a}\,0\,\bar{a}\}.$$

We give one other method for constructing orthogonal designs using complementary sequences.

Lemma 7. *Let* $X = \{A, B, Z, Z\}$ *be 4-complementary sequences of length* n *and weight* k. *Then writing* Z^* *for the reverse of* Z *and with* x, y, z *variables*

$$Y = \{\{yA, 0, yB\}, \{yA, 0, -yB\}, \{yZ, 0, yZ^*\}, \{yZ, x, -yZ^*\}\}$$

are 4-complementary sequences of length $2n + 1$ *which may be used to give an orthogonal design of type* $(1, 2k)$ *in every order* $4m$, $m \geqslant 2n + 1$.

Proof. Use the four sequences of Y to generate matrices which can be used in the Goethals – Seidel array.

Corollary 11. *There exist orthogonal designs of types:*

(i) $(1, 20)$ *in order* $4m$, $m \geqslant 7$;

(ii) $(1, 24)$ *in order* $4m$, $m \geqslant 7$;

(iii) $(1, 32)$ *in order* $4m$, $m \geqslant 13$;

(iv) $(1, 44)$ *in order* $4m$, $m \geqslant 13$;

(v) $(1, 52)$ *in order* $4m$, $m \geqslant 15$;

(vi) $(1, 36)$ *in order* $4m$, $m \geqslant 11$;

(vii) $(1, 2, 66)$ *in order* $4m$, $m \geqslant 19$.

Proof. From Table 2 there exist 4-complementary sequences $\{A, B, Z, Z\}$ for lengths and weights: 3 and 10; 3 and 12; 5 and 18; 5 and 20; 6 and 16; 6 and 22; 7 and 26; length 9 and type $(1, 33)$.

CONCLUSION

We have shown how to use some types of complementary sequences to construct orthogonal designs. The computer results quoted show that the small amount of theory known for these sequences comes nowhere near resolving their existence. We summarize some of the results of this note:

Summary 2. There exist orthogonal designs of type $(1, k)$ for the quoted k in every order $4m$, $m \geq n$ where n is indicated below:

(i) for $k \in \{1, 2, 3, 4, 5\}$, $n = 2$;

(ii) for $k \in \{6, 8\}$, $n = 3$;

(iii) for $k \in \{9, 12\}$, $n = 4$;

(iv) for $k \in \{16, 17\}$, $n = 5$;

(iv) for $k \in \{10, 11, 14\}$, $n = 6$;

(v) for $k \in \{20, 21, 24\}$, $n = 7$;

(vii) for $k \in \{32, 33\}$, $n = 9$;

(vii) for $k \in \{21, 24\}$, $n = 10$;

(viii) for $k \in \{36\}$, $n = 11$;

(viii) for $k \in \{40, 41, 44\}$, $n = 13$;

(ix) for $k \in \{13, 17, 26, 27, 30\}$, $n = 14$;

(x) for $k \in \{52\}$, $n = 15$.

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