# An infinite family of skew weighing matrices

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## Abstract

We verify the skew weighing matrix conjecture for orders $2^t.7$ , $t \geq 3$ a positive integer, by showing that orthogonal designs $(1, k)$ exist for all $k = 0, 1, \ldots, 2^t.7 - 1$ in order $2^t.7$ .

We discuss the construction of orthogonal designs using circulant matrices. In particular we construct designs in orders 20 and 28 .

The weighing matrix conjecture is verified for order 60 .

## §1. Introduction.

*An orthogonal design of order* $n$ *and type* $(u_1, u_2, \ldots, u_s)$ $(u_i>0)$ on the commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix $A$ with entries from $\{0, \pm x_1, \ldots, \pm x_s\}$ such that

$$AA^T = \sum_{i=1}^{s} (u_i x_i^2) I_n .$$

Alternatively, the rows of $A$ are formally orthogonal and each row has precisely $u_i$ entries of the type $\pm x_i$ .

In [2], where this was first defined and many examples and properties of such designs were investigated, we mentioned that

$$A^T A = \sum_{i=1}^{s} (u_i x_i^2) I_n$$

and so our alternative description of $A$ applies equally well to the columns of $A$ . We also showed in [2] that $s \le \rho(n)$ , where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

when

$$n = 2^a \cdot b \ , \quad b \text{ odd} \ , \quad a = 4c + d \quad 0 \le d < 4 .$$

A *weighing matrix of weight* $k$ *and order* $n$ , is a square $\{0, 1, -1\}$ matrix, $A$ , of order $n$ satisfying

$$AA^T = kI_n .$$

In [2] we showed that the existence of an orthogonal design of order $n$ and type $(u_1, \ldots, u_s)$ is equivalent to the existence of disjoint weighing matrices $A_1, \ldots, A_s$, of order $n$, where $A_i$ has weight $u_i$ and the matrices, $\{A_i\}_{i=1}^{s}$, satisfy the matrix equation

$$XY^T + YX^T = 0$$

in pairs. In particular, the existence of an orthogonal design of order $n$ and type $(1, k)$ is equivalent to the existence of a skew-symmetric weighing matrix of weight $k$ and order $n$.

It is conjectured that:

(i) for $n \equiv 2 \pmod 4$ there is a weighing matrix of weight $k$ and order $n$ for every $k < n - 1$ which is the sum of two integer squares;

(ii) for $n \equiv 0 \pmod 4$ there is a weighing matrix of weight $k$ and order $n$ for every $k \leq n$;

(iii) for $n \equiv 4 \pmod 8$ there is a skew-symmetric weighing matrix of order $n$ for every $k < n$, except possibly $k = n - 2$, where $k$ is the sum of $\leq$ three squares of integers (equivalently, there is an orthogonal design of type $(1, k)$ in order $n$ for every $k < n$, except possibly $k = n - 2$, which is the sum of $\leq$ three squares of integers).

(iv) for $n \equiv 0 \pmod 8$ there is a skew-symmetric weighing matrix of order $n$ for every $k < n$ (equivalently there is an orthogonal design of type $(1, k)$ in order $n$ for every $k < n$) .

Conjecture (ii) above is an extension of the Hadamard conjecture (i.e. for every $n \equiv 0 \pmod 4$ there is a $\{1, -1\}$ matrix, $H$ , of order $n$ satisfying $HH^T = nI_n$) while (iv) and (iii) generalize the conjecture that for every $n \equiv 0 \pmod 4$ there is a Hadamard matrix, $H$ , of order $n$ , with the property that $H = I_n + S$ where $S = -S^T$ .

Conjecture (ii) is established for $n \in \{4, 8, 12, \ldots, 32, 40, 60\}$ , $2^t$, $2^{t+1}.3$, $2^{t+1}.5$, $2^{t+1}.7$, $2^{t+1}.9$ for $t$ a positive integer in [3], [5], [11], [12] and this paper.

Conjecture (iv) and consequently conjecture (ii ) is established for $2^t.m$, $m \in \{1, 3, 5, 7, 9\}$ , $t \geq 3$ a positive integer in [2], [4], [10] and this paper. Note that the phrase "except possibly $k = n-2$" is necessary because there is no design of type $(1, 42)$ in order 44.

Conjecture (iii) has been verified for $n \in \{12, 20, 28\}$ in [5].

D. Shapiro, [10], and W. Wolfe, [15], have found powerful algebraic non-existence theorems for orthogonal designs which supercede those of Geramita, Geramita and Wallis [2]. In addition Geramita and Verner [4] and P.J. Robinson [9] have found strong combinatorial theorems. We quote those relevant to this paper.

**THEOREM 1.** **(Wolfe).** *Let* $n \equiv 4 \pmod 8$ *and* $(a_i, a_j)_p$ *be the Hilbert Norm residue symbol. There exists an orthogonal design in order* $n$ *and type*

(i) $(a_1, a_2, a_3, a_4)$ *only if* $\prod_{i=1}^{4} a_i$ *is a square*

*and* $\prod_{1 \le i < j \le 4} (a_i, a_j)_p = 1$ *at every prime* $p$ ;

(ii) $(a_1, a_2, a_3)$ *only if* $(-1, a_1 a_2 a_3)_p \prod_{1 \le i < j \le 3} (a_i, a_j)_p = 1$ *at every prime* $p$ ;

(iii) $(a_1, a_2)$ *only if* $a_1 a_2$ *is the sum of three squares.*

See Hall [8] for details of how to evaluate $(a_i, a_j)_p$ . There are similar results in orders $n \equiv 2 \pmod 4$ , $n \equiv 8 \pmod{16}$ and $n \equiv 16 \pmod{32}$ . See Wolfe [14].

**THEOREM 2.** **(Geramita-Verner).** *If* $n \equiv 0 \pmod 4$ , *then there exists an orthogonal design in order* $n$ *and of type* $(u_1, \ldots, u_s)$ *where* $\sum_{i=1}^{s} u_i = n - 1$ *if and only if there exists an orthogonal design in order* $n$ *and type* $(1, u_1, \ldots, u_s)$ .

Let $R$ be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to be *constructed from two circulant matrices* $A$ and $B$ if it is of the form

$$\begin{bmatrix} A & BR \\ BR & -A \end{bmatrix}$$

and to be of *Goethals-Seidel type* if it is of the form

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{bmatrix}$$

where A, B, C, D are circulant matrices.

In [11] methods are described for constructing orthogonal designs by filling in circulant matrices in the design of type (1, 1, 1, 1, 1, 1, 1, 1) in order 8 . A design made in this way is said to be ***constructed from*** 8 ***circulants.***

We will also use the following result from [2]:

THEOREM 3. (Geramita-Wallis). *If there exists an orthogonal design of type* (s, t) *in order* n *there exists an orthogonal design of type* (s, s, t, t) *in order* 2n .

## §2. Comments on Constructions Using Circulant Matrices.

Suppose X is the incidence matrix of an orthogonal design of order n and type $(s_1, s_2, s_3, s_4)$ in the variables $x_1, x_2, x_3, x_4$ . Further suppose X is constructed using 4 circulant matrices $Y_1, Y_2, Y_3, Y_4$ in the Goethals-Seidel array. Suppose the row and column sum of $Y_i$ is

$$r_i = a_ix_1 + b_ix_2 + c_ix_3 + d_ix_4 , \quad i = 1, 2, 3, 4$$

Let $e$ be the $1 \times n/4$ matrix of 1's then $eY_i = r_i e$.

Now since $X$ is an orthogonal design

$$\sum_{i=1}^{4} Y_i Y_i^T = \sum_{i=1}^{4} s_i x_i^2 .$$

Multiplying on the left by $e$ and the right by $e^T$ we have

$$ee^T \sum_{i=1}^{4} s_i x_i^2 = n/4 \sum_{i=1}^{4} s_i x_i^2 = \sum_{i=1}^{4} (eY_i)(eY_i)^T = \sum_{i=1}^{4} (r_i e)(r_i e)^T = n/4 \sum_{i=1}^{4} r_i^2 .$$

Thus we have

$$\begin{aligned} s_1 x_1^2 + s_2 x_2^2 + s_3 x_3^2 + s_4 x_4^2 = {} & x_1^2 \sum_{i=1}^{4} a_i^2 + x_2^2 \sum_{i=1}^{4} b_i^2 + x_3^2 \sum_{i=1}^{4} c_i^2 \\ & + x_4^2 \sum_{i=1}^{4} d_i^2 + 2x_1 x_2 \sum_{i=1}^{4} a_i b_i \\ & + 2x_1 x_3 \sum_{i=1}^{4} a_i c_i + 2x_1 x_4 \sum_{i=1}^{4} a_i d_i \\ & + 2x_2 x_3 \sum_{i=1}^{4} b_i c_i + 2x_2 x_4 \sum_{i=1}^{4} b_i d_i \\ & + 2x_3 x_4 \sum_{i=1}^{4} c_i d_i \end{aligned}$$

Hence we have four integer vectors $\underset{\sim}{a} = (a_1, a_2, a_3, a_4)$, $\underset{\sim}{b} = (b_1, b_2, b_3, b_4)$, $\underset{\sim}{c} = (c_1, c_2, c_3, c_4)$, $\underset{\sim}{d} = (d_1, d_2, d_3, d_4)$ which are pairwise orthogonal. Also $|\underset{\sim}{a}|^2 = s_1$, $|\underset{\sim}{b}|^2 = s_2$, $|\underset{\sim}{c}|^2 = s_3$, $|\underset{\sim}{d}|^2 = s_4$.

Form these vectors into an orthogonal integer matrix $M$ with $M^T = [\underset{\sim}{a}^T, \underset{\sim}{b}^T, \underset{\sim}{c}^T, \underset{\sim}{d}^T]$. Then $MM^T = \text{diag}(s_1, s_2, s_3, s_4)$ and $\det M = \sqrt{s_1 s_2 s_3 s_4}$. But $M$ is integer so $s_1 s_2 s_3 s_4$ is a square. Thus we have shown

**LEMMA 4.** *The Goethals-Seidel construction for an orthogonal design of order* $n \equiv 0 (\text{mod } 4)$ *and type* $(s_1, s_2, s_3, s_4)$ *can only be used if*

1) *There is an integer matrix* $M$ *satisfying* $MM^T = \text{diag}(s_1, s_2, s_3, s_4)$ *and hence*

2) $s_1 s_2 s_3 s_4$ *is a square.*

Since the $ij$ th entry of $M$ is the coefficient of $x_i$ in the row sum of $Y_i$ this lemma helps in the search for circulant matrices to fit into the Goethals-Seidel array; see [1].

Note the relation between Lemma 4 and Theorem 1. Wolfe's Theorem says (essentially) that if a design $X$ of type $(s_1, s_2, s_3, s_4)$ exists in order $n \equiv 4 (\text{mod } 8)$ then there is a $4 \times 4$ rational matrix $Q$ such that $QQ^T = \text{diag}(s_1, s_2, s_3, s_4)$. Lemma 4 says that if $X$ is constructed from 4 circulants, in any order $n \equiv 0 (\text{mod } 4)$ then $Q$ can be chosen to be an integer matrix.

Similarly we may show:

**LEMMA 5.** *An orthogonal design of type* $(s_1, s_2, \ldots, s_8)$ *and order* $n \equiv 0 (\text{mod } 8)$ *can be constructed from eight circulants only if*

1) *there is an* $8 \times 8$ *integer matrix* $M$ *such that*

$$MM^T = \text{diag}(s_1, s_2, \ldots, s_8)$$

*and hence*

2) $s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8$ *is a square.*

**LEMMA 6.** *An orthogonal design of type* $(s, t)$ *and order* $n \equiv 0 \pmod 2$ *can be constructed from two circulant matrices only if*

1) *there is a* $2 \times 2$ *integer matrix* $M$ *such that*

$MM^T = \text{diag}(s, t)$

*and hence*

2) $st$ *is a square.*

Again, Lemma 5 and Lemma 6 are similar to Wolfe's theorems in orders $n \equiv 8 \pmod{16}$ and $n \equiv 2 \pmod 4$ .

## §3. Some Results on the Conjectures.

**LEMMA 7.** *All orthogonal designs* $(1, k)$, $k = 0, 1, \ldots, 55$ *exist in order* 56 .

**Proof.** In [5] it is established that the orthogonal designs $(1, k)$ exist in order 56 for $k \neq 46, 47$ , $0 \leq k \leq 55$ . It remains to construct a $(1, 46)$ and a $(1, 47)$. For these we use the $(1, 1, 1, 1, 1, 1, 1, 1)$ orthogonal design in order 8 and replace the variables $x_i$ by the circulant matrices $X_i$ indicated below and the variables $x_j$ by the back circulant matrices $X_j$ indicated below.

For $(1, 46)$ use the matrices with first rows ($X_2$ backcirculant, the rest circulant)

$X_1$ : x 0 0 0 0 0 0 $\qquad$ $X_5$ : y y y -y -y y y

$X_2$ : y y y 0 y 0 0 $\qquad$ $X_6$ : y y -y -y -y -y y

$X_3$ : y -y y y y y -y $\qquad$ $X_7$ : y -y y -y -y y -y

$X_4$ : y y -y y y -y y $\qquad$ $X_8$ : y -y -y y y -y -y

and for $(1, 47)$ use the matrices with first rows ($X_1$ circulant, the rest backcirculant)

$X_1$ : x y 0 y -y 0 -y $\qquad$ $X_5$ : y -y -y y y -y -y

$X_2$ : 0 y y y y y y $\qquad$ $X_6$ : y -y y -y -y y -y

$X_3$ : 0 y -y y y -y y $\qquad$ $X_7$ : y y -y -y -y -y y

$X_4$ : 0 y y -y -y y y $\qquad$ $X_8$ : 0 y 0 -y -y 0 y .

<u>COROLLARY 8.</u> *All orthogonal designs* (1, k), k = 0,1, ...., $2^t.7-1$ *exist in orders* $2^t.7$ , $t \geq 3$ *a positive integer.*

<u>Proof.</u> We proceed by induction after first noting that the existence of an orthogonal design of type (a, b) in order n implies the existence of an orthogonal design of type (a, a, b, b) in order 2n .

<u>COROLLARY 9.</u> *A skew-symmetric weighing matrix,* $W(2^t.7, k)$ , *exists for all* k = 0, 1, ..., $2^t.7-1$ *when* $t \geq 3$ *is a positive integer* .

<u>COROLLARY 10.</u> *A weighing matrix,* $W(2^t.7, k)$ , *exists for all* k = 0, 1, ..., $2^t.7$ *when* $t \geq 3$ *is a positive integer* .

<u>LEMMA 11.</u> *There exists a* W(60, k) *for all* k = 0, 1, ..., 60 . *All but* W(60, j) *for* $j \in \{19, 35, 38, 41, 43, 47, 50, 51, 53, 57\}$ *are constructed from circulant matrices.*

<u>Proof.</u> We have the enunciation from [6, lemma 16] except for the W(60, 51) and W(60, 53) . It remains then to show these two weighing matrices exist.

The W(60, 51) is found by replacing the variables of the orthogonal design of type (2, 3, 6, 9) in order 20 by the matrices

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & - \\ 1 & - & 0 \\ - & 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} - & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix}$$

respectively. The W(60, 53) is found by replacing the variables of the orthogonal design of type (2, 5, 5, 8) in order 20 by the matrices B, J, B and A respectively.

## §4. Constructions Using Circulant Weighing Matrices.

It was shown in [14] that orthogonal designs of types $(1, 1, 1, q^2)$, $(1, 1, q^2, q^2)$, $(1, q^2, q^2, q^2)$, $(q^2, q^2, q^2, q^2)$, $(2q^2, 2(q^2+2q+2))$ exist in order $4(q^2+q+1)$ when $q$ is a prime power.

**THEOREM 12.** *Let* $q$ *be a prime power. Then there exist orthogonal designs of types* (i) $(1, 3, 6, 2q^2)$, (ii) $(2, 2, 4, 4q^2)$, (iii) $(2, 2, q^2+1, q^2+1)$, (iv) $(3, 4, 6, 2q^2)$, (v) $(3, 3, 3, 3q^2)$, (vi) $(4, 4, 4, 4q^2)$, (vii) $(4, 4, 2(q^2+1), 2(q^2+1))$, (viii) $(q^2+1, q^2+1, q^2+1, q^2+1)$, (ix) $(q^2+2, q^2+2\ q^2+2, q^2+2)$, (x) $(q^2+3, q^2+3, q^2+3, q^2+3)$, (xi) $(1, 1, q^2+1, q^2+1)$, (xii) $(1, 4, q^2+1, q^2+1)$, (xiii) $(1, 4, q^2, q^2)$, (xiv) $(q^2, q^2, q^2+1, q^2+1)$, (xv) $(1, q^2, q^2+1, q^2+1)$, (xvi) $(1, 6, 3q^2)$, (xvii) $(1, 2, q^2, 2q^2)$, (xviii) $(2, 2, 2(q^2+1), 2(q^2+1))$, *in order* $4(q^2+q+1)$.

**Proof.** Let $W = W(q^2+q+1, q^2)$ be the circulant weighing matrix found in [14]. Let $T$ be the shift matrix of order $q^2 + q + 1$. Since $P = J - W * W$ ($*$ the Hadamard product) is the circulant incidence matrix of the projective plane of order $q$, $P = \sum_i T^{d_i}$ where the set of the $d_i$, $D$, is a $(q^2+q+1, q+1, 1)$ planar difference set. Now there exist $x, y = x + 1$ in $D$. Let $X = T^x$, $Y = T^y$ and $Z = T^z$ for some $z \in D$, $z \neq x$, $z \neq y$. Let $m = (q^2+q)/2$.

Then if we use

(i) $aX + bY + cW$, $aX + bY - cW$, $bI-aT + bT^2$, $bI + cT - bT^2$

(ii) $aX + bY + dW$, $aX - bY + dW$, $aX + cY - dW$, $aX - cY - dW$

(iii) $aX + bY$, $aX - bY$, $cX + dW$, $dX - cW$

(iv) $aX + bY + cW$, $aX + bW - cW$, $dI + bT - bT^3 + dT^4$,

$-dI + bT - aT^2 + bT^3 + dT^4$

(v) $aX + bY + cW$, $aX + dZ - cW$, $bY - dZ - cW$, $aX - bY - dZ$

(vi) $aX + bY + cZ + dW$, $aX + bY - cZ - dW$, $aX - bY + cZ - dW$,

$aX - bY - cZ + dW$

(vii) $aT^{m-1} + bT^m + bT^{m+1} - aT^{m+2}$, $aT^{m-1} + bT^m - bT^{m+1} + aT^{m+2}$

$dX + cW$, $cX - dW$

(viii) $aX + bW$, $aX - bW$, $cX + dW$, $cX - dW$

(ix) $aX + bY + dW$, $-bX + aY - cW$, $-cX - dY + dW$, $-dX + cY + aW$

(x) $aX + bY + cZ + dW$, $-bX + aY + dZ - cW$, $-cX - dY + aZ + dW$,

$-dX + cY - dZ + aW$

(xi) $aX$, $bX$, $cX + dW$, $cX - dW$

(xii) $aI + bT^m - bT^{m+1}$, $bT^m + bT^{m+1}$, $dX + cW$, $cX - dW$

(xiii) $aI + bT^m - bT^{m+1}$, $bT^m + bT^{m+1}$, $cW$, $dW$

(xiv) $aW$, $bW$, $cX + dW$, $dX - cW$

(xv) $aX$, $bW$, $cX + dW$, $dX - cW$

(xvi) $aX + aY - bW$, $aX + bW$, $aY + bW$, $cI + aT^m - aT^{m+1}$

(xvii) $aX + bW$, $aX - bW$, $cW$, $dI$

(xviii) $aW + bX + cY$, $aW + bX - cY$, $W - aX + dY$, $bW - aX - dY$,

respectively, in the Goethals-Seidel array, we obtain the designs of the enunciation.

<u>COROLLARY 13</u>. *Since* $7 = 2^2 + 2 + 1$ *the following orthogonal designs exist in order* 28 :

(1, 1, 1, 4), (1, 1, 4, 4), (1, 1, 5, 5), (1, 2, 4, 8), (1, 3, 6, 8),
(1, 4, 4, 4), (1, 4, 5, 5), (2, 2, 4, 16), (2, 2, 10, 10), (3, 3, 3, 12),
(3, 4, 6, 8), (4, 4, 4, 4), (4, 4, 4, 16), (4, 4, 5, 5), (4, 4, 10, 10)
(5, 5, 5, 5), (6, 6, 6, 6), (7, 7, 7, 7) .

## Appendix I

We give the current status of known orthogonal designs in order 20 . We use as a base the results of [6] and [13].

The following 4, 3 and 2 tuples give orthogonal designs which may be found using theorem 13 of [1]. We give the first rows of the circulant matrices used:

types (1, 5, 5, 9) and (5, 5, 9)

| | | | |
|---|---|---|---|
| a b c -c -b | -c b d -d b | b d c c -d | -d d d d d |

type (3, 3, 6, 6)

| | | | |
|---|---|---|---|
| 0 b -d a -c | 0 b d -a -c | c a b b -a | d a b -b a |

type (1, 3, 14)

| | | | |
|---|---|---|---|
| a c -c c -c | -b c c c 0 | c c c b -c | c c -c 0 b |

type (2, 5, 7)

| | | | |
|---|---|---|---|
| c c -c 0 -a | a c b -b 0 | b b 0 c 0 | c -b 0 c 0 |

type (3, 6, 8)

| | | | |
|---|---|---|---|
| 0 b c -b -c | 0 b c a c | a b -c 0 -c | a -b c -b -c |

type (7, 10)

| | | | |
|---|---|---|---|
| 0 a a -b -b | b a a -a a | 0 a a -a b | b 0 b -b a |

At present it is still unknown whether the following tuples are the types of designs in order 20 (one of us has shown it is not possible to construct these designs using four circulant matrices):

(1, 3, 6, 8) (1, 4, 4, 9) (2, 2, 5, 5) (3, 7, 8)

## Appendix II.

We now update the status of the orthogonal design problem in order 28 . We use Wolfe's list [2] as a basis. Theorem 12 gives us designs of types (1, 3, 6, 8), (2,2,4,16), (2,2,10,10) (3, 4, 6, 8), (3, 3, 3, 12), (4, 4, 4, 16), (4, 4, 10, 10), (1, 6, 12) and (1, 2, 12) constructed from four circulant matrices. Theorem 13 of [1] may be used with the circulant matrices whose first rows are given below to form the indicated orthogonal designs:

type (1, 1, 2, 18)

a b b -b b -b -b , c b b -b b -b -b , d b b 0 b 0 0 , -d b b 0 b 0 0

type (1, 1, 4, 16)

a b b -b b -b -b , c b b -b b -b -b , b b b 0 b 0 0 , -d d d 0 d 0 0

type (1, 1, 8, 18)

-a b b a b a a , a b b -a b -a -a , c b b -b b -b -b , d b b -b b -b -b

type (1, 1, 10, 10)

d a a -a a -a -a , c b b -b b -b -b , -a b b 0 b 0 0 , b a a 0 a 0 0

type (1, 1, 13, 13)

-a b b a b a a , -b -a -a b -a b b , c a a -a a -a -a , d b b -b b -b -b

type (1, 2, 2, 16)

a d d -d d -d -d , b 0 d c 0 -d 0 , b 0 d -c 0 -d 0 , 0 d d -d d d d

type (2, 2, 2, 18)

a d -d 0 b d -d a d -d 0 -b d -d , c d d d d -d 0 , -c d d d d -d 0 type (2, 3, 4, 6)

0 d a 0 0 a -d     c d a 0 0 -a d ,   b c -d 0 0 0 0 ,   -b c -d 0 0 0 0

type (2, 8, 8, 8)

a a b -b c d -c ,   a a b -b -c -d c ,   a -a b b c 0 c ,   a -a b b -c 0 -c

type (3, 3, 6, 6)

b d a 0 0 a -d ,   c d a 0 0 -a d ,   a c -d 0 0 -b 0 ,   -a c -d 0 0 b 0

type (4, 4, 8, 8)

a a b -b c d 0 ,   a a b -b -c -d 0 ,   a -a b b c -d 0 ,   a -a b b -c d 0

type (6, 6, 6, 6)

a a b -b c d 0 ,   -b -b a -a d -c 0 , -c -c -d d a b 0 ,   -d -d c -c -b a 0

type (1, 2, 22)

a b 0 b b b -b ,   a -b 0 -b -b -b b , 0 -b b -b b b b ,   c b b -b b -b -b

type (1, 3, 24)

a c -c c -c c -c ,   b c -c -c -c -c -c , b -c -c c c c c ,   b -c c c -c c c

type (1, 4, 20)

a b b -b b -b -b ,   c -c b b b b 0 ,   0 b b -b -b b b ,   c 0 -b b -b b c

type (1, 6, 18)

c a a b -b -a -a ,   -b -b a -a -a a 0 , a a a -a a b 0 ,   a a -a a -b a 0

type (4, 4, 18)

a -b -c -b -b b -b , a b -c b b -b b ,   a -b -b b b c 0 ,   a b b -b -b c 0

type (3, 23)

-a 0 b b b -b b ,   a -b b b b -b b ,   a b b -b b -b 0 ,   b b b b b -b -b type (5, 23)

a a -a -b b b b , -a b -a b -b b -b , b b b b b -b -b , b b b -b b b -b

type (7, 10)

a 0 0 b 0 0 0 , b -a a 0 0 a 0 , a 0 -b b b 0 b , -b b b 0 b -a -a

**Four Variables.** We have the following possible 4-tuples for orthogonal designs in order 28 (those that are known to exist are marked ✓) - all but the (4, 4, 9, 9) design may be constructed using four circulant matrices in the Goethals-Seidel array:

(1, 1, 1, 1) ✓
(1, 1, 1, 4) ✓
(1, 1, 1, 9) ✓
(1, 1, 1, 16) ✓
(1, 1, 1, 25) ✓
(1, 1, 2, 2) ✓
(1, 1, 2, 8) ✓
(1, 1, 2, 18) ✓
(1, 1, 4, 4) ✓
(1, 1, 4, 9) ✓
(1, 1, 4, 16) ✓
(1, 1, 5, 5) ✓
(1, 1, 8, 8) ✓
(1, 1, 8, 18) ✓
(1, 1, 9, 9) ✓
(1, 1, 10, 10) ✓
(1, 1, 13, 13) ✓
(1, 2, 2, 4) ✓
(1, 2, 2, 9) ✓
(1, 2, 2, 16) ✓
(1, 2, 3, 6) ✓
(1, 2, 4, 8) ✓
(1, 2, 4, 18)
(1, 2, 6, 12)
(1, 2, 8, 9) ✓
(1, 3, 6, 8) ✓
(1, 3, 6, 18)
(1, 4, 4, 4) ✓
(1, 4, 4, 9) ✓
(1, 4, 4, 16)
(1, 4, 5, 5) ✓
(1, 4, 8, 8)
(1, 4, 9, 9)
(1, 4, 10, 10)
(1, 5, 5, 9) ✓
(1, 8, 8, 9)
(1, 9, 9, 9)
(2, 2, 2, 2) ✓
(2, 2, 2, 8) ✓
(2, 2, 2, 18) ✓
(2, 2, 4, 4) ✓
(2, 2, 4, 9)
(2, 2, 4, 16) ✓
(2, 2, 5, 5) ✓
(2, 2, 8, 8) ✓
(2, 2, 9, 9)
(2, 2, 10, 10) ✓
(2, 3, 4, 6) ✓
(2, 3, 6, 9)
(2, 4, 4, 8) ✓
(2, 4, 4, 18)
(2, 4, 6, 12)
(2, 4, 8, 9)
(2, 5, 5, 8) ✓
(2, 8, 8, 8) ✓
(2, 8, 9, 9)
(3, 3, 3, 3) ✓
(3, 3, 3, 12) ✓
(3, 3, 6, 6) ✓
(3, 4, 6, 8) ✓
(3, 6, 8, 9)
(4, 4, 4, 4) ✓
(4, 4, 4, 9)
(4, 4, 4, 16) ✓
(4, 4, 5, 5) ✓
(4, 4, 8, 8) ✓
(4, 4, 9, 9) ✓
(4, 4, 10, 10) ✓
(4, 5, 5, 9)
(5, 5, 5, 5) ✓
(5, 5, 8, 8)
(5, 5, 9, 9)
(6, 6, 6, 6) ✓
(7, 7, 7, 7) ✓

**Three Variables.** There are orthogonal designs of types (1, 9), (2, 2), (2, 8), (5, 5) and (13) each constructed from two circulant matrices in order 14 . Hence there are orthogonal designs of types

(1, 9, 13) (2, 2, 13) (2, 8, 13) (5, 5, 13)

in order 28 .

Using the non-existence results quoted above we have listed all 3-tuples in order 28 which could be the types of orthogonal designs. The designs marked √ may be constructed using four circulant matrices (from the three or four variable designs quoted above) and those marked ⊗ exist but it is unknown whether they can be constructed using four circulant matrices.

(1, 1, 1) ✓
(1, 1, 2) ✓
(1, 1, 4) ✓
(1, 1, 5) ✓
(1, 1, 8) ✓
(1, 1, 9) ✓
(1, 1, 10) ✓
(1, 1, 13) ✓
(1, 1, 16) ✓
(1, 1, 17) ✓
(1, 1, 18) ✓
(1, 1, 20) ✓
(1, 1, 25) ✓
(1, 1, 26) ✓
(1, 2, 2) ✓
(1, 2, 3) ✓
(1, 2, 4) ✓
(1, 2, 6) ✓
(1, 2, 8) ✓
(1, 2, 9) ✓
(1, 2, 11) ✓
(1, 2, 12) ✓
(1, 2, 16) ✓
(1, 2, 17) ✓
(1, 2, 18) ✓
(1, 2, 19) ✓
(1, 2, 22)
(1, 2, 25) ✓
(1, 3, 6) ✓
(1, 3, 8) ✓
(1, 3, 14) ✓

(1, 3, 18) ✓
(1, 3, 22)
(1, 3, 24) ✓
(1, 4, 4) ✓
(1, 4, 5) ✓
(1, 4, 8) ✓
(1, 4, 9) ✓
(1, 4, 10) ✓
(1, 4, 13) ✓
(1, 4, 16) ✓
(1, 4, 17) ✓
(1, 4, 18)
(1, 4, 20) ✓
(1, 5, 5) ✓
(1, 5, 6) ✓
(1, 5, 9) ✓
(1, 5, 14) ✓
(1, 5, 16) ✓
(1, 5, 19)
(1, 5, 20)
(1, 6, 8) ✓
(1, 6, 11) ✓
(1, 6, 12) ✓
(1, 6, 14)
(1, 6, 18)
(1, 6, 21)
(1, 8, 8) ✓
(1, 8, 9) ✓
(1, 8, 11) ✓
(1, 8, 12)
(1, 8, 16)

(1, 8, 17)
(1, 8, 18) ✓
(1, 8, 19) ✓
(1, 9, 9) ✓
(1, 9, 10) ✓
(1, 9, 13) ✓
(1, 9, 16)
(1, 9, 18) ✓
(1, 10, 10) ✓
(1, 10, 11) ✓
(1, 10, 14)
(1, 13, 13) ✓
(1, 13, 14) ✓
(2, 2, 2) ✓
(2, 2, 4) ✓
(2, 2, 5) ✓
(2, 2, 8) ✓
(2, 2, 9) ✓
(2, 2, 10) ✓
(2, 2, 13) ✓
(2, 2, 16) ✓
(2, 2, 17) ✓
(2, 2, 18) ✓
(2, 2, 20) ✓
(2, 3, 4) ✓
(2, 3, 6) ✓
(2, 3, 7) ✓
(2, 3, 9) ✓
(2, 3, 10) ✓
(2, 3, 15)
(2, 3, 16) ✓

(2, 4, 4) ✓
(2, 4, 6) ✓
(2, 4, 8) ✓
(2, 4, 9) ✓
(2, 4, 11)
(2, 4, 12) ✓
(2, 4, 16) ✓
(2, 4, 17)
(2, 4, 18) ✓
(2, 4, 19)
(2, 4, 22)
(2, 5, 5) ✓
(2, 5, 7) ✓
(2, 5, 8) ✓
(2, 5, 13) ✓
(2, 5, 15)
(2, 5, 18)
(2, 6, 7) ✓
(2, 6, 9)
(2, 6, 11)
(2, 6, 12)
(2, 6, 13)
(2, 6, 16) ✓
(2, 6, 17)
(2, 7, 10)
(2, 7, 12)
(2, 7, 13)
(2, 7, 19)
(2, 8, 8) ✓
(2, 8, 9) ✓
(2, 8, 10) ✓

(2, 8, 13) ✓
(2, 8, 16) ✓
(2, 8, 18)
(2, 9, 9) ✓
(2, 9, 11)
(2, 9, 12)
(2, 9, 17)
(2, 10, 10) ✓
(2, 10, 12) ✓
(2, 11, 11)
(2, 11, 13)
(2, 11, 15)
(2, 13, 13) ✓
(3, 3, 3) ✓
(3, 3, 6) ✓
(3, 3, 12) ✓
(3, 3, 15) ✓
(3, 4, 6) ✓
(3, 4, 8) ✓
(3, 4, 14) ✓
(3, 4, 18)
(3, 6, 6) ✓
(3, 6, 8) ✓
(3, 6, 9) ✓
(3, 6, 11)
(3, 6, 12) ✓
(3, 6, 16)
(3, 6, 17)

(3, 6, 18)
(3, 6, 19)
(3, 7, 8) ✓
(3, 7, 10)
(3, 7, 11)
(3, 7, 15)
(3, 7, 18)
(3, 8, 9) ✓
(3, 8, 10) ✓
(3, 8, 15)
(3, 9, 14)
(3, 10, 15)
(3, 11, 14)
(4, 4, 4) ✓
(4, 4, 5) ✓
(4, 4, 8) ✓
(4, 4, 9) ✓
(4, 4, 10) ✓
(4, 4, 13)
(4, 4, 16) ✓
(4, 4, 17)
(4, 4, 18) ✓
(4, 4, 20) ✓
(4, 5, 5) ✓
(4, 5, 6) ✓
(4, 5, 9) ✓
(4, 5, 14)
(4, 5, 16)

(4, 5, 19)
(4, 6, 8) ✓
(4, 6, 11) ✓
(4, 6, 12)
(4, 6, 14)
(4, 6, 18)
(4, 8, 8) ✓
(4, 8, 9) ✓
(4, 8, 11)
(4, 8, 12) ✓
(4, 8, 16) ✓
(4, 9, 9) ⊗
(4, 9, 10)
(4, 9, 13) ⊗
(4, 10, 10) ✓
(4, 10, 11)
(4, 10, 14) ✓
(5, 5, 5) ✓
(5, 5, 8) ✓
(5, 5, 9) ✓
(5, 5, 10) ✓
(5, 5, 13) ✓
(5, 5, 16)
(5, 5, 18)
(5, 6, 9) ✓
(5, 6, 14)
(5, 6, 15)
(5, 7, 8) ✓

(5, 7, 10)
(5, 7, 14)
(5, 8, 8)
(5, 8, 13)
(5, 9, 9)
(5, 9, 10)
(5, 9, 14)
(5, 10, 10)
(6, 6, 6) ✓
(6, 6, 12) ✓
(6, 7, 8) ✓
(6, 8, 9)
(6, 8, 11)
(6, 8, 12)
(6, 9, 11)
(7, 7, 7) ✓
(7, 7, 14) ✓
(7, 8, 10)
(7, 8, 13)
(7, 10, 11)
(8, 8, 8) ✓
(8, 8, 9)
(8, 8, 10) ✓
(8, 9, 9) ⊗
(8, 9, 11)
(8, 10, 10) ✓
(9, 9, 9)
(9, 9, 10)

✓ these matrices are constructed using four circulant matrices

⊗ constructed from the (4, 4, 9, 9) design

**Two Variables.** There are 196 pairs (j, k) such that $j + k \le 28$. Of these, 27 are eliminated as types of orthogonal designs by Theorem 1 (iii) and Theorem 2. Of the remaining 169 pairs, only the following 16 are not known to be the type of an orthogonal design in order 28 .

| | | | |
|---|---|---|---|
| (4, 19) | (6, 22) | (10, 15) | (12, 14) . |
| (5, 21) | (7, 15) | (11, 12) | |
| (6, 17) | (7, 19) | (11, 14) | |
| (6, 20) | (8, 17) | (11, 15) | |
| (6, 21) | (9, 16) | (11, 17) | |

All those that do exist except (9, 17) are constructed using four circulant matrices.

The 27 2-tuples that cannot be the types of orthogonal designs in order 28 are:

| | | | | |
|---|---|---|---|---|
| (1, 7) | (3, 20) | (5, 12) | (7, 17) | (11, 16) |
| (1, 15) | (3, 21) | (5, 19) | (7, 20) | (12, 13) |
| (1, 23) | (4, 7) | (5, 22) | (8, 14) | (12, 15) . |
| (2, 14) | (4, 15) | (6, 10) | (9, 15) | |
| (3, 5) | (4, 23) | (7, 9) | (10, 17) | |
| (3, 13) | (5, 11) | (7, 16) | (11, 13) | |

**One Variable.** All one variable designs exist in order 28 .

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