Some New Constructions for Orthogonal Designs

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Abstract

We give three new constructions for orthogonal designs using amicable orthogonal designs.

These are then used to show (i) all possible n-tuples, $n \leq 5$, are the types of orthogonal designs in order 16 and (ii) all possible n-tuples, $n \leq 3$ are the types of orthogonal designs in order 32, (iii) all 4-tuples, $(e, f, g, 32-e-f-g)$, $0 \leq e + f + g \leq 32$ are the types of orthogonal designs in order 32.

These results are used in a paper by Peter J. Robinson, "Orthogonal designs of order sixteen", in this same volume, to fully update the status of the existence of orthogonal designs in order 16.

§1. Introduction

An orthogonal design of order n *and type* $(u_1, u_2, \ldots, u_s)$ $(u_i > 0)$ on the commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_s\}$ such that

$$AA^T = \sum_{i=1}^{s} (u_i x_i^2) I_n .$$

Alternatively, the rows of A are formally orthogonal and each row has precisely u_i entries of the type $\pm x_i$.

In [1], where this was first defined and many examples and properties of such designs were investigated, we mentioned that

$$A^T A = \sum_{i=1}^{s} (u_i x_i^2) I_n$$

and so our alternative description of A applies equally well to the columns of A . We also showed in [1] that $s \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

when

$$n = 2^a \cdot b, \quad b \text{ odd}, \quad a = 4c + d, \quad 0 \leq d < 4 .$$

Two orthogonal designs, A and B , of order n and types $(a_1, a_2, \ldots, a_s)$ and $(b_1, b_2, \ldots, b_t)$ in the variables $x_1, \ldots, x_s$ and $y_1, \ldots, y_t$ will be called *amicable* if

$$AB^T = BA^T.$$

In this generality, amicable orthogonal designs were first systematically studied by Wolfe in [10]. In that paper infinite families of amicable orthogonal designs are constructed and exact bounds (similar to Radon's function) are given for the number of variables that may appear in each orthogonal design of an amicable pair of orthogonal designs. If, in addition $a_1 = 1$, $\sum_{i=1}^{s} a_i = \sum_{j=1}^{t} b_j = n$, we can use monomial matrices P and Q to ensure $PAQ = x_1 I + x_i S_i$, $S_i^T = -S_i$ and $PBQ = y_j B_j$, $B_j^T = B_j$. If we now set all the variables x_i and y_j equal to 1, we have two $(1, -1)$ orthogonal matrices, $M = I + W$ and N, of order n satisfying

$$MN^T = NM^T, \quad N^T = N, \quad W^T = -W.$$

M and N are *amicable Hadamard matrices* and in this specific case amicable orthogonal designs are well known and have been studied [6].

In our earlier work we studied existence and non-existence results for orthogonal designs. Many of our results have been superceded by the beautiful results of W. Wolfe [9] and D. Shapiro [5]. Wolfe's results, in particular, have pointed out the existence of two separate and independently interesting aspects of the question of existence for orthogonal designs. It is easy to see that an orthogonal design of order n and type $(a_1, \ldots, a_n)$ exists in order n iff there are k $\{0, 1, -1\}$ matrices $A_1, \ldots, A_n$, of order n such that

(i) $A_j A_j^T = a_j I_n$,

(ii) $A_i A_j^T + A_j A_i^T = 0$,

(iii) $A_i * A_j = 0$ (Hadamard product).

A rational family of order n and type $[a_1, \ldots, a_n]$ is a collection of k rational matrices of order n, $R_1, \ldots, R_k$ such that

(i) $R_j R_j^T = a_j I_n$,

(ii) $R_i R_j^T + R_j R_i^T = 0$.

Clearly, any theorem which precludes the existence of a rational family precludes the existence of an orthogonal design of the same order and type.

We refer to the questions concerning existence of rational families as "algebraic" and those that refer to the question of existence of orthogonal designs as "combinatorial".

Shapiro has made significant inroads into the algebraic problem. He has shown that a rational family of type $[a_1, \ldots, a_n]$ exists in order $2^t.n$ (n odd) iff a family of the same type exists in order 2^t . Thus, for the algebraic problem, all results rest on getting information in powers of two. We shall use the following result of Shapiro.

THEOREM (D. Shapiro). *If* $n \equiv 16 \pmod{32}$, *then there exists an orthogonal design of type* $(a_1, a_2, \ldots, a_9)$ *only if the Hasse invariant* $s_p(a_1, \ldots, a_9) = 1$ *at every prime* p .

We note that

$$s_p(a_1, \ldots, a_t) = \prod_{1 \le i < j \le t} (a_i, a_j)_p$$

where $(a_i, a_j)_p$ is the Hilbert norm residue symbol (see [3]).

The following results of Geramita and Verner [2] and P. Robinson show that algebraic existence is not enough to imply combinatorial existence. The question of combinatorial non-existence is still uncharted territory.

THEOREM (Geramita and Verner). *If there exists an orthogonal design of type* $(u_1, u_2, \ldots, u_s)$ *in order* $n \equiv 0 \pmod 4$ *and* $\sum_{i=1}^{s} u_i = n - 1$ *then there exists an orthogonal design of type* $(1, u_1, u_2, \ldots, u_s)$ *in order* n .

THEOREM (Peter J. Robinson). *The orthogonal design* $(1, 1, 1, n-4)$ *only exists in order* n *for* $n = 4, 8, 16$.

It has been shown [7] that

THEOREM (Jennifer Wallis). *All orthogonal designs of types* $(a, b, n-a-b)$ *and* (a, b) , $0 \le a + b \le n$, *exist in orders* n *which are a power of* 2 .

These results will be combined with those of this paper to discuss the existence of orthogonal designs in order 16 and 32 .

§2. Some useful matrices.

We note

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_3 \\ -x_2 & x_1 & x_3 & -x_3 \\ -x_3 & -x_3 & x_1 & x_2 \\ -x_3 & x_3 & -x_2 & x_1 \end{bmatrix} \qquad B = \begin{bmatrix} y_1 & y_2 & y_3 & y_3 \\ y_2 & -y_1 & y_3 & -y_3 \\ y_3 & y_3 & -y_2 & -y_1 \\ y_3 & -y_3 & -y_1 & y_2 \end{bmatrix} \tag{1}$$

$$= x_1A_1 + x_2A_2 + x_3A_3 \qquad\qquad = y_1B_1 + y_2B_2 + y_3B_3$$

are amicable orthogonal designs of types (1, 1, 2) and (1, 1, 2) in order 4 .

Three matrices C_1, C_2, C_3 of order n will be called an *amicable triple* if each C_i, $i = 1, 2, 3$ is an orthogonal design and $C_iC_j^T = C_jC_i^T$, $i \neq j$. These matrices were first studied by Wolfe [11].

Let

$$T_0 = \begin{bmatrix} x & y & y & y \\ y & -x & -y & y \\ y & -y & y & -x \\ y & y & -x & -y \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & + & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & - & + \\ - & + & 0 & 0 \\ - & - & 0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{bmatrix} \tag{2}$$

then $\{T_0, T_1, T_3\}$ is an amicable triple of types (1, 3) , (1) , (3) and order 4 , and $\{T_0, T_2, T_3\}$ is an amicable triple of types (1, 3) , (2) , (3) and order 4 .

§3. Three Constructions for Orthogonal Designs

THEOREM 1. *Suppose* $A = \sum_{i=1}^{t} a_i A_i$ *and* $B = \sum_{j=1}^{s} b_j B_j$ *are amicable orthogonal designs of types* $(u_1, u_2, \ldots, u_t)$ *and* $(v_1, v_2, \ldots, v_s)$ *in order* n . *Then*

$$C = A_1 \times (a_0 I + a_1 M) + \sum_{i=2}^{t} A_i \times a_i N \quad \text{and} \quad D = \sum_{j=1}^{s} B_j \times b_j N$$

are amicable orthogonal designs of types $(u_1, wu_1, mu_2, \ldots, mu_t)$ *and* $(mv_1, mv_2, \ldots, mv_s)$ *in order* pn *where* $I + M$ *and* N *are amicable orthogonal designs of types* $(1, w)$ *and* (m) *in order* p .

Proof. Clearly $M^T = -M$, $N^T = N$, $MN^T = NM^T$, $A_i A_i^T = u_i I$,

$B_j B_j^T = v_j I$, $A_i A_j^T + A_j A_i^T = 0$, $B_i B_j^T + B_j B_i^T = 0$, $i \neq j$

$A_i B_j^T = B_j A_i^T$, $1 \le i \le t$, $1 \le j \le s$.

Now it may be easily verified that C and D are amicable orthogonal designs.

COROLLARY 2. *Suppose there exist amicable orthogonal designs of types* $(u_1, u_2, \ldots, u_t)$ *and* $(v_1, v_2, \ldots, v_s)$ *in order* n . *Further suppose there exist amicable Hadamard matrices of order* m . *Then there exist amicable orthogonal designs of types* $(u_1, (m-1)u_1, mu_2, \ldots, mu_t)$ *and* $(mv_1, mv_2, \ldots, mv_s)$ *in order* mn .

Now we see from [6] there exist amicable Hadamard matrices of order

I	2 ;	
II	$p^r + 1$	p^r(prime power) $\equiv 3 \pmod 4$;
III	$2(q^s+1)$	q^s(prime power) $\equiv 1 \pmod 4$ and $2q + 1$ a prime power;
IV	$2(q^s+1)$	q^s(prime power) $\equiv p^2 + 4 \equiv 5 \pmod 8$;
V	$4(q^s+1)$	q^s(prime power) $\equiv p^2 + 36 \equiv 5 \pmod 8$;
VI	d	where d is the product of any of the above orders.

In particular, therefore, we have

COROLLARY 3. *Suppose there exist amicable orthogonal designs of types* $(u_1, u_2, \ldots, u_t)$ *and* $(v_1, v_2, \ldots, v_s)$ *in order* n . *Then there exist amicable orthogonal designs of types* $(u_1, u_1, 2u_2, \ldots, 2u_t)$ *and* $(2v_1, 2v_2, \ldots, 2v_s)$ *in order* $2n$.

COROLLARY 4. *There exist amicable orthogonal designs of types* $(1, 1, 2, 4)$ *and* $(2, 2, 4)$ *in order* 8 .

COROLLARY 5. *There exist amicable orthogonal designs of types* $(1, 1, 2, 4, \ldots, 2^{t-1})$ *and* $(2^{t-2}, 2^{t-2}, 2^{t-1})$ *in order* 2^t .

THEOREM 6. *Suppose there exist three matrices* R, P, S *of order* n *which give amicable orthogonal designs*

$$S, \quad x_2R + x_3P$$

of types $(s_1, s_2, \ldots, s_t)$ *and* (u_1, u_2) *respectively. Then*

$$\begin{bmatrix} y_1R+y_2P & y_3R+y_4P & S & y_6R+y_7P \\ -y_3R+y_4P & y_1R-y_2P & -y_6R-y_7P & S \\ -S & y_6R-y_7P & y_1R+y_2P & -y_3R+y_4P \\ -y_6R+y_7P & -S & y_3R+y_4P & y_1R-y_2P \end{bmatrix}$$

is an orthogonal design of order $4n$ *and type*

$(s_1, s_2, \ldots, s_t, u_1, u_1, u_1, u_2, u_2, u_2)$.

Proof. By direct verification after noting $RP^T + PR^T = 0$, $SR^T = RS^T$ and $SP^T = PS^T$.

COROLLARY 7. *There exist orthogonal designs of types*

(i) (1, 1, 2, 1, 3, 1, 3, 1, 3) , (ii) (1, 1, 2, 1, 2, 1, 2, 1, 2)

(iii) (1, 1, 2, 2, 2, 2, 2, 2, 2) and

(iv) (1, 1, 2, 1, 1, 1, 1, 1, 1)

in order 16.

Proof. Use $S = x_1A_1 + x_2A_2 + x_3A_3$ and (i) $R = B_1$, $P = B_2 + B_3$, (ii) $R = B_1$, $P = B_3$, (iii) $R = B_1 + B_2$, $P = B_3$, (iv) $R = B_1$, $P = B_2$, respectively in the theorem, where the A_i and B_i are defined in (1) of §2.

COROLLARY 8. *There exist orthogonal designs of types* (i) (2, 2, 4, 3, 5, 3, 5, 3, 5) , (ii) (2, 2, 4, 1, a, 1, a, 1, a), a = 1, 2, 3, 4, 5, 6 *or* 7 , (iii) (1, 1, 2, 4, 2, a, 2, a, 2, a), a = 2, 4 *or* 6 , (iv) (1, 7, 1, 7, 1, 7, 1, 7) *in order* 32.

Proof. For (i), (ii) and (iii) we use the existence of amicable orthogonal designs of types (1, 1, 2, 4) and (2, 2, 4) in order 8 . For (iv) we let S and $x_2R + x_3P$ be the amicable orthogonal designs of types (1, 7) and (1, 7) in order 8 .

THEOREM 9. *Suppose there exist matrices* S, R, P *of order* n *which give amicable orthogonal designs*

$$S \text{ and } x_1R + P$$

of types $(s_1, s_2, \ldots, s_t)$ *and* $(u_1, u_2, \ldots, u_r)$ *respectively. Then*

$$\begin{bmatrix} y_1R+P & y_3R+P & S & y_6R+P \\ -y_3R+P & y_1R-P & -y_6R-P & S \\ -S & y_6R-P & y_1R+P & -y_3R+P \\ -y_6R+P & -S & y_3R+P & y_1R-P \end{bmatrix}$$

is an orthogonal design of order 4n *and type*

$(s_1, s_2, \ldots, s_t, u_1, u_1, u_1, 3u_2, 3u_3, \ldots, 3u_r)$.

Proof. By direct verification.

COROLLARY 10. *There exists an orthogonal design of type*

(2, 2, 4, 1, 1, 1, 3, 6, 12)

in order 32 .

Proof. Let S be the (2, 2, 4) and $x_1R + P$ the (1, 1, 2, 4) design in order 8 .

COROLLARY 11. *There exist orthogonal designs of types*
(i) (1, 1, 2, 1, 1, 1, 3, 6) , (ii) (1, 1, 2, 2, 2, 2, 3, 3) , *in order* 16 .

Proof. We note there exist amicable orthogonal designs $x_1A_1 + x_2A_2 + x_3A_3$, $z_1B_1 + z_2B_2 + z_3B_3$ of types (1, 1, 2) and (1, 1, 2) in order 4 . We obtain the types of the enunciation by choosing (i) $S = x_1A_1 + x_2A_2 + x_3A_3$, $R = B_1$, $P = z_2B_2 + z_3B_3$, (ii) $S = x_1A_1 + x_2A_2 + x_3A_3$, $R = B_3$, $P = z_1B_1 + z_2B_2$.

We recall construction 19 of [1] which we generalize slightly

THEOREM 12. *Let* P_1, P_2, P_3, H *be orthogonal designs of order* n *satisfying* $P_i^T = -P_i$, $i = 1, 2, 3$, $H^T = H$ *and* $MN^T = NM^T$ *for* $M, N \in \{P_1, P_2, P_3, H\}$. *Suppose* P_i *is of type* $(p_{i1}, p_{i2}, \ldots .)$ *and* H *is of type* $(h_1, h_2, \ldots)$ *then*

$$\begin{bmatrix} x_1I_n+P_1 & x_3I_n+P_2 & x_5I_n+P_3 & H \\ -x_3I_n+P_2 & x_1I_n-P_1 & -H & x_5I_n+P_3 \\ -x_5I_n+P_3 & H & x_1I_n-P_1 & -x_3I_n-P_2 \\ -H & -x_5I_n+P_3 & x_3I_n-P_2 & x_1I_n+P_1 \end{bmatrix}$$

is an orthogonal design of order 4n *and type*
$(1, p_{11}, p_{12}, \ldots, 1, p_{21}, p_{22}, \ldots, 1, p_{31}, p_{32}, \ldots, h_1, h_2, \ldots)$.

COROLLARY 13. *There exist orthogonal designs of type* $(1, p_{11}, 1, p_{21}, 1, p_{31}, 1, 3)$ *where* $p_{i1} \in \{1, 3\}$ or $p_{i1} \in \{2, 3\}$ for $i = 1, 2, 3$, *in order* 16 . *That is there exist* *orthogonal designs* (1, 1, 1, 1, 3, j, j, j), $j \in \{1, 2, 3\}$, (1, 1, 1, 1, 3, 3, j, j) , $j \in \{1, 2\}$, *and* (1, 1, 1, 1, 3, 3, 3, j) , $j \in \{1, 2\}$ *in order* 16 .

Proof. We let $H = T_0$ defined in (2) of §2. Then since there are amicable triples (1, 3), (j), (3) , where j = 1, 2 , given in (2) of §2 we have orthogonal designs of the types given in the enunciation.

§4. Applications.

LEMMA 14. *All* 6-*tuples of the form* (a, b, c, d, e, 16-a-b-c-d-e), $0 \le a + b + c + d + e \le 16$, *are the types of orthogonal designs in order* 16 .

Proof. All these designs may be constructed using the (1, 1, 2, 2, 2, 2, 2, 2, 2) design found in [1], the (1, 1, 1, 1, 1, 2, 3, 3, 3) design found in Corollary 7 and the (1, 1, 2, 2, 2, 2, 3, 3) design found in Corollary 11.

COROLLARY 15. *All* n-*tuples*, n = 1, 2, 3, 4, 5 *are the types of orthogonal designs in order* 16 .

LEMMA 16. *All* 5-*tuples*, (a, b, c, d, 32-a-b-c-d), $0 \le a + b + c + d \le 32$, *are the types of orthogonal designs in order* 32 *except possibly those listed here which are unresolved*

(3, 9, 9, 9, 2)	(1, 3, 11, 11, 6)	(1, 3, 9, 9, 10)
(1, 5, 5, 17, 4)	(1, 5, 9, 11, 6)	(1, 5, 5, 11, 10)
(1, 5, 11, 11, 4)	(3, 3, 9, 11, 6)	(1, 5, 5, 5, 16)
(3, 3, 11, 11, 4)		(3, 3, 3, 3, 20) .

Proof. Since all five variable designs exist in order 16 every design in order 32 of the form (2x, 2y, 2z, 2u, 2v) or (x, 2y+x, 2z, 2u, 2w) exists.

So we only have to check the existence of designs (2x+1, 2y+1, 2z+1, 2u+1, 28-2x-2y-2z-2w) and all these, except those listed above, may be found from the (1, 1, 1, 2, 2, 3, 4, 6, 12) design constructed in Corollary 10 and the (1, 1, 1, 2, 2, 4, 7, 7, 7), (1, 1, 1, 1, 7, 7, 7, 7) and (2, 2, 3, 3, 3, 4, 5, 5, 5) designs constructed in Corollary 8.

LEMMA 17. *All* 4-*tuples* (a, b, c, 32-a-b-c), $0 \leq a + b + c \leq 32$, *are the types of orthogonal designs in order* 32 .

Proof. All designs of these types may be constructed using the designs quoted in the proof of the previous lemma.

COROLLARY 18. *All* n-*tuples*, n = 1, 2, 3 , *are the types of orthogonal designs in order* 32 .

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