ORTHOGONAL DESIGNS V: ORDERS DIVISIBLE BY EIGHT

Jennifer Seberry Wallis

ABSTRACT. Constructions are given for orthogonal designs in orders divisible by eight. These are then used to show all two variable orthogonal designs exist in orders 24, 32 and 48. The existence of two variable designs in order 40 and three variable designs in order 24 is discussed.

The conjectures on the existence of all orthogonal designs (1, k) and skew-symmetric weighing matrices for weights $k = 1, 2, \ldots, 2^t.9-1$ are resolved in the affirmative for orders $2^t.9$, $t \geq 3$ a positive integer.

1. *Introduction.*

An orthogonal design of order n *and type* $(u_1, u_2, \ldots, u_s)$ $(u_i > 0)$ on the commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_s\}$ such that

$$AA^t = \sum_{i=1}^{s} (u_i x_i^2) I_n \ .$$

Alternatively, the rows of A are formally orthogonal and each row has precisely u_i entries of the type $\pm x_i$.

In [2], where this was first defined and many examples and properties of such designs were investigated, we mentioned that

$$A^t A = \sum_{i=1}^{s} (u_i x_i^2) I_n \ ,$$

and so our alternative description of A applies equally well to the columns of A. We also showed in [2] that $s \leq \rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

UTILITAS MATHEMATICA Vol. 9 (1976), pp. 263-281.

when

$$n = 2^a \cdot b, \quad b \text{ odd}, \quad a = 4c + d, \quad 0 \le d \le 4 .$$

In [2] we also showed that if there is an orthogonal design of order $n \equiv 2 \pmod 4$ and type (a,b) then $\frac{b}{a}$ is a rational square.

While in [5] it was shown that if $n \equiv 4 \pmod 8$ and if X is an orthogonal design of order n and type

(i) (a, a, a, b), then $\frac{b}{a}$ is a rational square;

(ii) (a, a, b), then $\frac{b}{a}$ is the sum of at most two rational squares;

(iii) (a, b), then $\frac{b}{a}$ is the sum of at most three rational squares.

It has been established in [2] that conditions (i), (ii) and (iii) were necessary and sufficient for $n = 12$ but Geramita and Verner [9] proved

THEOREM (Geramita-Verner). *If there exists an orthogonal design of type* $(u_1, u_2, \ldots, u_s)$ *in order* $n \equiv 0 \pmod 4$ *and* $\sum_{i=1}^{s} u_i = n - 1$ *then there exists an orthogonal design of type* $(1, u_1, u_2, \ldots, u_s)$ *in order* n.

This meant the conditions (i), (ii), (iii) were not sufficient for $n = 20$.

D. Shapiro [6] has shown that if $n \equiv 8 \pmod{16}$ and if X is an orthogonal design of order n and type

(i) (a, a, a, a, a, a, a, b), then $\frac{b}{a}$ is a rational square;

(ii) (a, a, a, a, a, a, b), then $\frac{b}{a}$ is the sum of at most two rational squares;

(iii) (a, a, a, a, a, b), then $\frac{b}{a}$ is the sum of at most three rational squares.

We conjecture that:

A necessary and sufficient condition for the existence of an orthogonal design of the type (a, b) *and of order* $n \equiv 0 \pmod 8$ *is that* $a + b \le n$.

Definition.

A *weighing matrix of weight* k *and order* n, is a square {0, 1, -1} matrix, A, of order n satisfying

$$AA^t = kI_n .$$

In [2] we showed that the existence of an orthogonal design of order n and type $(u_1, \ldots, u_s)$ is equivalent to the existence of weighing matrices $A_1, \ldots, A_s$, of order n, where A_i has weight u_i and the matrices, $\{A_i\}_i^s = 1$, satisfy the matrix equations

$$XX^t + YX^t = 0 \quad \text{and} \quad X * Y = 0 \quad (\text{*the Hadamard product})$$

in pairs. In particular, the existence of an orthogonal design of order n and type (1, k) is equivalent to the existence of a skew-symmetric weighing matrix of weight k and order n.

It is conjectured that:

(I) for $n \equiv 0 \pmod 4$ there is a weighing matrix of weight k and order n for every $k \le n$;

(II) for $n \equiv 0 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$ (equivalently there is an orthogonal design of type (1, k) in order n for every $k < n$);

(III) for $n \equiv 4 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$, where k is the sum of $\le$ three squares of integers (equivalently, there is an orthogonal design of type (1, k) in order n for every $k < n$ which is the sum of $\le$ three squares of integers. In other words, the necessary condition for the existence of an orthogonal design of type (1, k) in order n, $n \equiv 4 \pmod 8$, is also sufficient);

(IV) for $n \equiv 2 \pmod 4$ there is a skew-symmetric weighing matrix for every weight $k < n - 1$ when k is a square (equivalently, the necessary condition for the existence of an orthogonal design of type (1, k) in order n is also sufficient).

Conjecture (I) is an extension of the Hadamard conjecture (i.e. for every $n \equiv 0 \pmod 4$ there is a {1, -1} matrix, H, of order n satisfying $HH^t = nI_n$), while (II) and (III) generalize the conjecture that for every $n \equiv 0 \pmod 4$ there is a Hadamard matrix, H, of order n with the property that $H = I_n + S$ where $S = -S^t$.

Conjecture (I) was established in [7] for $n \in \{4, 8, 12, \ldots, 32, 40\}$ and in [1] for $n = 2^t$ $(t \geq 3)$, while conjecture III was established in [2] for $n = 4, 12$ and in [3] for $n = 20$ and 28. Conjecture (II) (and as a consequence (I)) was established in [3] for $n = 2^{t+1}.3$, $n = 2^{t+1}.5$, $t \geq 2$ a positive integer. Conjecture IV was established in [2] for $n = 6, 10, 14$.

In this paper we establish conjecture (II) (and as a consequence (I)) for $n = 2^{t+1}.9$, $t \geq 2$ a positive integer.

Let R be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to be *constructed from two circulant matrices* A and B if it is of the form

$$\begin{bmatrix} A & B \\ B^t & -A^t \end{bmatrix}$$

and to be of *Goethals-Seidel type* if it is of the form

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^tR & -C^tR \\ -CR & -D^tR & A & B^tR \\ -DR & C^tR & -B^tR & A \end{bmatrix}$$

where A, B, C, D are circulant matrices.

2. *Known Results.*

In this section we list some of the results from [2] that we shall use.

LEMMA 1. [2, corollary to construction 22]. *If there is an orthogonal design of type* (a, b) *in order* n *then there is an orthogonal design* *of type* (a, a, b, b) *in order* $2n$ *and of type* $(a, a, 2a, b, b, 2b)$ *in order* $4n$.

The following easy corollary was mentioned in [3].

COROLLARY 1.1. *If there are orthogonal designs of type* $(1, k)$ $1 \le k \le \ell$ *in order* n *then there are orthogonal designs of type* $(1, m)$ *in order* $2n$ *for* $1 \le m \le 2\ell + 1$. *In particular, if there are orthogonal designs of type* $(1, k)$, $1 \le k \le n - 1$, *in order* n *then there are orthogonal designs of type* $(1, m)$, $1 \le m \le 2^t n - 1$, *in order* $2^t n$, t *a positive integer.*

LEMMA 2. *If* X *is an orthogonal design of order* n *and type* $(u_1, u_2, \ldots, u_s)$ *on the variables* $x_1, \ldots, x_s$ *then there is an orthogonal design of order* n *and type* $(u_1, \ldots, u_i + u_j, \ldots, u_s)$ *on the* $s - 1$ *variables* $x_1, \ldots, \bar{x}_j, \ldots, x_s$.

Part (i) of the following lemma appeared in [2] and will be used extensively. Part (ii) with $f = 1$ has been observed, independently, by Joan Murphy Geramita.

LEMMA 3. *If there exists an orthogonal design of order* n *and type* $(s_1, s_2, \ldots, s_\ell)$ *then there exist orthogonal designs of type*

(i) $(e_1 s_1, e_2 s_2, \ldots, e_\ell s_\ell)$ *where* $e_i = 1$ *or* 2,

(ii) $(s_1, s_1, fs_2, \ldots, fs_\ell)$ *where* $f = 1$ *or* 2,

in order $2n$.

Proof. (i) Replace each variable by

$\begin{pmatrix} x_i & 0 \\ 0 & x_i \end{pmatrix}$ if $e_i = 1$ and by $\begin{pmatrix} x_i & x_i \\ x_i & -x_i \end{pmatrix}$ if $e_i = 2$.

(ii) Replace the variable x_1 by

$$\begin{pmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{pmatrix}$$

and the variable x_i, $i \neq 1$, by

$$\begin{pmatrix} 0 & x_i \\ x_i & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x_i & x_i \\ x_i & -x_i \end{pmatrix}$$

according as f is 1 or 2.

The following also holds.

LEMMA 4. *There are orthogonal designs of order* $8n$ *and type* $(1, k)$ *where*

(i) $n = 3, 4, 5$ *or* 6, $k \in \{1, \ldots, 8n - 1\}$;

(ii) $n \geq 7$, $k \in \{1, \ldots, 46\}$.

3. *Orthogonal Designs of Order Divisible by Eight.*

In trying to find designs of order $n \equiv 0 \pmod 8$ it soon becomes clear that designs of the Goethal-Seidel type on eight variables are invaluable. But the difficulty of finding matrices to replace the variables has led to the following lemma using part Williamson and part Goethals-Seidel criteria.

LEMMA 5. *Suppose* $X_1, X_2, \ldots, X_8$ *are eight circulant matrices of order* n *satisfying*

$$\sum_{i=1}^{8} X_i X_i^t = fI .$$

Further suppose:

(i) $X_1, X_2, \ldots, X_8$ *are all symmetric or all skew; or*

(ii) $X_1 = X_2 = \ldots = X_i$ *and* $X_{i+1} = \ldots = X_8$, $1 \geq i \geq 8$; *or*

(iii) $X_1 = X_2 = \ldots = X_i$, *and* $X_{i+1}, \ldots, X_8$ *all symmetric or all skew*, $1 \leq i \leq 8$; *or*

(iv) $X_2 = X_3 = X_4$ *and* X_5, X_6, X_7, X_8 *are all symmetric (skew); or*

(v) $X_1 X_2^t = X_2 X_1^t, X_3 = X_4$ *and* X_5, X_6, X_7, X_8 *are all symmetric; or*

(vi) $X_1, \ldots, X_i$ *are all skew and* $X_{i+1}, \ldots, X_8$ *all symmetric; or*

(vii) X_2, X_3, X_4 *all skew and* X_5, X_6, X_7, X_8 *all symmetric; or*

(viii) $X_i X_{i+4}^t = X_{i+4} X_i^t, i = 1, 2, 3, 4.$

Then, with

$$f = \sum_{i=1}^{\ell} s_i x_i^2 I ,$$

there exists an orthogonal design of order 8n *and type* $(s_1, s_2, \ldots, s_\ell)$.

Proof. Use the following constructions:

(i) the matrices in design 1; (ii) X_1R and X_{i+1} in design 1; (iii) $X_1R, X_{i+1}, \ldots, X_8$ in design 1; (iv) letting $A = X_1$, $B = X_2$ in designs 2 and 3; (v) $X=X_1$, $A = X_2$, $B = X_3$ in design 4; (vi) design 1 and X_jR for $j \le i$; (vii) design 5; (viii) design 6 (this was discovered by E. Spence see [10]).

A	B	C	D	E	F	G	H
-B	A	D	-C	F	-E	-H	G
-C	-D	A	B	G	H	-E	-F
-D	C	-B	A	H	-G	F	-E
-E	-F	-G	-H	A	B	C	D
-F	E	-H	G	-B	A	-D	C
-G	H	E	-F	-C	D	A	-B
-H	-G	F	E	-D	-C	B	A

Design 1.

$$
\left[\begin{array}{cccc|cccc}
AR & B & B & B & E & F & G & H \\
-B & AR & B & -B & F & -E & -H & G \\
-B & -B & AR & B & G & H & -E & -F \\
-B & B & -B & AR & H & -G & F & -E \\
\hline
-E & -F & -G & -H & AR & -B^T & -B^T & -B^T \\
-F & E & -H & G & B^T & AR & B^T & -B^T \\
-G & H & E & -F & B^T & -B^T & AR & B^T \\
-H & -G & F & E & B^T & B^T & -B^T & AR
\end{array}\right]
$$

Design 2.

$$
\left[\begin{array}{cccc|cccc}
AR & B & B & B & E & F & G & H \\
-B & AR & B & -B & F & -E & -H & G \\
-B & -B & AR & B & G & H & -E & -F \\
-B & B & -B & AR & H & -G & F & -E \\
\hline
-E & -F & -G & -H & AR & B^T & B^T & B^T \\
-F & E & -H & G & -B^T & AR & -B^T & B^T \\
-G & H & E & -F & -B^T & B^T & AR & -B^T \\
-H & -G & F & E & -B^T & -B^T & B^T & AR
\end{array}\right]
$$

Design 3.

$$\left[\begin{array}{cccc|cccc}
XR & AR & B & B & C & D & E & F \\
-AR & XR & B & -B & D & -C & -F & E \\
-B & -B & XR & AR & E & F & -C & -D \\
-B & B & -AR & XR & F & -E & D & -C \\
\hline
-C & -D & -E & -F & XR & AR & B^T & B^T \\
-D & C & -F & E & -AR & XR & -B^T & B^T \\
-E & F & C & -D & -B^T & B^T & XR & -AR \\
-F & -E & D & C & -B^T & -B^T & AR & XR
\end{array}\right]$$

Design 4.

$$\left[\begin{array}{cccc|cccc}
AR & B & C & D & E & F & G & H \\
-B & AR & D & -C & F & -E & -H & G \\
-C & -D & AR & B & G & H & -E & -F \\
-D & C & -B & AR & H & -G & F & -E \\
\hline
-E & -F & -G & -H & AR & B^T & C^T & D^T \\
-F & E & -H & G & -B^T & AR & -D^T & C^T \\
-G & H & E & -F & -C^T & D^T & AR & -B^T \\
-H & -G & F & E & -D^T & -C^T & B^T & AR
\end{array}\right]$$

Design 5.

$$\begin{bmatrix} X_1 & X_2R & X_3R & X_4R & X_5 & X_6R & X_7R & X_8R \\ -X_2R & X_1 & X_4^TR & -X_3^TR & X_6R & -X_5 & -X_8^TR & X_7^TR \\ -X_3R & -X_4^TR & X_1 & X_2^TR & X_7R & X_8^TR & -X_5 & -X_6^TR \\ -X_4R & X_3^TR & -X_2^TR & X_1 & X_8R & -X_7^TR & X_6^TR & -X_5 \\ -X_5 & -X_6R & -X_7R & -X_8R & X_1 & X_2R & X_3R & X_4R \\ -X_6R & X_5 & -X_8^TR & X_7^TR & -X_2R & X_1 & -X_4^TR & X_3^TR \\ -X_7R & X_8^TR & X_5 & -X_6^TR & -X_3R & X_4^TR & X_1 & -X_2^TR \\ -X_8R & -X_7^TR & X_6^TR & X_5 & -X_4R & -X_3^TR & X_2^TR & X_1 \end{bmatrix}$$

Design 6.

LEMMA 6. *If every design* $(1, i, k)$ *where* $1+i+k \le n$ *exists in order* n *then every design* (ℓ, m) *where* $\ell+m \le 2n$ *exists in order* $2n$.

Proof. *Case 1.* To construct all the designs (ℓ, m) where ℓ is odd use the designs $(1, \frac{1}{2}(\ell-1), j)$ in order n where $j \in \{\frac{1}{2}(\ell-1),\ldots,n-1-\frac{1}{2}(\ell-1)\}$. Then using (ii) of lemma 3 (with $f = 2$) we have the designs $(1, 1, \ell-1, 2j)$ which give all the designs (ℓ, m) where ℓ is odd and $m \in \{\ell, \ldots, 2n-\ell\}$.

Case 2. The construction of all designs (ℓ,m) where ℓ is even is similar but starts by using all the designs $(1, \ell/2, j)$ in order n where $j \in \{\ell/2, \ldots, n-1-\ell/2\}$. Again using (ii) of lemma 3 (with $f = 2$) we have the designs $(1, 1, \ell, 2j)$ which give all the designs (ℓ,m) where ℓ is even and $m \in \{\ell, \ldots, 2n-\ell\}$.

COROLLARY 6.1. *Since all designs* $(1, i, j)$ *exist in order* 16 *all designs* (k, ℓ) *exist in order* 32.

Lemma 6 is a nice method for constructing two variable designs in order $2n$ given that all three variable designs $(1, i, j)$

are known in order n. Unfortunately (as we will see) all the designs (1, i, j) may not be known. So we ask when can the two variable designs in order 2n constructed from a given three variable design (a, b, c) be constructed using other three variable designs (which hopefully exist) in the same order.

Suppose a design (a, b, c) exists in order n, then much tedious work (using lemma 3) will allow us to construct many two variable designs in order 2n. If the design (a, b, c) is not known but other three variable designs in order n are known it may happen that most of the two variable designs in order 2n which could have been constructed from (a, b, c) can be constructed using other designs. It may in fact be shown that

THEOREM 7. *Suppose a three variable design* (a, b, c) *is unknown or does not exist in order* n. *Further suppose* (a, b, a + c), (a, a + b, c) *and* (2a, b, c) *designs do exist in order* n. *Then the existence of the following two variable designs is in doubt in order* 2n.

(a, b + c)	(a, a + 2b + 2c)	(2b, a + 2c)
(a, b + 2c)	(a + b, b + c)	(c, a + 2b)
(a, c + 2b)	(a + c, b + c)	(2c, a + 2b)
(a, a + b + c)	(b, a + 2c)	(a + 2b, a + 2c).
(a, 2b + 2c)		

If in addition an (a, b, b + c) *design is known in order* n *then the designs in doubt in order* 2n *are:*

(a, b + 2c)	(2b, a + 2c)	(2c, a + 2b)
(a + c, b + c)	(c, a + 2b)	(a + 2b, a + 2c).
(b, a + 2c)		

Example. Suppose we wish to reduce the number of cases in considering the two variable designs in order 48 which are in doubt because a design (1, 1, 21) has not yet been constructed in order 24.

First we note that both (1, 1, 21 + 1) and (1, 1 + 1, 21) are known and (1 + 1, 1, 21) can be used for (2a, b, c). So the conditions of the theorem apply. Hence the designs

(1, 43)	(3, 21)	(3, 43)
(2, 43)	(3, 42)	(22, 22)

are in doubt. In this case the design (1, 2, 21) also eliminates (1, 43), (2, 43), (3, 21), (3, 42) and (1, 1, 22) eliminates (22, 22). (3, 43) exists since a (1, 3, 20) design exists in order 24.

4. *Numerical Results in Orders 24 and 48.*

We use the following matrices:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & - \\ - & 0 & 1 \\ 1 & - & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} - & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix}$$

In [2] it is noted that all two variable designs exist in order 12 except (1, 7), (3, 5) and (4, 7) which are impossible. From theorem 3 of [3] we have that every design (1, k) for $k \in \{1, \ldots, 23\}$ exists in order 24.

LEMMA 8. *The following designs exist in order* 24:

1) (1,1,1,1,6,6,);	10) (2,2,5,5,8);	19) (1,1,4,4,5);
2) (1,1,1,1,2,10);	11) (1,1,1,2,4,10);	20) (1,2,5,5,8);
3) (1,1,2,2,5,8);	12) (1,1,1,3,4,9);	21) (1,2,2,8,8);
4) (1,2,6,6,9);	13) (1,3,5,6,9);	22) (1,2,2,4,13);
5) (1,2,4,5,10);	14) (1,2,3,5,13);	23) (1,2,2,5,14);
6) (1,2,5,6,10);	15) (1,2,3,4,12);	24) (1,1,2,2,9,9);
7) (1,1,1,4,6,6);	16) (1,3,4,5,9);	25) (1,2,2,2,8,9);
8) (1,4,5,6,6);	17) (1,2,2,3,16);	26) (1,1,2,4,4,8);
9) (1,2,5,5,9);	18) (1,2,2,8,11);	27) (3,3,3,3,3,3,3,3)

Proof. We note that there is a (1, 4) design in order 6 and so a (1, 1, 2, 4, 4, 8) design exists in order 24 giving 26). The design 27) is given by Plotkin in [8].

All the other designs are found by using various parts of lemma 5: part (i) for 1) and 2); part (iii) for 3), 4), ..., 11); part (iv) for 12), 13), ..., 17); part (v) for 18); part (vi) for 19); part (vii) for 20) and 21); part (viii) for 22), 23), 24) and 25). Appendix 1 contains the first rows of the circulant matrices which should be used in lemma 5 for each design.

THEOREM 9. *All two variable designs in order* 24 *exist.*

Proof. We have already noted that all designs (1, k) for $k \in \{1, \ldots, 23\}$ exist. All the other designs may be obtained by repeated use of lemma 2 on the designs in lemma 8.

THEOREM 10. *All three variable designs exist in order* 24 *with the possible exception of*

(1, 1, 21)	(2, 7, 11)	(4, 4, 15)
(1, 3, 17)	(3, 3, 11)	(5, 7, 11)
(1, 3, 19)	(3, 3, 17)	(7, 7, 7)
(1, 5, 17)	(3, 6, 11)	(7, 7, 9)
(1, 7, 15)	(3, 7, 11)	(7, 8, 8).
(1, 8, 14)	(3, 8, 12)	

Proof. All three variable designs in order 24 except those of the enunciation may be constructed either from the known four variable designs in order 12 (see [2]) by use of lemma 3 or by using lemma 2 on the designs in lemma 8.

THEOREM 11. *All two variable designs exist in order* 48.

Proof. Recalling that the existence of all three variable designs (1, i, j) where $1 + i + j \le 24$ in order 24 would give the result we consider these three variable designs.

Now we have, as yet, been unable to demonstrate the existence of the following three variable designs (1, i, j) in order 24:

(1, 1, 21)	(1, 5, 17)
(1, 3, 17)	(1, 7, 15)
(1, 3, 19)	(1, 8, 14)

Hence we use theorem 7 to decide which two variable designs in order 48 are left in doubt because these designs (1, i, j) have not yet been constructed in order 24. We then use lemma 3 with known designs in order 24 and find all the two variable designs left in doubt.

5. *Numerical Results in Order 40.*

LEMMA 12. *The following designs exist in order* 40:

1) (2, 10, 10, 13);	6) (1, 2, 2, 4, 25);	11) (1, 10, 10, 17);
2) (1, 2, 14, 23);	7) (1, 2, 2, 11, 24);	12) (1, 10, 10, 19);
3) (5, 9, 9, 15);	8) (1, 2, 12, 25);	13) (2, 4, 11, 16);
4) (1, 2, 6, 9, 20);	9) (1, 4, 9, 9, 9);	14) (1, 4, 8, 8, 16);
5) (1, 2, 2, 19);	10) (2, 8, 25);	15) (1, 8, 8, 8, 8).

Proof. All the designs are found by using various parts of lemma 5: part (i) for 1) and 2); part (iii) for 3), 4), ..., 13); and part (vii) for 14) and 15). Appendix 2 contains the first rows of the circulant matrices which should be used in lemma 5 for each design.

THEOREM 13. *All two variable designs in order* 40 *exist except possibly*

(6, 33), (7, 32), (8, 31), (9, 30) .

Proof. From [3; theorem 10] we have that all the designs (1,k) for $k \in \{1, \ldots, 39\}$ exist in order 40.

Using the known designs listed in [5] for order 20 and applying lemma 3 or using lemma 2 on the designs in lemma 12 gives the result.

For completeness we note the following result:

THEOREM 14. *All two variable designs in order* 80 *exist except possibly* (13,64), (14,65).

6. *Some Results on the* (1, k) *Conjecture.*

It has been conjectured "Let $n \equiv 0 \pmod 8$. Then there exists an orthogonal design (1, k) in order n for every $k = 1, 2, \ldots, n - 1$". The verity of this conjecture for n implies the verity of two other conjectures.

(i) Let $n \equiv 0 \pmod 8$. Then there exists a skew-symmetric weighing matrix of order n for every weight $w = 1, 2, \ldots, n - 1$;

(ii) Let $n \equiv 0 \pmod 8$. Then there exists a weighing matrix of order n for every weight $w = 1, 2, \ldots, n$.

We now prove

THEOREM 15. *Let* $n = 2^t \cdot 9$ *where* $t \geq 3$ *is a positive integer. Then there exists*

(i) *an orthogonal design* (1, k) *in order* n *for* $k = 1, 2, \ldots, n - 1$;

(ii) *a skew-symmetric weighing matrix of order* n *for every weight* $w = 1, 2, \ldots, n - 1$;

(iii) *a weighing matrix of order* n *for every weight* $w = 1, 2, \ldots, n$.

Proof. We observe that in [4; lemma 18] the existence of orthogonal designs (1, k) in order 72 is established for $k \in \{x : x \neq 31, 46, 47, 56, 60, 61, 62, 63, 68, 0 \leq x \leq 71\}$.

First we note that the four circulant matrices with first rows $x_1 {-x_3} x_3 x_3 {-x_3} x_3 {-x_3} {-x_3} x_3$, $x_2 x_3 x_3 {-x_3} x_3 {-x_3} x_3 {-x_3} {-x_3}$, $x_3 x_3 {-x_3} x_3 x_3 x_3 x_3 {-x_3} x_3$, $x_3 x_3 {-x_3} {-x_3} {-x_3} {-x_3} {-x_3} {-x_3} x_3$, may be used in the Goethals-Seidel array to form the design (1, 1, 34) in order 36 and this then gives, using lemma 3, the design (1, 1, 2, 68) in order 72.

Now the required designs may be obtained by replacing the variables of the indicated four variable designs in order 24 (found in lemma 8) by the variable matrices shown (see section 4 for definition of I, J, K, B, R, S):

For (1, 31) use x_1I, x_2S, x_2K, x_2BR in the (1, 1, 7, 7) design.

For (1, 46) use x_1I, x_2I, x_2K, x_2S in the (1, 1, 9, 9) design.

For (1, 47) use x_1I, x_2I, x_2K, x_2S in the (1, 2, 9, 9) design.

For (1, 56) use x_1I, x_2I, x_2K, x_2S in the (1, 1, 11, 11) design.

For (1, 60) use $x_1I + x_2B$, x_2J, x_2KR, x_2SR in the (1, 3, 5, 13) design.

For (1, 61) use $x_1I + x_2B$, x_2R, x_2J, x_2SR in the (1, 2, 5, 14) design.

For (1, 62) use $x_1I + x_2B$, x_2KR, x_2J, x_2SR in the (1, 3, 4, 14) design.

For (1, 63) use x_1I, x_2I, x_2J, x_2S in the (1, 3, 5, 15) design.

Hence all designs (1, k) for $k = 1, 2, \ldots, 71$ exist in order 72. Now by the corollary to proposition 1 of [3] we have part (i) of the theorem.

Clearly the rows and columns of an orthogonal design (1, k) may be rearranged until it is in the form $x_1I + x_2W$ where $W^T = -W$ and $WW^T = kI$. But W now satisfies part (ii) and W + I part (iii) of the enunciation and so we have the theorem.

The result on the existence of a design (1, 1, 34) in order 36 allows part (iv) of the summary in [4] to read

LEMMA 16. *There are orthogonal designs* (1, k) *for* $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, 17, 18, 20, 21, 22, 24, \ldots, 27, 29, 32, 33, 34\}$ *for every order* 4n *where* n (odd) ≥ 9.

APPENDIX 1.

Design	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
(1, 1, 1, 1, 6, 6)	$x_{1}00$	$x_{2}00$	$x_{3}00$	$x_{4}00$	$x_5x_6x_6$	$-x_6x_5x_5$	$-x_5x_5x_5$	$-x_6x_6x_6$
(1, 1, 1, 1, 2, 10)	$x_{1}00$	$x_{2}00$	$x_{3}00$	$x_{4}00$	$x_5x_6x_6$	$-x_5x_6x_6$	$-x_6x_6x_6$	$-x_6x_6x_6$
(1, 1, 2, 2, 5, 8)	$x_3x_6-x_6$	$x_3x_6-x_6$	$x_{1}00$	$x_{2}00$	$x_4x_6x_6$	$-x_4x_6x_6$	$0x_5x_5$	$-x_5x_5x_5$
(1, 2, 6, 6, 9)	$x_5x_3-x_3$	$x_1x_2x_2$	$-x_2x_1x_1$	$-x_1x_1x_1$	$-x_2x_2x_2$	$x_4x_3x_3$	$-x_4x_3x_3$	$-x_3x_3x_3$
(1, 2, 4, 5, 10)	$x_1-x_3x_3$	$x_2x_5x_5$	$-x_2x_5x_5$	$-x_5x_5x_5$	$-x_5x_5x_5$	$0x_3x_3$	$0x_4x_4$	$-x_4x_4x_4$
(1, 2, 5, 6, 10)	$x_1-x_3x_3$	$x_2x_5x_5$	$-x_2x_5x_5$	$-x_5x_5x_5$	$-x_5x_5x_5$	$x_4x_3x_3$	$-x_3x_4x_4$	$-x_4x_4x_4$
(1, 1, 1, 4, 6, 6)	$x_1-x_4x_4$	$x_{2}00$	$x_{3}00$	$0x_4x_4$	$x_5x_6x_6$	$-x_6x_5x_5$	$-x_5x_5x_5$	$-x_6x_6x_6$
(1, 4, 5, 6, 6)	$x_1-x_2x_2$	$0x_2x_2$	$0x_3x_3$	$-x_3x_3x_3$	$x_4x_5x_5$	$-x_5x_4x_4$	$-x_4x_4x_4$	$-x_5x_5x_5$
(1, 2, 5, 5, 9)	$x_1-x_5x_5$	$x_2x_5x_5$	$-x_2x_5x_5$	$-x_5x_5x_5$	$0x_3x_3$	$-x_3x_3x_3$	$0x_4x_4$	$-x_4x_4x_4$
(2, 2, 5, 5, 8)	$x_1-x_5x_5$	$x_1-x_5x_5$	$x_2x_5x_5$	$-x_2x_5x_5$	$0x_3x_3$	$-x_3x_3x_3$	$0x_4x_4$	$-x_4x_4x_4$
(1, 1, 1, 2, 4, 10)	$x_1-x_5x_5$	$0x_5x_5$	$x_{2}00$	x_3[illegible]	$x_4x_6x_6$	$-x_4x_6x_6$	$-x_6x_6x_6$	$-x_6x_6x_6$
(1, 1, 1, 3, 4, 9)	$x_1-x_5x_5$	$x_4-x_6x_6$	$x_4-x_6x_6$	$x_4-x_6x_6$	$x_6x_6x_6$	$0x_5x_5$	$x_{2}00$	$x_{3}00$
(1, 3, 5, 6, 9)	$x_1-x_4x_4$	$x_2-x_5x_5$	$x_2-x_5x_5$	$x_2-x_5x_5$	$x_5x_5x_5$	$x_3x_4x_4$	$-x_4x_3x_3$	$-x_3x_3x_3$
(1, 2, 3, 5, 13)	$x_1-x_4x_4$	$x_3-x_5x_5$	$x_3-x_5x_5$	$x_3-x_5x_5$	$x_2x_5x_5$	$-x_2x_5x_5$	$x_4x_5x_5$	$-x_5x_4x_4$
(1, 2, 3, 4, 12)	$x_1-x_4x_4$	$x_3-x_5x_5$	$x_3-x_5x_5$	$x_3-x_5x_5$	$x_2x_5x_5$	$-x_2x_5x_5$	$0x_5x_5$	$0x_4x_4$
(1, 2, 3, 4, 12)	$x_1-x_4x_4$	$x_3-x_5x_5$	$x_3-x_5x_5$	$x_3-x_5x_5$	$x_2x_5x_5$	$-x_2x_5x_5$	$0x_5x_5$	$0x_4x_4$
(1, 3, 4, 5, 9)	$x_1-x_4x_4$	$x_2-x_5x_5$	$x_2-x_5x_5$	$x_2-x_5x_5$	$x_5x_5x_5$	$0x_4x_4$	$0x_3x_3$	$-x_3x_3x_3$
(1, 2, 2, 3, 16)	$x_1-x_5x_5$	$x_4-x_5x_5$	$x_4-x_5x_5$	$x_4-x_5x_5$	$x_2x_5x_5$	$-x_2x_5x_5$	$x_3x_5x_5$	$-x_3x_5x_5$
(1, 2, 2, 8, 11)	$x_1-x_5x_5$	$x_5x_5x_5$	$x_2-x_4x_4$	$x_2-x_4x_4$	$x_3x_4x_4$	$-x_3x_4x_4$	$-x_5x_5x_5$	$-x_5x_5x_5$
(1, 1, 4, 4, 5)	$0-x_3x_3$	$0-x_4x_4$	$x_{1}00$	$x_{2}00$	$0x_3x_3$	$0x_4x_4$	$0x_5x_5$	$-x_5x_5x_5$
(1, 2, 5, 5, 8)	$x_1-x_4x_4$	$0-x_3x_3$	$0-x_5x_5$	$0-x_5x_5$	$x_3x_4x_4$	$-x_4x_3x_3$	$x_2x_5x_5$	$-x_2x_5x_5$
(1, 2, 2, 8, 8)	$x_1-x_4x_4$	$0-x_4x_4$	$0-x_5x_5$	$0-x_5x_5$	$x_2x_4x_4$	$-x_2x_4x_4$	$x_3x_5x_5$	$-x_3x_5x_5$
(1, 2, 2, 4, 13)	$x_2-x_5x_5$	$x_1-x_4x_4$	$x_3-x_5x_5$	$0x_4x_4$	$x_2-x_5x_5$	$x_5x_5x_5$	$x_3-x_5x_5$	$0x_5x_5$
(1, 2, 2, 5, 14)	$x_2-x_5x_5$	$x_1-x_4x_4$	$x_3-x_5x_5$	$x_5x_4x_4$	$x_2-x_5x_5$	$x_5x_5x_5$	$x_3-x_5x_5$	$-x_4x_5x_5$
(1, 1, 2, 2, 9, 9)	$x_2-x_5x_5$	$x_1-x_6x_6$	$x_3-x_5x_5$	$x_4-x_6x_6$	$x_5x_5x_5$	$x_6x_6x_6$	$x_3-x_5x_5$	$x_4-x_6x_6$
(1, 2, 2, 2, 8, 9)	$x_2-x_5x_5$	$-x_3x_5x_5$	$x_1-x_6x_6$	$x_4-x_6x_6$	$x_2-x_5x_5$	$x_3x_5x_5$	$x_6x_6x_6$	$x_4-x_6x_6$.

APPENDIX 2.

Design	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
(2, 10, 10, 13)	$x_1x_4-x_4-x_4x_4$	$x_1-x_4x_4x_4-x_4$	$x_4 0x_4x_4 0$	$00x_4x_4 0$	$x_3x_2-x_2-x_2x_2$	$-x_2x_3-x_3-x_3x_3$	$-x_2x_2x_2x_2x_2$	$-x_3x_3x_3x_3x_3$
(1, 2, 14, 23)	$x_1x_4x_4-x_4-x_4$	$-x_2x_3-x_3-x_3x_3$	$x_2x_3-x_3-x_3x_3$	$-x_4x_4-x_4-x_4x_4$	$-x_4x_4x_4x_4x_4$	$-x_4-x_4x_3x_3-x_4$	$x_3x_4x_3x_3x_4$	$-x_3x_4-x_4-x_4x_4$
(5, 9, 9, 15)	$x_1x_4x_1-x_1x_4$	$x_4-x_4x_1x_1-x_4$	$-x_4x_4x_4x_4x_4$	$x_4x_4-x_4-x_4x_4$	$0x_2-x_2-x_2x_2$	$-x_2x_2x_2x_2x_2$	$0x_3-x_3-x_3x_3$	$-x_3x_3x_3x_3x_3$
(1, 2, 6, 9, 20)	$x_1-x_5x_3-x_3x_5$	$0-x_5x_3x_3-x_5$	$x_3x_5-x_5-x_5x_5$	$x_3x_5x_5x_5x_5$	$x_2x_5-x_5-x_5x_5$	$-x_2x_5-x_5-x_5x_5$	$0x_4-x_4-x_4x_4$	$-x_4x_4x_4x_4x_4$
(1, 2, 16, 19)	$x_1x_4x_4-x_4-x_4$	$-x_4x_4x_4x_4x_4$	$-x_4x_4x_4x_4x_4$	$x_4-x_4x_4x_4-x_4$	$x_2x_3-x_3-x_3x_3$	$x_2-x_3x_3x_3-x_3$	$0x_3-x_3-x_3x_3$	$0x_3x_3x_3$
(1, 2, 2, 4, 25)	$x_1 0x_4-x_4 0$	$00x_4x_4 0$	$x_2x_5-x_5-x_5x_5$	$-x_2x_5-x_5-x_5x_5$	$x_3x_5-x_5-x_5x_5$	$-x_3x_5-x_5-x_5x_5$	$0x_5x_5x_5$	$-x_5x_5x_5x_5x_5$
(1, 2, 2, 11, 24)	$x_1x_4x_4-x_4-x_4$	$x_2x_5-x_5-x_5x_5$	$-x_2x_5-x_5-x_5x_5$	$x_3x_5-x_5-x_5x_5$	$-x_3x_5-x_5-x_5x_5$	$-x_4x_5x_5x_5x_5$	$x_4-x_4x_5x_5-x_4$	$x_4x_5x_4x_4x_5$
(1, 2, 12, 25)	$x_1x_4x_4-x_4-x_4$	$x_3x_3-x_3-x_3x_3$	$-x_4x_4-x_4-x_4x_4$	$-x_3x_4x_4x_4x_4$	$x_3-x_3x_4x_4-x_3$	$x_3x_4x_3x_3x_4$	$x_2x_4-x_4-x_4x_4$	$-x_2x_4-x_4-x_4x_4$
(1, 4, 4, 9, 9, 9)	$x_1 0x_2-x_2 0$	$00x_2x_2 0$	$0x_3-x_3-x_3x_3$	$-x_3x_3x_3x_3x_3$	$0x_4-x_4-x_4x_4$	$-x_4x_4x_4x_4x_4$	$0x_5-x_5-x_5x_5$	$-x_5x_5x_5x_5x_5$
(2, 8, 25)	$x_2x_3x_2x_2-x_3$	$x_2x_3x_2x_2-x_3$	$-x_2x_3x_2x_2x_3$	$-x_2-x_3x_2x_2-x_3$	$x_2-x_2 00-x_2$	$0x_2 00x_2$	$x_1x_2-x_2-x_2x_2$	$-x_1x_2-x_2-x_2x_2$
(1, 10, 10, 17)	$x_1x_4x_4-x_4-x_4$	$x_4-x_4x_4x_4-x_4$	$0x_4x_4x_4x_4$	$0x_4-x_4-x_4x_4$	$x_2x_3-x_3-x_3x_3$	$-x_3x_2-x_2-x_2x_2$	$-x_2x_2x_2x_2x_2$	$-x_3x_3x_3x_3x_3$
(1, 10, 10,19)	$x_1x_4x_4-x_4-x_4$	$-x_4x_4x_4x_4x_4$	$-x_4x_4x_4x_4x_4$	$x_4-x_4x_4x_4-x_4$	$x_2x_3-x_3-x_3x_3$	$-x_3x_2-x_2-x_2x_2$	$-x_2x_2x_2x_2x_2$	$-x_3x_3x_3x_3x_3$
(2, 4, 11, 16)	$x_3-x_3x_2-x_2-x_3$	$00x_2x_2 0$	$x_3 0x_3x_3 0$	$x_3x_3-x_3-x_3x_3$	$x_1x_4-x_4-x_4x_4$	$-x_1x_4-x_4-x_4x_4$	$0x_4-x_4-x_4x_4$	$0x_4x_4x_4x_4$
(1, 4, 8, 8, 16)	$x_1x_3x_4-x_4-x_3$	$0x_3-x_4x_4-x_3$	$0x_3x_5-x_5-x_3$	$0x_3-x_5x_5-x_3$	$-x_2x_3x_4x_4x_3$	$-x_2x_3-x_4-x_4x_3$	$x_2x_3x_5x_5x_3$	$x_2x_3-x_5-x_5x_3$
(1, 8, 8, 8, 8)	$x_1x_2x_3-x_3-x_2$	$0x_2-x_3x_3-x_2$	$0x_4x_5-x_5-x_4$	$0x_4-x_5x_5-x_4$	$0x_2x_3x_3x_2$	$0x_2-x_3-x_3x_2$	$0x_4x_5x_5x_4$	$0x_4-x_5-x_5x_4$

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Institute of Advanced Studies
Australian National University
Canberra, A.C.T.

Received December 17, 1974; revised April 23, 1975.