

Orthogonal Designs

ANTHONY V. GERAMITA,[†] JOAN MURPHY GERAMITA and
JENNIFER SEBERRY WALLIS

Department of Mathematics, Queen's University at Kingston

(Received May 14, 1974)

INTRODUCTION

DEFINITION An orthogonal design of order n and type (s_1, \dots, s_l) ($s_i > 0$) on the commuting variables x_1, \dots, x_l is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \dots, \pm x_l\}$ such that

$$AA^t = \left(\sum_{i=1}^l s_i x_i^2 \right) I_n. \quad (*)$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$.

Remark A may be considered as a matrix with entries in the field of quotients of the integral domain $Z[x_1, \dots, x_l]$. Thus we obtain

$$A^t A = \left(\sum_{i=1}^l s_i x_i^2 \right) I_n$$

and so our alternative description above applies equally well to the columns of A .

Orthogonal designs of special type have been extensively studied, and it is the existence of these special types that has motivated our study of the general problem of the existence of orthogonal designs.

This paper is organized in the following way. In the first section we give some easily obtainable necessary conditions for the existence of orthogonal designs of various order and type. In Section 2 we briefly survey the examples of such designs that we have found in the literature. In the third section we describe several methods for constructing orthogonal designs. In the fourth

[†] The research of this author was supported in part by the National Research Council of Canada under grant 8488.

section we obtain some sharper necessary conditions for the existence of orthogonal designs. In the fifth section we apply the results obtained to calculate designs of small order and also improve some of the results of [14]. We conclude this paper with a collection of open questions and conjectures.

1. SOME EXISTENCE THEOREMS

In this section we shall describe some necessary conditions for the existence of an orthogonal design. The most fundamental restriction concerns the maximum number of variables that can occur in an orthogonal design of order n . It is an immediate consequence of a theorem of Radon [8].

DEFINITION Let $n = 2^a \cdot b$, b odd, and write $a = 4c + d$ where $0 \leq d < 4$. Radon's function is the arithmetic function

$$\rho(n) = 8c + 2^d.$$

Note $\rho(n) = n$ iff $n = 1, 2, 4, 8$. $\rho(16) = 9$, and in general $\rho(2^a \cdot b) = \rho(2^a)$ if b is odd.

THEOREM 1 Let A be an orthogonal design of order n and type (s_1, \dots, s_l) on the variables x_1, \dots, x_l . Then $l \leq \rho(n)$.

Proof Write $A = A_1x_1 + \dots + A_lx_l$ where A_i are $(0, 1, -1)$ matrices of order n . Now

$$AA^t = \left(\sum_{i=1}^l s_i x_i^2 \right) I_n \Leftrightarrow A_i A_i^t = s_i I_n, \quad i = 1, 2, \dots, l,$$

and $A_i A_j^t + A_j A_i^t = 0$ for $i \neq j$. If we replace A_i by the real matrix $(1/\sqrt{s_i})A_i = B_i$, then the B_i are real orthogonal matrices satisfying $B_i B_j^t + B_j B_i^t = 0$ for $i \neq j$, and Radon has shown that there do not exist more than $\rho(n)$ such real matrices. This completes the proof.

So, with the initial limitations of this theorem in mind we propose as a problem finding all orthogonal designs of order n . We note that this theorem gives only a weak necessary condition and that conditions for sufficiency are far deeper.

To consider further the question of necessity we introduce weighing matrices.

DEFINITION A weighing matrix of weight k and order n , is a $(0, 1, -1)$ matrix, A , of order n satisfying

$$AA^t = kI_n.$$

We shall denote such a matrix by $W(n, k)$. Thus, our definition of an orthogonal design could be reformulated in terms of the existence of certain weighing matrices satisfying the equation $XY^t + YX^t = 0$ in pairs.

If n is odd then $\rho(n) = 1$ and Theorem 1 shows we can only find designs on one variable. However, as observed above, finding an orthogonal design of order n and type (s) is equivalent to finding a $W(n, s)$.

If n is odd the type of a design is severely restricted as we have:

PROPOSITION 2 *If n is odd and a $W(n, s)$ exists, then s is a square.*

Proof If A is a $W(n, s)$ i.e. $AA^t = sI_n$, then the proposition follows immediately from the observation that $\det A$ must be an integer.

We note that a $W(5, 4)$ does *not* exist but a $W(13, 9)$ does. (We shall exhibit the $W(13, 9)$ later.) Thus, Proposition 2 is a weak necessary condition. In fact, the existence question for weighing matrices of odd order has some very interesting consequences. We shall discuss this further in Section 4.

If $n \equiv 2 \pmod{4}$ we have $\rho(n) = 2$ and so we need only consider orthogonal designs on ≤ 2 variables. The following proposition is relevant.

PROPOSITION 3 [9] *If $n \equiv 2 \pmod{4}$ and A is a $W(n, k)$ then k is a sum of two squares.*

COROLLARY *If $n \equiv 2 \pmod{4}$ and A is an orthogonal design of order n and type (s_1, s_2) then s_1, s_2 and $s_1 + s_2$ must each be the sum of two squares.*

Proof If A is an orthogonal design of order n and type (s_1, s_2) on the variables x_1, x_2 then we have noted that this implies the existence of a $W(n, s_1)$ and $W(n, s_2)$, thus by Proposition 3, s_1 and s_2 are each a sum of two squares. Now, if we set $x_1 = x_2 = 1$ in A then A becomes a $W(n, s_1 + s_2)$, and so again, by Proposition 3, $s_1 + s_2$ is a sum of two squares.

Finally if $n \equiv 0 \pmod{4}$ then $\rho(n) \geq 4$. It is in these orders that questions of necessary conditions for the existence of orthogonal designs become more delicate. If $n = 2^t$ for any integer t , we have been unable to find any condition (apart from Theorem 1) which precludes the existence of an orthogonal design of order n . If $n = 4t$, where t is odd, we have had some success in finding some necessary conditions which assert the nonexistence of designs of order n (see Section 4). However, if $n = 8t$, t odd, we have been unable to eliminate any l -tuple (s_1, \dots, s_l) , (where $l \leq 8$ and $\sum_{i=1}^l s_i \leq n$) as the type of an orthogonal design of order n .

2. KNOWN CLASSES OF ORTHOGONAL DESIGNS

Baumert-Hall arrays

Let $n = 4t$, t odd, then $\rho(n) = 4$. The Baumert-Hall arrays are orthogonal designs of order n and type (t, t, t, t) . These exist for a large number of odd t and it has been conjectured that they exist for all odd t .

For a discussion of these arrays see [16; Part 4, Chapter VII].

Designs of type $(1, 1, \dots, 1)$

Let n be any integer. In [2] it was shown that there is an orthogonal design of order n and type $(1, 1, \dots, 1)$ on the variables $x_1, \dots, x_{\rho(n)}$.

Plotkin's array

Let $n = 8t$, t odd, then $\rho(n) = 8$. A *Plotkin array* is an orthogonal design of order n and type (t, t, t, t, t, t, t, t) . If $t = 1$ such an array is a classical one derived from the multiplication of the Cayley numbers. Plotkin has exhibited such an array for $t = 3$ [7]. These appear to be the only such arrays (for t odd) in the literature. It would be interesting to have more.

In his paper [7] Plotkin also proves the following about orthogonal designs: in our terminology, he shows that if n is the order of an Hadamard matrix then

- a) there is an orthogonal design of type $\left(\frac{n}{2}, \frac{n}{2}\right)$ on the variables x_1, x_2 .
- b) There is an orthogonal design of order $2n$ and type $\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}\right)$, and
- c) an orthogonal design of order $4n$ and type $\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}\right)$.

Plotkin makes the very strong conjecture that every Hadamard matrix of order $8n$ can be obtained from specializing some orthogonal design of order $8n$ and type (n, n, n, n, n, n, n, n) , i.e. every Hadamard matrix of order $8n$ may be obtained from an orthogonal design of the order and type above by setting the variables all equal to 1.

Designs on one variable

As we have noted these are weighing matrices. It has been conjectured [14] that weighing matrices $W(n, k)$ exist for every $k \leq n$ when $n \equiv 0 \pmod{4}$. (This includes the Hadamard conjecture: that for $n \equiv 0 \pmod{4}$ there exists a $W(n, n)$ for all integers $n \geq 4$.) This conjecture has been verified for $n = 2^t$ [4] and for $n \in \{12, 20, 24, 28, 32, 40\}$ [14].

3. PROLIFERATING ORTHOGONAL DESIGNS

LEMMA 4 *If A is an orthogonal design of order n and type (s_1, \dots, s_l) on the variables x_1, \dots, x_l then there is an orthogonal design of order n and type $(s_1, \dots, s_i + s_j, \dots, s_l)$ on the $l - 1$ variables $x_1, \dots, \hat{x}_j, \dots, x_l$.*

COROLLARY 1 *For any integer n , an orthogonal design of order n and type (s_1, \dots, s_l) exists if $\sum_{i=1}^l s_i \leq \rho(n)$ and so if $n = 2, 4, 8$ orthogonal designs of order n and any type exist.*

Proof The proof follows by using the designs of type $(1, 1, \dots, 1)$ in the previous section.

COROLLARY 2 *If $n = 4t$ and there exists a Baumert-Hall array of order $4t$, then there exist orthogonal designs of order n and type*

$$(t), (2t), (3t), (4t), (t, t, 2t), (t, 2t), (t, 3t) \\ (t, t, t), (t, t), (2t, 2t)$$

Proof These designs result from setting the variables in the Baumert-Hall array equal to each other or to 0.

LEMMA 5 *If A is an orthogonal design of order n and type (s_1, \dots, s_l) on x_1, \dots, x_l , then there exists an orthogonal design of order mn and of type (s_1, \dots, s_l) on x_1, \dots, x_l for any integer $m \geq 1$.*

PROPOSITION 6 *Let A_1, \dots, A_s be matrices of order m satisfying*

i) $A_i A_i^t = a_i I_m \quad i = 1, \dots, s,$

ii) $A_i A_j^t + A_j A_i^t = 0 \quad (i \neq j),$

and let N_1, \dots, N_l be matrices of order n satisfying

iii) $N_i N_i^t = b_i I_n \quad i = 1, \dots, l,$

iv) $N_i N_j^t = N_j N_i^t \quad i \neq j.$

Then if $B_1 = A_1 \otimes N_{k_1}, \dots, B_s = A_s \otimes N_{k_s} (k_i \in \{1, \dots, l\}, k_i$ not necessarily distinct), we have

a) $B_i B_i^t = a_i b_{k_i} I_{nm},$

b) $B_i B_j^t + B_j B_i^t = 0 \quad i \neq j.$

Proof By straightforward verification.

COROLLARY *If A is an orthogonal design of order n and type (s_1, \dots, s_l) then there exists an orthogonal design of order $2n$ and type $(\varepsilon_1 s_1, \varepsilon_2 s_2, \dots, \varepsilon_l s_l), \varepsilon_i = 1$ or 2 .*

Proof Let $N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $N_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and let A_1, \dots, A_l be the $n \times n$ matrices given by the existence of A . We now use these matrices in the Proposition above.

PROPOSITION 7 *Let F_1, F_2, \dots, F_r be weighing matrices of order n and weights $s_1, s_2, \dots, s_r (s_k \geq 0, 1 \leq k \leq r)$ where*

a) F_1 is skew-symmetric and F_i is symmetric, for all $i > 1$.

b) $F_i F_j^t = F_j F_i^t$ for all $1 \leq i, j \leq r$.

Let m be any integer for which $\rho(m) \geq r$. Then there is an orthogonal design of order mn and type $(1, s_1, \dots, s_r)$ on the $1 + r$ variables x_0, x_1, \dots, x_r .

Proof Let D be an orthogonal design of order m and type $(1, 1, \dots, 1)$ on the variables $z_1, z_2, \dots, z_{\rho(m)}$ as described in Section 2. Set $z_1 = x_0 I + x_1 F_1$ and $z_j = x_j F_j (1 \leq j \leq r)$ and set all the remaining variables (if any)

equal to the zero matrix. The resulting matrix, of order mn , is the required design since $z_i z_j^t = z_j z_i^t$ for $1 \leq i, j \leq \rho(m)$.

Remarks Note that F_2, \dots, F_r may all be chosen to be the same matrix. If $r \geq 4$ the matrices $x_0 I + x_1 F_1, x_2 F_2, x_3 F_3, x_4 F_4$ may be used in any Baumert-Hall array; while if $r \geq 8$, these matrices, along with $x_5 F_5, x_6 F_6, x_7 F_7, x_8 F_8$ may be used in any Plotkin array.

DEFINITION *The $l \times l$ matrices W and M are called amicable Hadamard matrices if*

- i) $W = I + S$ where $S^t = -S$ and $M = M^t$ are both Hadamard matrices i.e. weighing matrices of weight l , and
- ii) $WM^t = MW^t$.

These pairs of matrices exist for many orders (see [16; Part 4, Chapter II] for a thorough discussion and also [15] for more recent results).

COROLLARY 1 *Let W and M (as above) be amicable Hadamard matrices of order l and let n be any integer. Then there is an orthogonal design of order nl of type $(1, l-1, l, l, \dots, l)$ on the variables $x_0, x_1, x_2, \dots, x_{\rho(n)}$.*

COROLLARY 2 *If there is a Baumert-Hall array of order $4n$ and amicable Hadamard matrices of order l then there is an orthogonal design of order $4nl$ and of type $(n, n(l-1), nl, nl, nl)$.*

COROLLARY 3 *Let A_1, \dots, A_s be an $n \times n$ $(0, 1, -1)$ matrices as in Proposition 6. Let $W = I_l + S$ and M be amicable Hadamard matrices of order l . Then*

$B_1 = A_1 \otimes I_l, B_2 = A_1 \otimes S, B_3 = A_2 \otimes M, \dots, B_{s+1} = A_s \otimes M$ are matrices of order nl which give an orthogonal design of type $(a_1, a_1(l-1), a_2 l, \dots, a_s l)$ on the variables x_1, \dots, x_s, x_{s+1} .

COROLLARY 4 *Let W, M be amicable Hadamard matrices of order l . If there is a Plotkin array of order $8n$ then there is an orthogonal design of order $8nl$ and type $(n, n(l-1), nl, \dots, nl)$ on the variables $x_1, x_2, x_3, \dots, x_9$.*

The following results are well known but we do not know their first appearances.

PROPOSITION 8 *Suppose there exist two circulant matrices A, B of order n satisfying $AA^t + BB^t = fI_n$. Then*

$$\begin{bmatrix} A & B \\ -B^t & A^t \end{bmatrix} \text{ is a } W(2n, f)$$

when A, B are $(0, 1, -1)$ matrices and an orthogonal design of order $2n$ and type (s_1, s_2, \dots, s_l) on x_1, \dots, x_l when $f = \sum_{i=1}^l s_i x_i^2$.

Proof Straightforward verification.

PROPOSITION 9 Suppose there exist two circulant matrices A, B of order n satisfying

$$AA^t + BB^t = fI_n.$$

Further suppose that R is the back diagonal matrix. Then

$$G = \begin{bmatrix} A & BR \\ -BR & A \end{bmatrix} \text{ is a } W(2n, f)$$

when A, B are $(0, 1, -1)$ matrices and an orthogonal design of order $2n$ and type (s_1, s_2, \dots, s_l) on x_1, \dots, x_l when $f = \sum_{i=1}^l s_i x_i^2$.

Proof A straightforward verification.

Remark We note here that these propositions remain true if A and B are type 1 matrices and R is an appropriately chosen matrix. (See [16; Part 4, Chapter I] for a discussion of these matrices which generalize the notion of a circulant matrix.) A similar remark holds for Constructions 10–16.

DEFINITION We say that the weighing matrix $W = W(2n, k)$ is constructed from two circulant matrices M, N of order n , if

$$W = \begin{bmatrix} M & N \\ -N^t & M^t \end{bmatrix}.$$

CONSTRUCTION 10 Suppose there is an $W(n, k)$ constructed from two circulant matrices M, N of order $n/2$ with the property that $M * N = 0$ ($*$ denotes Hadamard product). Then $A = x_1 M + x_2 N, B = x_1 N - x_2 M$ may be used in Proposition 9 to obtain an orthogonal design of order n and type (k, k) on x_1, x_2 .

Proof Is a straightforward verification. One need only observe that since M, N are circulant MR, NR are back circulant; and if X is circulant and Y is back circulant then $XY^t = YX^t$.

CONSTRUCTION 11 Suppose there exist $W(n, k_i), i = 1, 2$, constructed from circulant matrices $M_i, N_i, i = 1, 2$ of order $n/2$ where $M_1 * M_2 = N_1 * N_2 = 0$ and $M_1 M_2^t + M_2 M_1^t = N_1 N_2^t + N_2 N_1^t = 0$, then

$$A = x_1 M_1 + x_2 M_2, \quad B = x_1 N_1 + x_2 N_2$$

may be used in Proposition 9 to give an orthogonal design of order n and type (k_1, k_2) on the variables x_1, x_2 .

LEMMA 12 If

$$N = \begin{bmatrix} x_1 A_1 + x_2 A_2 & (x_1 A_3 + x_2 A_4)R \\ -(x_1 A_3 + x_2 A_4)R & (x_1 A_1 + x_2 A_2) \end{bmatrix}$$

is an orthogonal design of type (s_1, s_2) where $A_i, i = 1, 2, 3, 4$, are $(0, 1, -1)$ matrices which are circulant and $A_i J = y_i J, (J \text{ is the matrix of ones}), i = 1, \dots, 4$, Then

$$\sum_{i=1}^4 y_i^2 = s_1 + s_2.$$

Proof By definition,

$$(x_1A_1 + x_2A_2)(x_1A_1^t + x_2A_2^t) + (x_1A_3 + x_2A_4)(x_1A_3 + x_2A_4)^t = (s_1x_1^2 + s_2x_2^2)I.$$

So,

$$x_1^2(A_1A_1^t + A_3A_3^t) + x_2^2(A_2A_2^t + A_4A_4^t) = (s_1x_1^2 + s_2x_2^2)I$$

and setting $x_1 = x_2 = 1$ we have

$$\sum_{i=1}^4 A_iA_i^t = (s_1 + s_2)I.$$

Post multiplying by J gives $(y_1^2 + \dots + y_4^2)J = (s_1 + s_2)J$.

Remark This lemma shows that for order $N = 2n$, n odd, this method can only construct orthogonal designs of order $2n$ and type $(1, s)$ when s is a square. For under those circumstances we may assume $A_1 = I$, $A_2 = -A_2^t$ is circulant, and $A_3 = 0$. Hence $y_1 = 1$, $y_2 = 0$ (since n is odd), and $y_3 = 0$. Thus $1 + y_4^2 = 1 + s$. So, for example, this method could not be used to construct an orthogonal design of type $(1, 8)$ in order $2n$, n odd. In fact we had shown that type $(1, 8)$ in order 10 is impossible by any method, although this was not precluded by our original considerations.

This suggests the following conjecture:

If $n \equiv 2 \pmod{4}$ there does *not* exist a $(0, 1, -1)$ matrix X of order n such that $X = -X^t$ and $XX^t = kI_n$ unless (possibly) if k is a square. We verify this conjecture in Section 4.

THEOREM 13 (Goethals and Seidel, see [16, p. 355]). *Suppose there exist four circulant matrices A, B, C, D of order n satisfying*

$$AA^t + BB^t + CC^t + DD^t = fI_n.$$

Let R be the back-diagonal matrix. Then

$$GS = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^tR & -C^tR \\ -CR & -D^tR & A & B^tR \\ -DR & C^tR & -B^tR & A \end{bmatrix}$$

is a $W(4n, f)$ when A, B, C, D are $(0, 1, -1)$ matrices, and an orthogonal design of order $4n$ and type (s_1, s_2, \dots, s_l) on x_1, x_2, \dots, x_l when

$$f = \sum_{j=1}^l s_jx_j^2.$$

Further GS is skew or skew-type if A is skew or skew-type.

We shall, of course, try to use Theorem 13 to construct orthogonal designs, but we first remark on the limitations of this method.

LEMMA 14 *Let A_i , $i = 1, 2, 3, 4$ be circulant matrices of order n where*

$$\sum_{i=1}^4 A_iA_i^t = \left(\sum_{j=1}^l s_jx_j^2 \right) I_n.$$

Suppose $A_i = \sum_{j=1}^l x_j A_{ij}$ and that $A_{ij} J = y_{ij} J$. Then

$$\sum_{i=1}^4 \left(\sum_{j=1}^l y_{ij}^2 \right) = \sum_{j=1}^l s_j.$$

Proof See Lemma 12 above.

Remark If we have four circulants A_1, A_2, A_3, A_4 such that $\sum_{i=1}^4 A_i A_i^t = (x_1^2 + s x_2^2) I_n$, n odd, then s must be the sum of three squares. For we may assume $A_1 = x_1 I + x_2 A_{12}$, $A_2 = x_2 A_{22}$, $A_3 = x_2 A_{32}$ and $A_4 = x_2 A_{42}$. Then, by Lemma 14 we have $1 + y_{12}^2 + y_{22}^2 + y_{32}^2 + y_{42}^2 = 1 + s$. But $A_{12} = -A_{12}^t$ and the order of A_{12} is n (odd). So $y_{12} = 0$, and consequently s is a sum of three squares. We note that no s of the form $4^a(8b + 7)$ can be written as the sum of three squares [6].

This remark led us to conjecture that if $n = 4t$, t odd and $X = -X^t$ is a $W(n, k)$ then k must be the sum of three squares. We verify this conjecture also in Section 4.

CONSTRUCTION 15 Suppose there exist $W(n, k_i)$, $i = 1, 2$ constructed from circulant matrices M_i, N_i , $i = 1, 2$ of order $n/2$. Then there exists an orthogonal design of order $2n$ and type (k_1, k_2) .

Proof Set $A = x_1 M_1, B = x_1 N_1, C = x_2 M_2, D = x_2 N_2$ in Theorem 13.

CONSTRUCTION 16 Suppose there exist orthogonal designs X_1, X_2 of order $2n$ and type $(s_{11}, s_{12}, \dots, s_{1m_1})$ on the variables $x_{i1}, x_{i2}, \dots, x_{im_i}$, $i = 1, 2$ each of which is constructed by using Proposition 9. Then there exists an orthogonal design of order $4n$ and type $(s_{11}, s_{12}, \dots, s_{1m_1}, s_{21}, s_{22}, \dots, s_{2m_2})$ on the variables $x_{11}, x_{12}, \dots, x_{1m_1}, x_{21}, x_{22}, \dots, x_{2m_2}$.

Proof Let A_i, B_i be the matrices used in Proposition 9 to form the orthogonal design X_i . Then use A_1, B_1, A_2, B_2 in the Goethals-Seidel array to get the result.

DEFINITION An L-family of order n is a collection of $n \times n$ matrices L_1, L_2, \dots, L_k satisfying the following properties:

- i) $L_i^t = -L_i$ $1 \leq i \leq k - 1$
- ii) $L_k^t = L_k$
- iii) $L_j L_j^t = jI$ $1 \leq j \leq k$
- iv) $L_i L_k^t = L_k L_i^t$ $1 \leq i \leq k - 1$.

THEOREM 17

a) If there is an L-family of order n having k members then there is an L-family of order $2n$ having $2k$ members.

b) If $n = 2^t$ then there is an L -family of order n having n members.

Proof (b) follows from (a) and the observation that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is an L -family of order 2.

To prove (a) we let L_1, \dots, L_k be an L -family of order n and define $\bar{L}_1, \bar{L}_2, \dots, \bar{L}_k, \dots, \bar{L}_{2k}$ to be matrices of order $2n$ where

$$\begin{aligned} \bar{L}_i &= \begin{bmatrix} L_i & 0 \\ 0 & L_i \end{bmatrix} \quad \text{for } 1 \leq i \leq k-1 \\ \bar{L}_k &= \begin{bmatrix} 0 & L_k \\ -L_k & 0 \end{bmatrix} \\ \bar{L}_{k+i} &= \begin{bmatrix} L_i & L_k \\ -L_k & L_i \end{bmatrix} \quad \text{for } 1 \leq i \leq k-1 \end{aligned}$$

and

$$\bar{L}_{2k} = \begin{bmatrix} L_k & L_k \\ L_k & -L_k \end{bmatrix}.$$

It is an easy verification that $\bar{L}_1, \dots, \bar{L}_{2k}$ is an L -family of order $2n$.

CONSTRUCTION 18 Let R, S be $(0, 1, -1)$ matrices of order n satisfying $RR^t = rI_n, SS^t = sI_n, R^t = -R, S^t = -S$ and $RS^t = SR^t$. Then

$$\begin{bmatrix} x_1I + x_2R & x_3I + x_4S \\ -x_3I + x_4S & x_1I - x_2R \end{bmatrix}$$

is an orthogonal design of order $2n$ and type $(1, r, 1, s)$.

Proof Straightforward verification.

CONSTRUCTION 19 Let P, Q, R, H be $(0, 1, -1)$ matrices of order n satisfying $P^t = -P, Q^t = -Q, R^t = -R, H^t = H$ and $MN^t = NM^t$ for $M, N \in \{P, Q, R, H\}$. If $PP^t = pI_n, QQ^t = qI_n, RR^t = rI_n$ and $HH^t = hI_n$ then

$$\begin{bmatrix} x_1I_n + x_2P & x_3I_n + x_4Q & x_5I_n + x_6R & x_7H \\ -x_3I_n + x_4Q & x_1I_n - x_2P & -x_7H & x_5I_n + x_6R \\ -x_5I_n + x_6R & x_7H & x_1I_n - x_2P & -x_3I_n - x_4Q \\ -x_7H & -x_5I_n + x_6R & x_3I_n - x_4Q & x_1I_n + x_2P \end{bmatrix}$$

is an orthogonal design of order $4n$ and type $(1, p, 1, q, 1, r, h)$.

Remark We note that if there are amicable Hadamard matrices A, B of order n where $A = I_n + C, C = -C^t$, then $P = Q = R = C$ and $H = B$ may be used in Construction 19. Also, if there is an L -family of order n then we may choose $P = Q = R$ to be any member of the L -family (except the final member) and H equal to the final member and use them in Construction 19.

CONSTRUCTION 20 For $n = 2$ the following pairs of matrices may be used in Construction 19 (by choosing $P = Q = R$ to be the first member of the pair and H to be the second).

$$\left\{ \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}, \begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}, \begin{bmatrix} + & + \\ + & - \end{bmatrix} \right\}.$$

CONSTRUCTION 21 For $n = 4$, the following triplets of matrices may be used in Construction 19 (where P, Q, R are chosen from among the first two members of the triplet and H is chosen as the last member).

Let $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_4 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, then

- I. $\{A_1 \otimes A_2, (A_2 \otimes I_2) + (A_4 \otimes A_1), A_4 \otimes A_2\}$.
- II. $\{A_1 \otimes A_2, A_4 \otimes A_1, A_4 \otimes A_2\}$.
- III. $\{A_1 \otimes A_2, (A_1 \otimes I_2) + (A_4 \otimes A_1), (A_3 \oplus A_2) + (I_2 \otimes A_3) + (A_1 \otimes A_1)\}$.
- IV. $\{(I_2 \otimes A_1) + (A_1 \otimes A_3), (A_1 \otimes I_2) + (A_4 \otimes A_1), (A_3 \oplus A_2) + (I_2 \otimes A_3) + (A_1 \otimes A_1)\}$.
- V. $\{A_1 \otimes A_2, (A_1 \otimes I_2) + (A_4 \otimes A_1), (A_2 \otimes I_2) + (A_3 \otimes A_2), (I_2 \otimes A_3) + (A_1 \otimes A_1)\}$.
- VI. $\{(I_2 \otimes A_1) + (A_1 \otimes A_3), (A_1 \otimes I_2) + (A_4 \otimes A_1), (A_2 \otimes I_2) + (A_3 \otimes A_2) + (I_2 \otimes A_3) + (A_1 \otimes A_1)\}$.

(where \oplus denotes direct sum and \otimes denotes the Kronecker product).

CONSTRUCTION 22 The following pairs of orthogonal designs satisfy the equation $XY^t = YX^t$,

$$\begin{aligned} \text{I. } X &= \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}, & Y &= \begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix} \\ \text{II. } X &= \begin{bmatrix} x_1 & x_2 & x_3 & x_3 \\ -x_2 & x_1 & x_3 & -x_3 \\ x_3 & x_3 & -x_1 & -x_2 \\ x_3 & -x_3 & x_2 & -x_1 \end{bmatrix}, & Y &= \begin{bmatrix} y_1 & y_2 & y_3 & y_3 \\ y_2 & -y_1 & y_3 & -y_3 \\ -y_3 & -y_3 & y_2 & y_1 \\ -y_3 & y_3 & y_1 & -y_2 \end{bmatrix} \end{aligned}$$

COROLLARY *If there is an orthogonal design of order n and type (s_1, s_2) then*

- i) *there is an orthogonal design of order $2n$ and type (s_1, s_1, s_2, s_2)*
- ii) *there is an orthogonal design of order $4n$ and type $(s_1, s_1, 2s_1, s_2, s_2, 2s_2)$.*

4. SOME MORE NECESSARY CONDITIONS

PROPOSITION 23 *If n is odd then a necessary condition that $W(n, k)$ exist is that*

$$(n - k)^2 - (n - k) + 2 > n.$$

Proof We have already noted that k must be a square. We consider two cases:

Case 1 k even.

We may as well assume the first row of A is

$$[1 \dots 1 \quad 0 \dots 0].$$

k $n-k$

Since n is odd and k is even, $n - k$ is odd. Since every row contains an even number of nonzero entries and an even number of nonzero entries must appear under the 1's of the first row we find that there must be an odd number of zeroes (and hence at least one) under the zeroes of the first row. Thus, if there are $(n - k)(n - k - 1) + 2$ rows then at least one of the last $n - k$ columns contains more than $n - k$ zeroes. Thus, in order that $W(n, k)$ exist we must have $(n - k)(n - k - 1) + 2 > n$.

Case 2 k odd.

As in Case 1, we may assume the first row is

$$[1 \dots 1 \quad 0 \dots 0].$$

k $n-k$

Since k is odd, $n - k$ is even. But, any other row must have an even number of nonzero elements under the k ones of the first row. This leaves an odd number of nonzero elements to go in the last $n - k$ columns. Thus, there must be at least one zero in every row among the last $n - k$ columns. The argument now proceeds as in Case 1. So, we conclude that $W(n, k)$ does not exist if $n \geq (n - k)^2 - (n - k) + 2$.

Remark We conclude from this theorem that $W(5, 4)$, $W(11, 9)$, $W(19, 16)$, $W(29, 25)$, $W(36, 41)$ etc. do not exist. This does not cover $W(13, 9)$ or $W(31, 25)$, . . . , $W(111, 100)$ among others.

We are indebted to Professor David Gregory for the construction of $W(13, 9)$ which we exhibit below.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & - & - & - & + & + & + & + & + & + \\ 0 & + & + & + & 0 & 0 & 0 & - & - & - & + & + & + \\ 0 & + & + & + & - & - & - & 0 & 0 & 0 & - & - & - \\ 0 & - & - & - & - & - & - & - & - & 0 & 0 & 0 & 0 \\ + & 0 & + & - & + & - & 0 & + & - & 0 & + & - & 0 \\ + & 0 & + & - & - & 0 & + & - & 0 & + & - & 0 & + \\ + & 0 & + & - & 0 & + & - & 0 & + & - & 0 & + & - \\ + & - & 0 & + & + & - & 0 & 0 & + & - & - & 0 & + \\ + & - & 0 & + & - & 0 & + & + & - & 0 & 0 & + & - \\ + & - & 0 & + & 0 & + & - & - & 0 & + & + & - & 0 \\ + & + & - & 0 & + & - & 0 & - & 0 & + & 0 & + & - \\ + & + & - & 0 & - & 0 & + & 0 & + & - & + & - & 0 \\ + & + & - & 0 & 0 & + & - & + & - & 0 & - & 0 & + \end{bmatrix}.$$

We have also been able to construct a circulant $W(13, 9)$. Its first row is

$$[0 \ 0 \ 1 \ 0 \ -1 \ 1 \ 1 \ 1 \ 1 \ 0 \ -1 \ 1 \ -1].$$

The “boundary” values from this Proposition are of special interest, i.e. if $n = (n - k)^2 - (n - k) + 1$. An inspection of the proof shows that if a $W(n, k)$ exists for such an n and k then the incidence of zeroes between any two rows is exactly one. So, if we let $A = W(n, k)$ and let $B = J - A * A$ then B satisfies $BB^t = (n - k - 1)I_n + 1 \cdot J_n$, i.e. B is the incidence matrix of a projective plane of order $n - k - 1$. So, the existence of $W(111, 100)$ would imply the existence of a projective plane of order 10. Thus, the Bruck-Ryser Theorem [5 or 10] on the nonexistence of projective planes of various orders imply nonexistence for the appropriate $W(n, k)$. However, even when the projective plane of order $(n - k - 1)$ exists the construction of $W(n, k)$ remains a formidable task.

We have previously mentioned two conjectures on the nonexistence of skew-symmetric $W(n, k)$ for various k and n . We repeat these conjectures here for definiteness.

CONJECTURE 1 *If $n \equiv 2 \pmod{4}$ and X is a $W(n, k)$ for which $X = -X^t$, then k is a square.*

CONJECTURE 2 *If $n = 4t$, t odd and X is a $W(n, k)$ for which $X = -X^t$, then k must be a sum of three squares.*

As far as conjecture 1 is concerned, we prove more generally

Proposition 24 *If $n \equiv 2 \pmod{4}$ and X is a skew-symmetric rational matrix of order n satisfying $XX^t = kI_n$, then $k = q^2$ for some rational number q . If, in addition, k is an integer then $k = a^2$ for some integer “ a ”.*

Proof Since $XX^t = kI_n$ we have $\det X = (k^{n/2})^{1/2} = k^{n/4}$ where $n/2 = s$ is odd. On the other hand, since X is skew-symmetric, $\det X = r^2$ where $r = \text{Pfaffian of } X$ (see [1, pages 141–142]). Thus, $r^2 = k^s$, and since s is odd,

k must be a square. To complete the proof we just note that if an integer is the square of a rational number it is also the square of an integer.

COROLLARY *If $n \equiv 2 \pmod{4}$ and there is an orthogonal design of type (a, b) in order n then b/a is a square.*

Proof We let X be the orthogonal design and write $X = x_1A_1 + x_2A_2$. Then we have shown that $A_1A_1^t = aI_n$, $A_2A_2^t = bI_n$ and $A_1A_2^t + A_2A_1^t = 0$.

Consider the matrices $B_1 = (1/a)A_1^tA_1$, $B_2 = (1/a)A_1^tA_2$. Then $B_1 = I_n$ and $B_1B_2^t + B_2B_1^t = 0$ and so B_2 is skew-symmetric. Furthermore, $B_2B_2^t = (b/a)I_n$. Thus by Proposition 24, b/a must be a square.

Example There is no orthogonal design of type $(4, 5)$ or $(1, 8)$ in order 10 and none of type $(5, 8)$ in order 14. The first two are clear; to see the result for $(5, 8)$ we note that $\frac{8}{5} = \frac{4 \cdot 0}{2 \cdot 5}$ and 40 is not a square of an integer. Note that the existence of these designs was not prohibited by the corollary to Proposition 3.

The proof of Conjecture 2 closely parallels the interesting proof of a theorem of Raghavarao (our Proposition 3) which was given by van Lint and Seidel in [13]. We shall make extensive use of the linearity properties of the Pfaffian and we refer the reader to Artin's monograph [1, p. 141] for the salient facts about this matrix function. Our original proof has been vastly improved by the following lemma, which was supplied by N. J. Pullman. We are very grateful to him for it.

LEMMA *If M and A are skew-symmetric matrices of order n and \mathcal{D} is the set of all diagonal matrices of order n whose diagonal entries are all in $\{1, -1\}$ then*

$$\text{a) } \sum_{\substack{D \in \mathcal{D} \\ \det D = 1}} \text{Pf}(DMD + A) = 2^{n-1}(\text{Pf}(A) + \text{Pf}(M))$$

and

$$\text{b) } \sum_{\substack{D \in \mathcal{D} \\ \det D = -1}} \text{Pf}(DMD + A) = 2^{n-1}(\text{Pf}(A) - \text{Pf}(M)).$$

Proof Part (b) is equivalent to (a) for if we use (a) with D_0MD_0 instead of M and $\det D_0 = -1$ then (b) follows directly from (a); similarly (b) implies (a). We shall prove (a) by induction on n which we may assume to be even. If $n = 2$ then

$$\begin{aligned} \sum_{\substack{D \in \mathcal{D} \\ \det D = 1}} \text{Pf}(DMD + A) &= \text{Pf}(IMI + A) + \text{Pf}((-I)M(-I) + A) \\ &= 2\text{Pf}(M + A) = 2(m_{12} + a_{12}) \\ &= 2(\text{Pf}(M) + \text{Pf}(A)). \end{aligned}$$

Suppose (a) is true for $n - 2 \geq 0$. Let X_{ij} denote the matrix obtained from

X be deleting rows i and j and columns i and j . Note that $[DXD]_{1j} = D_{1j}X_{1j}D_{1j}$ when D is a diagonal matrix $dg(d_1, d_2, \dots, d_n)$.

$$\begin{aligned} & \sum_{\substack{D \in \mathcal{D} \\ \det D = 1}} Pf(DMD + A) \\ &= \sum_{\substack{D \in \mathcal{D} \\ \det D = 1}} \sum_{j=2}^n (-1)^j (d_1 m_{1j} d_j + a_{1j}) Pf(DMD + A)_{1j} \\ &= \sum_{j=2}^n (-1)^j \left\{ \sum_{\substack{d_1 d_j = 1 \\ \det D = 1}} (m_{1j} + a_{1j}) Pf(DMD + A)_{1j} + \right. \\ & \quad \left. \sum_{\substack{d_1 d_j = -1 \\ \det D = -1}} (a_{1j} - m_{1j}) Pf(DMD + A)_{1j} \right\} \\ &= \sum_{j=2}^n 2(-1)^j \left\{ \sum_{\det D_{1j} = 1} (m_{1j} + a_{1j}) Pf(D_{1j}M_{1j}D_{1j} + A_{1j}) + \right. \\ & \quad \left. \sum_{\det D_{1j} = -1} (a_{1j} - m_{1j}) Pf(D_{1j}M_{1j}D_{1j} + A_{1j}) \right\} \\ &= \sum_{j=2}^n 2(-1)^j \left\{ (m_{1j} + a_{1j}) \sum_{\det D_{1j} = 1} Pf(D_{1j}M_{1j}D_{1j} + A_{1j}) + \right. \\ & \quad \left. (a_{1j} - m_{1j}) \sum_{\det D_{1j} = -1} Pf(D_{1j}M_{1j}D_{1j} + A_{1j}) \right\} \\ &= \sum_{j=2}^n 2(-1)^j \left\{ (m_{1j} + a_{1j}) 2^{n-3} (Pf(A_{1j}) + Pf(M_{1j})) + \right. \\ & \quad \left. (a_{1j} - m_{1j}) 2^{n-3} (Pf(A_{1j}) - Pf(M_{1j})) \right\} \\ &= 2^{n-1} \sum_{j=2}^n (-1)^j ((a_{1j} Pf(A_{1j}) + m_{1j} Pf(M_{1j})) \\ &= 2^{n-1} (Pf(A) + Pf(M)). \end{aligned}$$

COROLLARY 1

- a) $Pf(A) = 2^{-n} \sum_{D \in \mathcal{D}} Pf(DMD + A)$
- b) $Pf(M) = 2^{-n} \sum_{D \in \mathcal{D}} (\det D) Pf(DMD + A)$.

COROLLARY 2 *If A or M is nonsingular then there exists a $D \in \mathcal{D}$ for which $DMD + A$ is nonsingular.*

PROPOSITION 25 *Let X be a rational skew-symmetric matrix of order $n = 4s$, s odd and suppose $XX^t = kI_n$. Then $k = q_1^2 + q_2^2 + q_3^2$, $q_i \in \mathcal{Q}$. If, in addition $k \in \mathcal{Z}$ then q_1, q_2, q_3 may also be chosen in \mathcal{Z} .*

Proof Our proof is by induction on s . If $s = 1$, then

$$X = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{bmatrix}$$

and if $XX^t = kI_4$, then $k = x_1^2 + x_2^2 + x_3^2$, $x_i \in \mathcal{Q}$, proving the first assertion of the Proposition. Now, if $k \in \mathcal{Z}$ then by the theorem of Cassels-Davenport [see 11, p. 46] we obtain $k = a^2 + b^2 + c^2$ for three integers a, b, c .

So, we now assume s odd, $s > 1$. By the theorem of Lagrange we may certainly write

$$k = k_1^2 + k_2^2 + k_3^2 + k_4^2, \text{ where } k_i \in \mathcal{Q}.$$

Let

$$\tilde{M} = \begin{bmatrix} k_1 & -k_2 & -k_3 & -k_4 \\ k_2 & k_1 & -k_4 & k_3 \\ k_3 & k_4 & k_1 & -k_2 \\ k_4 & -k_3 & k_2 & k_1 \end{bmatrix}$$

and let

$$M = \begin{bmatrix} 0 & \tilde{M} \\ -\tilde{M}^t & 0 \end{bmatrix}.$$

Then, M has order 8, $M = -M^t$ and $MM^t = kI_8$. Furthermore, if D is any diagonal matrix of order 8, with diagonal entries ± 1 , then $\overline{M} = DMD$ is still skew-symmetric, has order 8 and satisfies $\overline{MM}^t = kI_8$.

We partition the matrix $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A has order 8 and note that A is skew-symmetric. Thus, since M and A are skew-symmetric and M is nonsingular there is, by Corollary 2 above, a diagonal matrix $D \in \mathcal{D}$ such that $DMD + A$ is nonsingular. (For convenience we shall still refer to DMD as M).

Now, let $P = D - C(M + A)^{-1}B$. Clearly P is a matrix of order $n - 8$ and one checks easily that P is skew-symmetric. We further claim that $PP^t = kI_{n-8}$. This is easily verified by calculating the following matrix product in two ways:

If $N = M + A$, consider

$$[-B^t(N^t)^{-1} \mid I_{n-8}] \begin{bmatrix} A^t & C^t \\ B^t & D^t \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} -N^{-1}B \\ I_{n-8} \end{bmatrix}$$

first by grouping the product as $(ST)(UV)$ and then as $S(TU)V$.

Thus, the induction hypothesis may be applied to P to complete the proof.

COROLLARY *If there is an orthogonal design of type (a, b) in order $n = 4s$, s odd, then b/a is a sum of ≤ 3 squares of rational numbers.*

Proof Exactly the same as the proof of the Corollary to Proposition 24.

Example There is no orthogonal design of type (1, 7), (4, 7) or (3, 5) in order 12. It is well known that 7 is not the sum of three integer squares. Now $\frac{5}{3} = \frac{1^2}{3} + \frac{1^2}{3} + \frac{1^2}{3}$ and 15 is not the sum of three integer squares and so $\frac{5}{3}$ is not the sum of three rational squares.

5. SOME CALCULATIONS

If $n \equiv 1$ or $3 \pmod{4}$

$n = 5$ By Proposition 23 no weighing matrix exists (except for weight 1 which we will consistently ignore).

$n = 7$ The only possibility is a $W(7, 4)$. The row $[- + + 0 + 0 0]$ generates a circulant $W(7, 4)$, which we will call A . $J - A * A$ is the incidence matrix of the projective plane of order 2.

$n = 9$ The only possibility is a $W(9, 4)$. A tedious, but not difficult, check on what the upper left hand corner of such a matrix must be, shows that $W(9, 4)$ does not exist.

$n = 11$ Weight 9 is precluded by Proposition 23, so we need only discover a $W(11, 4)$. If A is a $W(7, 4)$ and H_4 an Hadamard matrix of order 4 then $A \oplus H_4$ is a $W(11, 4)$.

$n = 13$ The possible weights are 4 and 9. We have already exhibited a $W(13, 9)$. In [14] a $W(6, 4)$ was found, so $W(6, 4) \oplus W(7, 4) = W(13, 4)$.

We mention, that $W(n, 4)$ exists for all $n \geq 4$ except for $n = 5, 9$. The proof of this follows readily from the fact that if two integers are relatively prime then eventually every integer can be expressed as a *positive* linear combination of these two integers. Thus, from the existence of $W(4, 4)$, $W(6, 4)$ and $W(7, 4)$ we assert that the “eventually” we mentioned occurs when $n = 10$. Hence by taking the appropriate direct sums of $W(4, 4)$, $W(6, 4)$ and $W(7, 4)$ we get $W(n, 4)$ for all $n \geq 10$. In fact, if $k = 4^s$ then there exists N such that for all $n \geq N$ $W(n, k)$ exists. We suspect this may be true for any square weight. We have exhibited a $W(13, 9)$ and in [14] or [4] a $W(10, 9)$, $W(12, 9)$ and $W(16, 9)$ are constructed and so $W(n, 9)$ exists for $n \geq 32$. So, our remarks above apply equally well to powers of 9. It would be interesting to have least upper bounds in these cases also, as we had for 4.

$n \equiv 2 \pmod{4}$

In this case $\rho(n) = 2$ and so we will consider the existence of designs on ≤ 2 variables.

$n = 2$ All designs exist.

$n = 6$ By Proposition 3 an orthogonal design of type (s) or of type (s_1, s_2) can exist only if $s, s_1, s_2, s_1 + s_2$ are each a sum of less than or equal to two squares. The numbers less than or equal to 6 that are the sum of ≤ 2 squares are 1, 2, 4, 5.

By the corollary to Proposition 3 the only designs on two variables that might exist are types (1, 1), (1, 4) and (2, 2). Using Proposition 9, we have

(type)	1st row of A	1st row of B
(1, 1)	$x_1 \quad 0 \quad 0$	$x_2 \quad 0 \quad 0$
(2, 2)	$x_1 \quad x_2 \quad 0$	$x_2 \quad -x_1 \quad 0$
(1, 4)	$x_1 \quad x_2 \quad -x_2$	$0 \quad x_2 \quad x_2$

Now, by setting the variables in these designs equal to one or zero, we obtain $W(6, k)$ for $k = 1, 2, 4, 5$.

Thus, for $n = 6$, Proposition 3 gives necessary and sufficient conditions for existence.

$n = 10$ Proposition 3 eliminates many 2-tuples (s_1, s_2) ($s_1 \leq s_2, s_1 + s_2 \leq 10$) as the type of an orthogonal design of order 10. The 2-tuples (1, 8) and (4, 5) are eliminated by the corollary to Proposition 24. The remaining are the types of an orthogonal design and can be constructed using Proposition 9, viz.

(type)	1st row of A	1st row of B
(1, 1)	$x_1 \quad 0 \quad 0 \quad 0 \quad 0$	$x_2 \quad 0 \quad 0 \quad 0 \quad 0$
(1, 4)	$0 \quad 0 \quad x_2 \quad x_2 \quad 0$	$x_1 \quad 0 \quad x_2 \quad -x_2 \quad 0$
(2, 2)	$x_1 \quad x_2 \quad 0 \quad 0 \quad 0$	$x_2 \quad -x_1 \quad 0 \quad 0 \quad 0$
(4, 4)	$0 \quad x_1 \quad x_2 \quad -x_2 \quad x_1$	$0 \quad x_1 \quad x_2 \quad x_2 \quad -x_1$

The numbers ≤ 10 that are the sum of ≤ 2 squares are 1, 2, 4, 5, 8, 9. The designs above give $W(10, k)$ for $k = 1, 2, 4, 5, 8$. We obtain a $W(10, 9)$ by using Proposition 9 and

(type)	1st row of A	1st row of B
(9)	$1 \quad -1 \quad -1 \quad -1 \quad -1$	$0 \quad 1 \quad -1 \quad -1 \quad 1$

Thus, Proposition 3 and the corollary to Proposition 24 give necessary and sufficient conditions for the existence of orthogonal designs of order 10.

$n = 14$ Again, Proposition 3 and the corollary to Proposition 24 exclude many 2-tuples (s_1, s_2) (where $0 < s_1 \leq s_2, s_1 + s_2 \leq 14$) as the types of

orthogonal designs of order 14. The tuple (4, 9) is not excluded and we cannot decide if it is the type of an orthogonal designs of order 14. All the other 2-tuples are the types of orthogonal designs. We use Proposition 9, and

(type)	1st row of <i>A</i>							1st row of <i>B</i>						
(1, 1)	x_1	0	0	0	0	0	0	x_2	0	0	0	0	0	0
(1, 4)	x_1	0	0	x_2	$-x_2$	0	0	0	0	0	x_2	x_2	0	0
(1, 9)	x_1	x_2	x_2	$-x_2$	x_2	$-x_2$	$-x_2$	0	x_2	x_2	0	x_2	0	0
(2, 2)	x_1	x_2	0	0	0	0	0	x_1	$-x_2$	0	0	0	0	0
(2, 8)	$-x_2$	x_2	x_2	0	x_2	0	x_1	$-x_2$	x_2	x_2	0	x_2	0	$-x_1$
(4, 4)	x_1	x_1	x_2	$-x_2$	0	0	0	x_1	$-x_1$	x_2	x_2	0	0	0
(5, 5)	$-x_2$	x_2	x_2	x_1	x_2	0	0	$-x_1$	x_1	x_1	$-x_2$	x_1	0	0

The numbers ≤ 14 that are the sum of ≤ 2 squares are 1, 2, 4, 5, 8, 9, 10, 13. The designs above give $W(14, k)$ for $k = 1, 2, 4, 5, 8, 9, 10$. A $W(14, 13)$ is given in [14], thus the necessary condition given in Proposition 3, for the existence of a weighing matrix, is also sufficient for $n = 14$.

$n \equiv 0 \pmod{4}$

We shall first consider $n = 2^t, t \geq 2$.

$n = 4, 8$ is covered by Corollary 1 to Lemma 4.

$n = 16$ Now $\rho(16) = 9$ and we must consider designs on ≤ 9 letters.

In [4] it was shown that if $n = 2^t$ then there exist $W(n, k)$ for all $k \leq n$. Thus we shall not make any further remarks about designs on one variable. From Theorem 17 we conclude that designs of type (1, s) exist when $n = 2^t$ and $s < n$.

To obtain the other designs on two variables and of order 16 we use Proposition 6 and the corollary to it. We note that there is an orthogonal design of type (1, 1, 1, 1, 1, 1, 1, 1) and order 8. This gives orthogonal designs of order 16 and type $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_8)$ where $\varepsilon_i = 1, 2$. Setting the variables in these designs equal to each other, or to zero we obtain (as the reader can easily check) 61 of the 64 possible designs on two variables in order 16. The remaining designs are of type (3, 13), (5, 11) and (7, 9).

Now, using Corollary 1 to Proposition 7 and the fact that there are amicable Hadamard matrices of order 4 we obtain an orthogonal design of order 16 and type (1, 3, 4, 4, 4). Thus by setting the variables in this design equal to each other, as necessary, we recover designs of type (3, 13), (5, 11) and (7, 9). Thus all orthogonal designs of order 16 on two variables exist.

We now consider designs on three variables. By using Proposition 6, as we did above, we obtain all but 23 of the 123 possible designs on 3 variables. The missing 23 are the following types

(1, 1, 13)	(1, 3, 12) *	(1, 6, 9) *
(1, 1, 14) *	(1, 4, 11) *	(1, 7, 7)
(1, 2, 13) *	(1, 5, 9)	(1, 7, 8) *
(1, 3, 11)	(1, 5, 10) *	(2, 3, 11) *
(2, 5, 9) *	(3, 4, 9) *	(4, 5, 7) *
(2, 7, 7) *	(3, 5, 7)	(5, 5, 5)
(3, 3, 9)	(3, 5, 8) *	(5, 5, 6) *
(3, 3, 10) *	(3, 6, 7) *	

Using Corollary 1 to Proposition 7 we obtain an orthogonal design of order 16 and type $(1, 1, 2, 2, 2, 2, 2, 2)$, since there are amicable Hadamard matrices of order 2. By setting some of the variables in this design equal to each other or to zero we obtain all those designs on three letters marked by *.

Now use Construction 19 and the fact that there are amicable Hadamard matrices of order 4 to obtain an orthogonal design of type $(1, 1, 1, 3, 3, 3, 4)$. If we use Construction 21 (VI) we obtain an orthogonal design of order 16 and type $(1, 1, 1, 2, 3, 3, 4)$. Thus all 3-tuples (s_1, s_2, s_3) ($1 \leq s_1 \leq s_2 \leq s_3$, $s_1 + s_2 + s_3 \leq 16$) are the types of orthogonal designs of order 16.

We next proceed to designs on four variables. Again, the use of Proposition 6 gives many of these designs, i.e. of the 155 possible 4-tuples (s_1, s_2, s_3, s_4) ($\sum_{i=1}^4 s_i \leq 16$, $1 \leq s_1 \leq s_2 \leq s_3 \leq s_4$) all but 51 are obtained as orthogonal designs by Proposition 6. The orthogonal designs $(1, 1, 2, 2, 2, 2, 2, 2)$, $(1, 1, 1, 3, 3, 3, 4)$ and $(1, 1, 1, 2, 3, 3, 4)$ that we have already obtained give many of these 51. The designs not yet obtained are

$$(1, 1, 1, 11) \quad \text{and} \quad (1, 5, 5, 5)$$

If we let

$$P = Q = R = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} + & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & - & - \\ 0 & 0 & - & + \end{bmatrix}$$

in Construction 19 we obtain an orthogonal design of type $(1, 1, 1, 2, 3, 3, 3)$ and order 16. This gives an orthogonal design of type $(1, 1, 1, 11)$ and order 16. We have been unable to construct an orthogonal design of order 16 and type $(1, 5, 5, 5)$.

We now consider designs on five variables. A tabulation shows that there are 149 possible 5-tuples, $(s_1, s_2, s_3, s_4, s_5)$ where $\sum_{i=1}^5 s_i \leq 16$ and $1 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq s_5$. This time Proposition 6 gives only 68 of these tuples as the type of an orthogonal design. However, the designs $(1, 1, 2, 2, 2, 2, 2, 2)$, $(1, 1, 1, 3, 3, 3, 4)$, $(1, 1, 1, 2, 3, 3, 3)$ and $(1, 1, 1, 2, 3, 3, 4)$ give all but the following 16:

(1, 1, 1, 1, 9)	(√)	(1, 1, 1, 5, 5)	(√)	(1, 1, 4, 5, 5)
(1, 1, 1, 1, 10)	(√)	(1, 1, 1, 5, 8)		(1, 2, 2, 5, 5)
(1, 1, 1, 1, 11)		(1, 1, 2, 2, 9)		(1, 2, 3, 5, 5)
(1, 1, 1, 1, 12)		(1, 1, 2, 3, 9)	(√)	(3, 3, 3, 3, 3)
(1, 1, 1, 2, 11)		(1, 1, 2, 5, 5)	(√)	
(1, 1, 1, 4, 7)	(√)	(1, 1, 2, 5, 7)		

Use Construction 21 (V) to obtain (1, 1, 1, 1, 3, 3, 4) and (1, 1, 1, 1, 1, 3, 4); Construction 21 (III) to obtain (1, 1, 1, 1, 3, 3, 3) and (1, 1, 1, 1, 1, 3, 3) and Construction 21 (IV) to obtain (1, 1, 1, 2, 2, 3, 3).

If we let

$$P = Q = R = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & + & + & + \\ + & 0 & - & + \\ + & - & + & 0 \\ + & + & 0 & 0 \end{bmatrix}$$

we obtain (from Construction 19) type (1, 1, 1, 3, 3, 3, 3). Also, using Construction 21 (VI) we obtain (1, 1, 1, 2, 2, 3, 4).

Now use Construction 22 (II) and the design (1, 3) in order 4 to obtain (1, 1, 2, 3, 3, 6).

The designs that can be obtained from the constructions above have been marked with a (√). We have been unable to decide if the 9 remaining tuples are the types of orthogonal designs.

Our results get markedly worse here. Of the 125 6-tuples that might be the types of orthogonal designs we can construct only 88. For the reader's information we list the 37 6-tuples for which the situation is undecided

1, 1, 1, 1, 1, 8	1, 1, 1, 1, 5, 5	1, 1, 2, 2, 2, 7
1, 1, 1, 1, 1, 9	1, 1, 1, 1, 5, 6	1, 1, 2, 2, 3, 6
1, 1, 1, 1, 1, 10	1, 1, 1, 1, 5, 7	1, 1, 2, 2, 3, 7
1, 1, 1, 1, 1, 11	1, 1, 1, 1, 6, 6	1, 1, 2, 2, 4, 5
1, 1, 1, 1, 2, 7	1, 1, 1, 2, 2, 8	1, 1, 2, 2, 5, 5
1, 1, 1, 1, 2, 8	1, 1, 1, 2, 2, 9	1, 1, 2, 3, 4, 5
1, 1, 1, 1, 2, 9	1, 1, 1, 2, 3, 8	1, 2, 2, 2, 3, 5
1, 1, 1, 1, 2, 10	1, 1, 1, 2, 4, 7	1, 2, 2, 3, 3, 5
1, 1, 1, 1, 3, 8	1, 1, 1, 2, 5, 5	1, 3, 3, 3, 3, 3
1, 1, 1, 1, 3, 9	1, 1, 1, 2, 5, 6	2, 2, 2, 3, 3, 3
1, 1, 1, 1, 4, 5	1, 1, 1, 3, 5, 5	2, 2, 3, 3, 3, 3
1, 1, 1, 1, 4, 7	1, 1, 1, 4, 4, 4	
1, 1, 1, 1, 4, 8	1, 1, 1, 4, 4, 5	

Of the 94 possible 7-tuples we can only prove that 37 are the types of orthogonal designs. (We have indicated these constructions above.) Again, for the reader's benefit we list the undecided tuples.

1, 1, 1, 1, 1, 1, 5	1, 1, 1, 1, 1, 3, 5	1, 1, 1, 1, 2, 2, 6
1, 1, 1, 1, 1, 1, 6	1, 1, 1, 1, 1, 3, 6	1, 1, 1, 1, 2, 2, 7
1, 1, 1, 1, 1, 1, 7	1, 1, 1, 1, 1, 3, 7	1, 1, 1, 1, 2, 2, 8
1, 1, 1, 1, 1, 1, 8	1, 1, 1, 1, 1, 3, 8	1, 1, 1, 1, 2, 3, 3
1, 1, 1, 1, 1, 1, 9	1, 1, 1, 1, 1, 4, 4	1, 1, 1, 1, 2, 3, 4
1, 1, 1, 1, 1, 1, 10	1, 1, 1, 1, 1, 4, 5	1, 1, 1, 1, 2, 3, 5
1, 1, 1, 1, 1, 2, 5	1, 1, 1, 1, 1, 4, 6	1, 1, 1, 1, 2, 3, 6
1, 1, 1, 1, 1, 2, 6	1, 1, 1, 1, 1, 4, 7	1, 1, 1, 1, 2, 3, 7
1, 1, 1, 1, 1, 2, 7	1, 1, 1, 1, 1, 5, 5	1, 1, 1, 1, 2, 4, 4
1, 1, 1, 1, 1, 2, 8	1, 1, 1, 1, 1, 5, 6	1, 1, 1, 1, 2, 4, 5
1, 1, 1, 1, 1, 2, 9	1, 1, 1, 1, 2, 2, 5	1, 1, 1, 1, 2, 4, 6
1, 1, 1, 1, 2, 5, 5	1, 1, 1, 2, 2, 3, 5	1, 1, 2, 2, 2, 3, 5
1, 1, 1, 1, 3, 3, 5	1, 1, 1, 2, 2, 3, 6	1, 1, 2, 2, 3, 3, 3
1, 1, 1, 1, 3, 3, 6	1, 1, 1, 2, 2, 4, 4	1, 1, 2, 2, 3, 3, 4
1, 1, 1, 1, 3, 4, 4	1, 1, 1, 2, 2, 4, 5	1, 1, 2, 3, 3, 3, 3
1, 1, 1, 1, 3, 4, 5	1, 1, 1, 2, 3, 3, 5	1, 2, 2, 2, 2, 3, 3
1, 1, 1, 1, 4, 4, 4	1, 1, 2, 2, 2, 2, 5	1, 2, 2, 2, 3, 3, 3
1, 1, 1, 2, 2, 2, 5	1, 1, 1, 2, 3, 4, 4	
1, 1, 1, 2, 2, 2, 6	1, 1, 2, 2, 2, 3, 3	
1, 1, 1, 2, 2, 2, 7	1, 1, 2, 2, 2, 3, 4	

Of the 67 8-tuples that might be the types of orthogonal designs we can only prove that 11 are. The orthogonal designs on eight variables that we know are

1, 1, 1, 1, 1, 1, 1, 1	1, 1, 2, 2, 2, 2, 2, 2
1, 1, 1, 1, 1, 1, 1, 2	1, 1, 2, 2, 2, 2, 2, 4
1, 1, 1, 1, 1, 1, 2, 2	1, 2, 2, 2, 2, 2, 2, 2
1, 1, 1, 1, 1, 2, 2, 2	1, 2, 2, 2, 2, 2, 2, 3
1, 1, 1, 1, 2, 2, 2, 2	2, 2, 2, 2, 2, 2, 2, 2
1, 1, 1, 2, 2, 2, 2, 2	

Of the 45 possible 9-tuples that might be the types of orthogonal designs we can only verify that two are! These two are

1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 2, 2, 2, 2, 2, 2, 2.

$n = 32$ While our methods give many designs in order 32 we have not yet calculated exactly which designs are missing as so much more work has yet to be done in deciding exactly what is happening in order 16.

$n \equiv 0 \pmod{4}$, $n \neq 2^t$

$n = 12$ In this case $\rho(n) = 4$, so we need only consider designs on ≤ 4 variables. We note that of all the numbers ≤ 12 , seven is the only one which is not the sum of ≤ 3 squares.

one variable In [14] it was shown that a $W(12, k)$ exists for every $k \leq 12$. Thus, all designs on one variable exist.

We next consider designs on *four* variables. There are 53 four-tuples (s_1, s_2, s_3, s_4) where $\sum s_i \leq 12, 1 \leq s_1 \leq s_2 \leq s_3 \leq s_4$. Of these, 32 are eliminated (by Proposition 24 and its corollary) as being the possible types of an orthogonal design of order 12. Of the 21 remaining we can construct the following by using Theorem 13:

(type)	1st row of A	1st row of B	1st row of C	1st row of D
(1, 1, 1, 1)	$x_1 \ 0 \ 0$	$x_2 \ 0 \ 0$	$x_3 \ 0 \ 0$	$x_4 \ 0 \ 0$
(1, 1, 1, 4)	$x_1 \ 0 \ 0$	$x_2 \ 0 \ 0$	$x_3 \ x_4 \ -x_4$	$0 \ x_4 \ x_4$
(1, 1, 1, 9)	$x_1 \ x_4 \ -x_4$	$x_2 \ x_4 \ -x_4$	$x_3 \ x_4 \ -x_4$	$x_4 \ x_4 \ x_4$
(1, 1, 2, 2)	$x_1 \ 0 \ 0$	$x_2 \ 0 \ 0$	$x_3 \ x_4 \ 0$	$x_3 \ -x_4 \ 0$
(1, 1, 2, 8)	$x_1 \ x_4 \ -x_4$	$x_2 \ x_4 \ -x_4$	$x_3 \ x_4 \ x_4$	$-x_3 \ x_4 \ x_4$
(1, 1, 4, 4)	$x_1 \ x_3 \ -x_3$	$0 \ x_3 \ x_3$	$x_2 \ x_4 \ -x_4$	$0 \ x_4 \ x_4$
(1, 1, 5, 5)	$x_1 \ x_3 \ -x_3$	$x_2 \ x_4 \ -x_4$	$x_3 \ x_4 \ x_4$	$-x_4 \ x_3 \ x_3$
(1, 2, 2, 4)	$x_1 \ x_4 \ -x_4$	$0 \ x_4 \ x_4$	$x_2 \ x_3 \ 0$	$x_2 \ -x_3 \ 0$
(1, 2, 3, 6)	$x_1 \ x_4 \ -x_4$	$x_3 \ x_4 \ x_4$	$x_3 \ -x_4 \ x_2$	$x_3 \ -x_2 \ -x_4$
(2, 2, 2, 2)	$x_1 \ x_2 \ 0$	$x_1 \ -x_2 \ 0$	$x_3 \ x_4 \ 0$	$x_3 \ -x_4 \ 0$
(2, 2, 4, 4)	$x_1 \ x_3 \ x_4$	$x_1 \ -x_4 \ -x_3$	$x_2 \ x_3 \ -x_4$	$x_2 \ -x_3 \ x_4$

In addition the Baumert-Hall array of order 12 gives type (3, 3, 3, 3).

For the reader's convenience we list those 4-tuples for which we have been unable to decide if an orthogonal design of that type exists.

- | | | |
|--------------|--------------|--------------|
| (1, 1, 1, 2) | (1, 1, 2, 3) | (2, 2, 2, 3) |
| (1, 1, 1, 3) | (1, 2, 2, 2) | (2, 2, 2, 4) |
| (1, 1, 1, 8) | (1, 3, 3, 3) | (2, 2, 2, 6) |

We now consider designs on *three* variables. There are 53 three-tuples (s_1, s_2, s_3) where $s_1 \leq s_2 \leq s_3$ and $\sum_{i=1}^3 s_i \leq 12$. We eliminate 16, because of Proposition 24 and its corollary. Many of the remaining 37 result from setting some of the variables in the designs on four variables we have found above equal to each other or to zero. In fact, the only orthogonal designs on three variables that we know exist arise in this fashion.

Again, for the reader's convenience we list all those three-tuples which we cannot prove correspond to the type of an orthogonal design.

- | | |
|-----------|-----------|
| (1, 1, 3) | (2, 2, 6) |
| (1, 3, 3) | (3, 3, 4) |
| (2, 2, 3) | |

There are 36 two-tuples (s_1, s_2) where $s_1 + s_2 \leq 12, 1 \leq s_1 \leq s_2$. Of these, 3 cannot be the types of an orthogonal design, in view of Proposition 24 and its corollary.

We can easily obtain the remaining 33 by setting the variables in the designs above equal to each other or to 0. Thus, Proposition 24 and its

corollary give necessary and sufficient conditions for the existence of designs on two variables in order 12.

Although our methods give many designs in order 20 we have not made such extensive calculations there.

We would now like to briefly indicate how some of the results we have obtained can be used in studying weighing matrices. It was conjectured in [14] that if $n \equiv 0 \pmod{4}$ then there is a $W(n, k)$ for every $k \leq n$. The first value of n for which this is not settled is $n = 36$. The missing weights there are $k = 23, 24, 26, 28, 29, 30, 31$.

We have constructed an orthogonal design of order 12 and type $(2, 5, 5)$ on the variables x_1, x_2, x_3 . Let $x_1 = I_3$,

$$x_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad x_3 = \begin{bmatrix} 1 & 1 & - \\ 1 & - & 1 \\ - & 1 & 1 \end{bmatrix}.$$

The resulting matrix is a $W(36, 26)$, as can be easily checked.

We have observed that there is an orthogonal design of order 12 and type $(6, 6)$. If in this design we let

$$x_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} 1 & 1 & - \\ 1 & - & 1 \\ - & 1 & 1 \end{bmatrix}$$

we obtain a $W(36, 30)$. On the other hand, if we let

$$x_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} 0 & 1 & - \\ - & 0 & 1 \\ 1 & - & 0 \end{bmatrix}$$

we obtain a $W(32, 24)$.

We have also found an orthogonal design of order 12 and type $(1, 5, 6)$ on the variables x_1, x_2, x_3 . If we let

$$x_1 = \begin{bmatrix} 0 & 1 & - \\ 1 & - & 0 \\ - & 0 & 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} - & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

then it can be easily checked that the resulting matrix is a $W(36, 29)$.

From the orthogonal design of order 12 and type $(2, 4, 6)$ on the variables x_1, x_2, x_3 we obtain a $W(36, 28)$ by letting

$$x_1 = \begin{bmatrix} 0 & 1 & - \\ 1 & - & 0 \\ - & 0 & 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} - & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Thus, the only $W(36, k)$ yet to be found are for $k = 23, 31$.

We mention, without proof, that the circulant $W(13, 9)$ that we have

found may be used with the results of [14] to obtain a $W(26, 18)$ and a $W(26, 20)$. These, in turn, will yield a $W(52, 36)$ and a $W(52, 43)$.

6. SOME OPEN QUESTIONS AND PROBLEMS

- a) If $n = 4t$, t odd, the only designs of type $(1, 1, 1, s)$ that we have found have $s = a^2$ from some integer a . We conjecture that this is always the case.
- b) We suspect that if $n = 4t$, t odd, then any design of type (a, a, a, x) must have $(x/a) = s^2$, $s \in Q$.
- c) If $n \equiv 0 \pmod{4}$, $n \neq 4t$, t odd, ($t > 1$), find some condition which says that there is a design on $\leq \rho(n)$ variables which is impossible to construct. (i.e. a theorem like our Proposition 25).
- d) More modestly, show that one of the designs of order 16, that we have been unable to find, is impossible.
- e) If $n = 4t$, t odd, the only designs that we have been able to find of type $(1, 1, s)$ have $s = a^2 + b^2$. We conjecture that this is always the case. More generally, if $n = 4t$, t odd then a design of type (a, a, x) will exist if and only if $x/a = q_1^2 + q_2^2$, $q_i \in Q$.
- f) Find families of "Williamson-type" matrices to use in these designs to construct new weighing matrices and new Hadamard matrices.
- g) Show that the only $n \equiv 0 \pmod{4}$ for which every possible $\rho(n)$ -tuple is the type of an orthogonal design is $n = 1, 2, 4, 8$. (Many of our previous questions would be answered by answering this one.)
- h) We have shown that there is a $W(n, 4)$, for all $n \geq 10$ and a $W(n, 9)$ for all $n \geq 32$. Find the least integer N such that a $W(n, 9)$ exists for all $n \geq N$. Obtain similar results for $W(n, s^2)$ by finding odd n 's for which $W(n, s^2)$ exists. As a first step, find a $W(31, 25)$.
- i) Find other pairs of orthogonal designs of order 4 (or tuples of designs of order 8) which, pairwise, satisfy the matrix equation $XY^t = YX^t$ (see Construction 22).

Note added in proof

Since the submission of this paper there has been considerable progress on the problems mentioned in Section 6. The interested reader is referred to the following papers and their bibliographies.

1. A. V. Geramita and J. S. Wallis, A Survey of Orthogonal Designs, Proc. *Fourth Manitoba Conference on Numerical Math.*, 1974, pp. 121-168 (Congressus Numerantium XII, University of Manitoba, Winnipeg).
2. A. V. Geramita and J. S. Wallis, Orthogonal Designs IV; Existence Questions; *J. Comb. Th., Ser. A.*, Vol. 19, No. 1, July 1975.

3. A. V. Geramita and J. Verner, Orthogonal Designs with Zero Diagonal, (to appear), *Canad. J. Math.*
4. Warren W. Wolfe, Clifford Algebras and Amicable Orthogonal Designs (Queen's Univ. Math. Preprint No. 1974-22).
5. Warren W. Wolfe, Rational Quadratic Forms and Orthogonal Designs (Queen's Univ. Math. Preprint No. 1975-22).

References

- [1] E. Artin, *Geometric Algebra*, Interscience Tracts in Pure and Applied Mathematics, 3, Wiley, New York, London, 3rd Printing, 1964.
- [2] A. V. Geramita and N. J. Pullman, Radon's function and Hadamard arrays, *Linear and Multilinear Algebra* (to appear).
- [3] A. V. Geramita and N. J. Pullman, A theorem of Radon-Hurwitz and orthogonal projective modules, *Proc. Amer. Math. Soc.* **42**, 1 (Jan. 1974).
- [4] Anthony V. Geramita, Norman J. Pullman and Jennifer S. Wallis, Families of weighing matrices, *Bull. Austral. Math. Soc.* **10** (1974), 119-122.
- [5] Marshall Hall, Jr., *Combinatorial Theory*, Blaisdell, Waltham, Mass., 1967.
- [6] E. Landau, *Elementary Number Theory*, Chelsea, New York, 1958.
- [7] M. Plotkin, Decomposition of Hadamard matrices, *J. Combinatorial Th.*, ser A, **12** (1972), 127-130.
- [8] J. Radon, Lineare Scharen Orthogonaler Matrizen, *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universitat*, Vol. 1 (1922), 1-14.
- [9] D. Raghavarao, Some aspects of weighing designs, *Ann. Math. Stat.* **31** (1960), 878-884.
- [10] H. J. Ryser, *Combinatorial Mathematics*, Carus Math. Monograph, 14, Wiley, New York, 1963.
- [11] Jean-Pierre Serre, *A course in Arithmetic*, Graduate texts in Mathematics, 7, Springer-Verlag, New York, Heidelberg, Berlin, 1973.
- [12] Olga Taussky, (1, 2, 4, 8)—sums of squares and Hadamard matrices, *Combinatorics*, Proc. Symp. in Pure Mathematics, Amer. Math. Soc., Providence, R.I., 1971, pp. 229-233.
- [13] J. H. Van Lint and J. J. Seidel, Equilateral point sets in elliptic geometry, *Kon. Ned. Akad. Wetensch. Amst. Proceeding, Series A* 69 and *Indag Math.* 28 (1966), 335-348.
- [14] Jennifer Wallis, Orthogonal (0, 1, -1)-matrices, *Proceedings of the First Australian Conference on Combinatorial Mathematics* (ed. Jennifer Wallis and W. D. Wallis) TUNRA Ltd., Newcastle, Australia, 1972, pp. 61-84.
- [15] Jennifer Wallis, A Note on Amicable Hadamard Matrices, *Utilitas Math.* **3** (1973), 119-125.
- [16] W. D. Wallis, Anne Penfold Street and Jennifer Seberry Wallis, *Combinatorics: Room Squares, Sum-free Sets, Hadamard Matrices*, Lecture Notes in Mathematics, Vol. 292, Springer-Verlag, Berlin-Heidelberg-New York, 1972.