

Orthogonal designs II

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Abstract

Orthogonal designs are a natural generalization of the Baumert-Hall arrays which have been used to construct Hadamard matrices. We continue our investigation of these designs and show that orthogonal designs of type $(1, k)$ and order n exist for every $k < n$ when $n = 2^{t+2} \cdot 3$ and $n = 2^{t+2} \cdot 5$ (where t is a positive integer). We also find orthogonal designs that exist in every order $2n$ and others that exist in every order $4n$.

Coupled with some results of earlier work, this means that the *weighing matrix conjecture* 'For every order $n \equiv 0 \pmod{4}$ there is, for each $k \leq n$, a square $\{0, 1, -1\}$ matrix $W = W(n, k)$ satisfying $WW^t = kI_n$ ' is resolved in the affirmative for all orders $n = 2^{t+1} \cdot 3$, $n = 2^{t+1} \cdot 5$ (t a positive integer).

The fact that the matrices we find are skew-symmetric for all $k < n$ when $n \equiv 0 \pmod{8}$ and because of other considerations we pose three other conjectures about weighing matrices having additional structure and resolve these conjectures affirmatively in a few cases.

In an appendix we give a table of the known results for orders ≤ 64 .

§0. Introduction

An orthogonal design of order n and type (s_1, s_2, \dots, s_l) ($s_i > 0$) on the commuting variables x_1, x_2, \dots, x_l is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \dots, \pm x_l\}$ such that

$$AA^t = \sum_{i=1}^l (s_i x_i^2) I_n.$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$.

In [2], where this was first defined and many examples and properties of such designs were investigated, we mentioned that

$$A^t A = \sum_{i=1}^l (s_i x_i^2) I_n$$

and so our alternative description of A applies equally well to the columns of A . We also showed in [2] that $l \leq \varrho(n)$, where $\varrho(n)$ (Radon's function) is defined by

$$\varrho(n) = 8c + 2^d$$

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when

$$n = 2^a \cdot b, \quad b \text{ odd}, \quad a = 4c + d \quad 0 \leq d < 4.$$

In [2] we also showed that if there is an orthogonal design of order n and type (a, b) then

- (i) $n \equiv 2 \pmod{4} \Rightarrow b/a = c^2$ for some rational number c
- (ii) $n = 4t, t \text{ odd} \Rightarrow b/a$ is a sum of \leq three rational squares.

DEFINITION. A *weighing matrix of weight k and order n* , is a square $\{0, 1, -1\}$ matrix, A , of order n satisfying

$$AA^t = kI_n.$$

In [2] we showed that the existence of an orthogonal design of order n and type (s_1, \dots, s_t) is equivalent to the existence of weighing matrices A_1, \dots, A_t , of order n , where A_i has weight s_i and the matrices, $\{A_i\}_{i=1}^t$, satisfy the matrix equation

$$XY^t + YX^t = 0$$

in pairs. In particular, the existence of an orthogonal design of order n and type $(1, k)$ is equivalent to the existence of a skew-symmetric weighing matrix of weight k and order n .

It is conjectured that:

(I) for $n \equiv 0 \pmod{4}$ there is a weighing matrix of weight k and order n for every $k \leq n$.

(II) for $n \equiv 0 \pmod{8}$ there is a skew-symmetric weighing matrix of order n for every $k < n$ (equivalently there is an orthogonal design of type $(1, k)$ in order n for every $k < n$).

(III) for $n \equiv 4 \pmod{8}$ there is a skew-symmetric weighing matrix of order n for every $k < n$, where k is the sum of \leq three squares of integers (equivalently, there is an orthogonal design of type $(1, k)$ in order n for every $k < n$ which is the sum of \leq three squares of integers. In other words, the necessary condition, given above in (ii), for the existence of an orthogonal design of type $(1, k)$ in order $n, n \equiv 4 \pmod{8}$, is also sufficient).

(IV) for $n \equiv 2 \pmod{4}$ there is a skew-symmetric weighing matrix for every weight $k < n - 1$ when k is a square (equivalently, the necessary condition given above for the existence of an orthogonal design of type $(1, k)$ in order n (see (i) above), is also sufficient.)

Conjecture (I) is an extension of the Hadamard conjecture (i.e. for every $n \equiv 0 \pmod{4}$ there is a $\{1, -1\}$ matrix, H , of order n satisfying $HH^t = nI_n$), while (II) and (III) generalize the conjecture that for every $n \equiv 0 \pmod{4}$ there is a Hadamard matrix, H , of order n with the property that $H = I_n + S$ where $S = -S^t$.

Conjecture (I) was established in [4] for $n \in \{4, 8, 12, \dots, 32, 40\}$ and in [1] for $n = 2^t$. Conjecture (II) was established in [2, Theorem 17] for $n = 2^t$ ($t \geq 3$), while Conjecture III was established in [2] for $n = 4, 12$. Conjecture IV was established in [2] for $n = 6, 10, 14$.

In this paper we establish conjectures (II) and (III) (and as a consequence (I)) for $n = 2^{t+1} \cdot 3, n = 2^{t+1} \cdot 5, t$ a positive integer. We also establish conjecture (III) for $n = 28$ separately. In addition, for every order $n \equiv 0 \pmod{4}$ we give 'segments' on which the conjecture is true. These segments grow with n .

Let R be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to be *constructed from two circulant matrices* A and B if it is of the form

$$\begin{bmatrix} A & B \\ B^t & -A^t \end{bmatrix}$$

and to be of Goethals-Seidel type if it is of the form

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^tR & -C^tR \\ -CR & -D^tR & A & B^tR \\ -DR & C^tR & -B^tR & A \end{bmatrix}$$

where A, B, C, D are circulant matrices.

§1. Known results and new applications

In this section we would like to list some of the results from [2] that we shall use and also give some new applications of them.

PROPOSITION 1. [2, Corollary to Construction 22]. *If there is an orthogonal design of type $(1, l)$ in order n then there is an orthogonal design of type $(1, 1, l, l)$ in order $2n$ and of type $(1, 1, 2, l, l, 2l)$ in order $4n$.*

The following is an easy corollary and was not mentioned in [2]. We shall use it extensively in this paper.

COROLLARY. *If there are orthogonal designs of type $(1, k), 1 \leq k \leq l$, in order n then there are orthogonal designs of type $(1, m)$ in order $2n$ for $1 \leq m \leq 2l + 1$. In particular, if there are orthogonal designs of type $(1, k), 1 \leq k \leq n - 1$, in order n then there are orthogonal designs of type $(1, m), 1 \leq m \leq 2^t n - 1$, in order $2^t n, t$ a positive integer.*

PROPOSITION 2. [2, Corollary to Proposition 6]. *If there is an orthogonal design of order n and type (s_1, \dots, s_i) then there are orthogonal designs of type $(\varepsilon_1 s_1, \dots, \varepsilon_i s_i)$ in order $2n$, where $\varepsilon_i = 1$ or 2 .*

We can now establish

THEOREM 3. *Let n be any number of the form $2^t \cdot 3$, t a positive integer. Then*

- (a) *If $t=1$, then conjecture IV is true.*
- (b) *If $t=2$, then conjectures I and III are true.*
- (c) *If $t \geq 3$, then conjecture II (and consequently conjecture I) is true.*

Proof. As we mentioned in the introduction, (a) was verified in [2] while (b) was verified in [1] and [2]. Thus we need only consider (c).

From (b) we have orthogonal designs of type $(1, k)$ in order 12 for $1 \leq k \leq 11$, $k \neq 7$. Proposition 1 then gives designs of type $(1, m)$ in order 24 for $1 \leq m \leq 23$, $m \neq 14, 15$. From (a) we have a design of type $(1, 4)$ in order 6 and so by Proposition 1 a design of type $(1, 1, 2, 4, 4, 8)$ in order 24 . By setting the variables in this design equal to each other, or to zero, we obtain designs of type $(1, 14)$, $(1, 15)$ in order 24 . Thus, conjecture II (and I) are true for $n=24$. Now the corollary to Proposition 1 gives the full result.

THEOREM 4. (a) *There are orthogonal designs of type $(1, 1)$ and $(1, 4)$ in order $2n$ for every integer $n \geq 3$.*

(b) *There is an orthogonal design of type $(1, 9)$ in order $2n$ for every integer $n \geq 6$ (except possibly for $n=9, 11$).*

(c) *There is an orthogonal design of type $(1, 1, 1, 4)$ in every order $4n$, $n \geq 2$.*

(d) *There are orthogonal designs of type $(1, 1, 2, 8)$ and $(1, 1, 1, 9)$ in every order $4n$, $n \geq 3$.*

Proof. The proofs of these statements all follow the same pattern. They will follow from the observation that if A and B are orthogonal designs of type (s_1, \dots, s_i) having orders n and m respectively, then $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is an orthogonal design of the same type having order $n+m$.

(a) In [2] we showed that orthogonal designs of type $(1, 1)$ and $(1, 4)$ exist in orders $6, 8$ and 10 . Since every even integer ≥ 6 is a non-negative linear combination of these three integers the result will follow from our remark above.

(b) In [2] we showed that an orthogonal design of type $(1, 9)$ exists for orders $12, 14, 16$. Since there is an orthogonal design of type $(1, 4)$ in order 10 , we have, by Proposition 1, an orthogonal design of type $(1, 1, 4, 4)$ in order 20 . Thus, there is an orthogonal design of type $(1, 9)$ in order 20 . Now observe that every even integer $2n$, $n \geq 12$, is a non-negative linear combination of $12, 14, 16$ and 20 to finish off the proof here.

(c) We have, in [2], exhibited a design of type (1, 1, 1, 4) in orders 8 and 12. Since every integer $4n, n \geq 2$, is a non-negative linear combination of these two integers we are finished with this part.

(d) In [2] we exhibited designs of type (1, 1, 2, 8) and (1, 1, 1, 9) in orders 12 and 16. If we can exhibit designs of this type in order 20 we will be done. We construct four circulant matrices A, B, C, D such that $AA^t + BB^t + CC^t + DD^t = kI_5$ where k is, in the first instance $x_1^2 + x_2^2 + x_3^2 + 9x_4^2$, and in the second instance $k = x_1^2 + x_2^2 + 2x_3^2 + 8x_4^2$. We then use these matrices in the Goethals-Seidel array:

k	(1, 1, 1, 9)	(1, 1, 2, 8)
1st Row of A	$x_1 \ 0 \ x_4 \ -x_4 \ 0$	$x_1 \ 0 \ x_4 \ -x_4 \ 0$
1st Row of B	$x_2 \ 0 \ x_4 \ -x_4 \ 0$	$x_2 \ 0 \ x_4 \ -x_4 \ 0$
1st Row of C	$x_3 \ -x_4 \ 0 \ 0 \ x_4$	$x_3 \ 0 \ x_4 \ x_4 \ 0$
1st Row of D	$x_4 \ x_4 \ x_4 \ 0 \ 0$	$-x_3 \ 0 \ x_4 \ x_4 \ 0$

Remark. (1) We strongly suspect that there are designs of type (1, 9) in orders 18 and 22.

(2) Our method of proof shows that any orthogonal design that appears in orders 12 and 16 also appears in order $4n, n \geq 6$. Thus, in addition to the designs mentioned above there are designs of type (1, 1, 5, 5), (1, 2, 2, 4), (1, 2, 3, 6) and (2, 2, 2, 2) in every order $4n, n = 3, 4, n \geq 6$. [See [2] for the description of these designs in orders 12 and 16].

COROLLARY 1. *There are orthogonal designs of type (1, k) in order $4n, n \geq 3$, for $1 \leq k \leq 11, k \neq 7$.*

Proof. From (c) and (d) above we have designs of type (1, 1, 2, 8) and (1, 1, 1, 4) in all the orders $4n, n \geq 3$. Now set the variables in these designs equal to each other or to zero to obtain the statement in this Corollary.

COROLLARY 2. *There is an orthogonal design of type (1, k) in every order $8n, n \geq 3$, for $k \in \{1, \dots, 23\}$.*

Proof. By using Proposition 1 and the designs of type (1, 1) and (1, 4) in order $2n, n \geq 3$, we obtain designs of type (1, 1, 2, 1, 1, 2) and (1, 1, 2, 4, 4, 8) in every order $8n, n \geq 3$. These two designs give designs of type (1, k) for $1 \leq k \leq 19$. By Corollary 1 we have designs of type (1, 10) and (1, 11) in order $4n, n \geq 3$ and an application of Proposition 1 then gives designs of type (1, 1, 10, 10) and (1, 1, 11, 11) in order $8n, n \geq 3$ and so the proof is complete.

COROLLARY 3. *There is an orthogonal design of type $(1, k)$ in order $8n, n=6, 7, 8, 10$ or $n \geq 12$ for $k \in \{1, \dots, 23, 27, \dots, 30, 36, \dots, 39\}$.*

Proof. For $1 \leq k \leq 23$ we appeal to Corollary 2. By Theorem 4, part (b), we have a design of type $(1, 9)$ in order $2n, n=6, 7, 8, 10, n \geq 12$ and so by Proposition 1 we have a design of type $(1, 1, 2, 9, 9, 18)$ in these orders. Setting the variables in this design equal to each other, or to zero will then prove the corollary.

§2. Golay sequences and orthogonal designs

Let $X = \{[a_{11}, \dots, a_{1n}], [a_{21}, \dots, a_{2n}], \dots, [a_{m1}, \dots, a_{mn}]\}$ be m sequences of integers of length n .

DEFINITION. (1) The *non-periodic auto-correlation function of the family of sequences X* (denoted N_X) is a function from the set of integers $\{1, 2, \dots, n-1\}$ to Z (the integers) where

$$N_X(j) = \sum_{i=1}^{n-j} (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \dots + a_{m,i}a_{m,i+j}).$$

Note that if the following collection of m matrices of order n is formed

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{11} & & a_{1,n-1} \\ & & \ddots & \\ \mathbf{0} & & & \ddots \\ & & & a_{11} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ & a_{21} & & a_{2,n-1} \\ & & \ddots & \\ \mathbf{0} & & & \ddots \\ & & & a_{21} \end{bmatrix}, \dots, \begin{bmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \\ & & & a_{m,n-1} \\ & & \ddots & \\ \mathbf{0} & & & \ddots \\ & & & a_{m1} \end{bmatrix},$$

then $N_X(j)$ is simply the sum of the inner products of rows 1 and $j+1$ of these matrices.

(2) The *periodic auto-correlation function of the family of sequences X* (denoted P_X) is a function from the set of integers $\{1, 2, \dots, n-1\}$ to Z where

$$P_X(j) = \sum_{i=1}^n (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \dots + a_{m,i}a_{m,i+j})$$

where we assume the second subscript is actually chosen from the complete set of residues mod $(n), \{1, 2, \dots, n\}$.

We can interpret the function P_X in the following way: Form the m circulant matrices which have first rows respectively, $[a_{11} \ a_{12} \ \dots \ a_{1n}], [a_{21} \ a_{22} \ \dots \ a_{2n}], \dots,$

$[a_{m1} a_{m2} \dots a_{mn}]$, then $P_X(j)$ is the sum of the inner products of rows 1 and $j+1$ of these matrices.

LEMMA 5. *Let X be a family of sequences as above, then*

$$P_X(j) = N_X(j) + N_X(n-j), \quad j = 1, \dots, n-1.$$

COROLLARY. *If $N_X(j) = 0$ for all $j = 1, \dots, n-1$ then $P_X(j) = 0$ for all $j = 1, \dots, n-1$.*

Note. $P_X(j)$ may equal 0 for all $j = 1, \dots, n-1$ even though the $N_X(j)$ are not.

DEFINITION. If $X = \{[a_1, \dots, a_n], [b_1, \dots, b_n]\}$ are two sequences where $a_i, b_j \in \{1, -1\}$ and $N_X(j) = 0$ for $j = 1, \dots, n-1$ then the sequences in X are called *Golay complementary sequences of length n* .

We note that if X is as above and A is the circulant formed by $[a_1, \dots, a_n]$ and B the circulant formed by $[b_1, \dots, b_n]$ then

$$AA^t + BB^t = \sum (a_i^2 + b_i^2) I_n.$$

Consequently, such matrices may be used in the Goethals-Seidel array to obtain Hadamard matrices.

EXAMPLE. $X = \{[1, -1], [1, 1]\}$ are Golay complementary sequences of length 2.

By results of R. J. Turyn [3], Golay complementary sequences exist having length r for

$$r = 2^a \cdot 10^b \cdot 26^c, \quad a, b, c$$

non-negative integers.

Since our interest is in orthogonal designs we shall not be restricted to sequences with entries only ± 1 , but shall allow 0's also. One very simple remark is in order. If we have a collection of sequences, X , (each having length n) such that $N_X(j) = 0, j = 1, \dots, n-1$, then we may augment each sequence at the beginning with k zeroes and at the end with l zeroes so that the resulting collection, (say \bar{X}), of sequences having length $k+n+l$ still has $N_{\bar{X}}(j) = 0, j = 1, \dots, k+n+l-1$. More interesting is the following result of Turyn.

PROPOSITION 6. *Let $X = \{[a_1, \dots, a_n], [b_1, \dots, b_n]\}$ be Golay complementary sequences. Then the sequences in*

$$X' = \left\{ \left[\frac{a_1 + b_1}{2}, \dots, \frac{a_n + b_n}{2} \right], \left[\frac{a_1 - b_1}{2}, \dots, \frac{a_n - b_n}{2} \right] \right\}$$

satisfy

- (i) $N_{X'}(j)=0, 1 \leq j \leq n-1$.
- (ii) exactly half of the $(a_i+b_i)/2$ are equal to 0 and exactly half of the $(a_i-b_i)/2$ equal 0, all others $= \pm 1$.

Thus, if we let $X = \{g_r, h_r\}$ represent Golay complementary sequences of length r we obtain a new pair of sequences of length r , which we denote g'_r, h'_r each having exactly $r/2$ non-zero members which can be chosen from $\{1, -1\}$ and such that if $X' = \{g'_r, h'_r\}$ then $N_{X'}(j)=0, 1 \leq j \leq r-1$.

Some more notation will be necessary. If g_r denotes a sequence of integers of length r then by xg_r we mean the sequence of length r obtained from g_r by multiplying each member of g_r by x . We let $\bar{0}_r$ denote a sequence consisting of r zeroes and $|g_r|$ denote the sum of the absolute values of the elements of g_r .

THEOREM 7. *Let r be any number of the form $2^a \cdot 10^b \cdot 26^c$, a, b, c non-negative integers, and let n be any integer $> r$. Then*

- (i) *There are orthogonal designs of order $4n$ and types $(1, 1, 2r)$ and $(1, 1, r)$. If, in addition n is odd, then*
- (ii) *there are orthogonal designs of order $4n$ and types $(1, 4, r)$ and $(1, 4, 2r)$.*

Proof. (i) Let r be as above and let $X = \{g_r, h_r\}$ be Golay complementary sequences of length r . Consider the four circulant matrices $A_i, i = 1, 2, 3, 4$ of order n having first rows respectively

$$[x_1 \bar{0}_{n-1}], [x_2 \bar{0}_{n-1}], [\bar{0}_{n-r} x_3 g_r], [\bar{0}_{n-r} x_3 h_r].$$

If

$$Y = \{[x_1, \bar{0}_{n-1}], [x_2, \bar{0}_{n-1}], [\bar{0}_{n-r}, x_3 g_r], [\bar{0}_{n-r}, x_3 h_r]\}$$

then $N_j(Y)=0, 1 \leq j \leq n-1$ and so we have

$$\sum_{i=1}^4 A_i A_i^t = (x_1^2 + x_2^2 + 2r x_3^2) I_n.$$

Thus, the A_i may be used in the Goethals-Seidel array to give an orthogonal design of order $4n$ and type $(1, 1, 2r)$.

If we now replace g_r and h_r by g'_r and h'_r (as in Proposition 6) we obtain an orthogonal design of order $4n$ and type $(1, 1, r)$.

- (ii) Let n be odd, $n > r$ and consider the sequences of length n in

$$Y = \{[x_1, \bar{0}_s, x_2, -x_2, \bar{0}_s], [0, \bar{0}_s, x_2, x_2, \bar{0}_s]\}$$

(where $s = (n-3)/2$). We claim that $P_Y(j)=0, 1 \leq j \leq n-1$. This is easy to see since the circulant matrix formed by the first sequence in Y has the form $x_1 I_n + U$, where $U = -U^t$ (note that this is the only place we use the fact that n is odd).

Now let g_r, h_r be Golay complementary sequences of length r and let $X = [\bar{0}_{n-r}, x_3 g_r], [\bar{0}_{n-r}, x_3 h_r]$. We thus use the sequences in X and Y , as in (i), to obtain orthogonal designs of order $4n$ and types $(1, 4, 2r)$ and $(1, 4, r)$.

THEOREM 8. *Let $X = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a collection of four $\{0, 1, -1\}$ -sequences of length n for which $P_X(j) = 0, 1 \leq j \leq n-1$. Suppose further that the circulant matrices generated by $\alpha_1, \alpha_2, \alpha_3$ are skew-symmetric. Then, (i) there is an orthogonal design of order $4n$ and type $(1, 1, 1, |\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4|)$.*

If, in addition, $N_X(j) = 0, 1 \leq j \leq n-1$, then (ii) this design exists in every order $4(n+2k), k \geq 0$.

Proof. In order that the hypothesis on $\alpha_1, \alpha_2, \alpha_3$ be satisfied we must have

$$\begin{aligned} \alpha_1 &= [0, a_1, \dots, a_{n-1}] = [0, g_{n-1}], \\ \alpha_2 &= [0, b_1, \dots, b_{n-1}] = [0, h_{n-1}] \\ \alpha_3 &= [0, c_1, \dots, c_{n-1}] = [0, l_{n-1}] \end{aligned}$$

where $a_{1+j} = -a_{(n-1)-j}, b_{1+j} = -b_{(n-1)-j}, c_{1+j} = -c_{(n-1)-j}, 0 \leq j \leq n-1$. Form four circulant matrices, $A_i, i = 1, 2, 3, 4$, whose first rows are, respectively,

$$[x_1 \ x_4 g_{n-1}], \ [x_2 \ x_4 h_{n-1}], \ [x_3 \ x_4 l_{n-1}], \ [x_4 \alpha_4].$$

Now $A_i = x_i I_n + B_i, i = 1, 2, 3$ and $B_i = -B_i^t$; thus,

$$\sum_{i=1}^4 A_i A_i^t = (x_1^2 + x_2^2 + x_3^2) I_n + \sum_{i=1}^3 B_i B_i^t + A_4 A_4^t.$$

Since $P_X(j) = 0$, we thus obtain

$$\sum_{i=1}^3 B_i B_i^t + A_4 A_4^t = (|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4|) x_4^2 \cdot I_n.$$

We may thus use these matrices, A_i in the Goethals-Seidel array to obtain the first part of the theorem.

The final assertion of the theorem follows from the observation we made earlier, that a collection of sequences whose non-periodic auto-correlation function was identically zero could be augmented, front or back (or both) by zero sequences to obtain longer sequences with the same property. If, in our case, we add sequences of *equal* length to the front *and* back of the given sequences then we preserve their skew-symmetric character. These remarks, coupled with the proof of (i) will then constitute a proof of (ii).

Table 1

α_1	α_2	α_3	α_4	Remarks
i) $[0, 1, 1, -, 1, -, -]$	$\alpha_2 = \alpha_1$	$[1, 1, 1, 0, 1, 0, 0]$	$[-, 1, 1, 0, 1, 0, 0]$	$P(j) = 0, 1 \leq j \leq 6$ $N(2) \neq 0$
ii) $[0, 1, 1, -, 1, -, -]$	$\alpha_2 = \alpha_1$	$\alpha_3 = \alpha_1$	$[-, 1, 1, 1, 1, 1, 1]$	$P(j) = 0, 1 \leq j \leq 6$ $N(1) \neq 0$
iii) $[0, 1, 1, -, 1, -, -]$	$[0, 1, 1, -, -, 1, 1]$	$[0, 1, 1, 1, -, 1, -]$	$[0, 1, -, 1, 1, 1, 1]$	$N(j) = 0, 1 \leq j \leq 6$
iv) $[0, 1, -, 1, -, -]$	$[0, 1, 1, -, -, -]$	$[0, 1, -, -, 1]$	$[0, 1, 1, 1, 1]$	$N(j) = 0, 1 \leq j \leq 4$
v) $[0, 0, 1, -, 0]$	$[0, 0, 1, 1, 0]$	$[-, 1, 1, 1, 1]$	$[0, 1, -, -, 1]$	$P(j) = 0, 1 \leq j \leq 4$ $N(1) \neq 0$
vi) $[0, 1, -, 1, -, -]$	$[-, 0, 1, 1, 0]$	$[-, 1, 1, 1, 1]$	$[0, 0, 1, 1, 0]$	$P(j) = 0, 1 \leq j \leq 4$ $N(1) \neq 0$
vii) $[0, 1, -, 1, -, -]$	$[0, 1, 1, -, -, -]$	$[-, 1, 1, 1, 1]$	$\alpha_4 = \alpha_3$	$P(j) = 0, 1 \leq j \leq 4$ $N(1) \neq 0$
viii) $[0, 1, 1, -, 1, -, -]$	$[0, 0, 1, 0, 0, -, 0]$	$\alpha_3 = \alpha_2$	$[0, 1, 1, -, 1, 1, 1]$	$N(j) = 0, 1 \leq j \leq 6$
ix) $[0, -, 1, -, -, 1, 1, -, 1]$	$[0, 1, -, -, -, 1, 1, 1, -]$	$[0, -, 1, 1, 1, 1, 1, 1, -]$	$[0, 1, -, 1, 1, 1, 1, -, 1]$	$N(j) = 0, 1 \leq j \leq 8$
x) $[0, 1, -, 1, 1, 1, -, -, 1, -]$	$[-, 1, 0, 1, 1, 1, 0, 0, 1, 0]$			$P(j) = 0, 1 \leq j \leq 10$ $N(2) \neq 0$

Remarks. (a) We would have an analogous theorem if only α_1 (or only α_1 and α_2) generated a skew-symmetric circulant.

(b) There is a completely analogous theorem if there are only two sequences of length n and the first has the skew-symmetric character described above. We just use the array $\begin{bmatrix} A & B \\ -B^t & A^t \end{bmatrix}$, mentioned in the introduction, in place of the Goethals-Seidel array. To facilitate the references we shall explicitly state:

COROLLARY. *Let $X = \{\alpha_1, \alpha_2\}$ be two $\{0, 1, -1\}$ sequences of length n for which $P_X(j) = 0, 1 \leq j \leq n-1$. Suppose that the circulant matrix generated by α_1 is skew-symmetric. Then, there is an orthogonal design of order $2n$ and type $(1, |\alpha_1| + |\alpha_2|)$.*

If, in addition, $N_X(j) = 0, 1 \leq j \leq n-1$, then this design exists in every order $2(n+2k)$, $k \geq 0$.

We now give some examples illustrating the use of Theorem 8 and its corollary.

These examples then give the following:

PROPOSITION 9. *The following orthogonal designs exist in the orders stated:*

- i) $(1, 1, 20)$ in order 28
- ii) $(1, 1, 1, 25)$ in order 28
- iii) $(1, 24)$ in orders $4(7+2k)$, $k \geq 0$
- iv) $(1, 1, 16)$ in orders $4(5+2k)$, $k \geq 0$
- v) $(1, 13)$ in order 20
- vi) $(1, 14)$ in order 20
- vii) $(1, 1, 18)$ in order 20
- viii) $(1, 1, 1, 16)$ in orders $4(7+2k)$, $k \geq 0$
- ix) $(1, 1, 32)$ in orders $4(9+2k)$, $k \geq 0$
- x) $(1, 16)$ in order 22.

Proof. Use the sequences in Table 1 as indicated in Theorem 8 or its corollary.

§3. Some applications

THEOREM 10. *Let $n = 2^t \cdot 5$, $t > 0$.*

(a) *If $t = 1$ then conjecture IV is true.*

(b) *If $t = 2$ then conjectures III and I are true.*

(c) *If $t \geq 3$ then conjectures II and I are true.*

Proof. (a) As mentioned in the introduction, this was established in [2].

(b) Corollary 1 to Theorem 4 gives designs of type $(1, k)$ in order 20 for $1 \leq k \leq 11$, $k \neq 7$. By Theorem 7 (ii), there is an orthogonal design of type $(1, 4, 2 \cdot 4)$ in order 20 which gives $(1, k)$ for $k = 12$. Proposition 9 then gives $(1, k)$ in order 20 for $k = 13$,

14, 16, 17, 18, 19. Since 7 and 15 are not the sum of \leq three squares this proves conjecture III for 20 and also conjecture I.

(c) From Propositions 1 and 2 and from (b) of this Theorem we obtain designs of type $(1, k)$ in order 40 for $1 \leq k \leq 39, k \neq 15, 30, 31$. By Corollary 2 to Theorem 4 we have the result for $k = 15$. If we can find orthogonal designs of type $(1, 30)$ and $(1, 31)$ in order 40 then the proof of (c) will be completed by repeated application of the corollary to Proposition 1.

To do this we need the following well-known fact:

LEMMA 11. *Let X, Y be two back-circulant matrices of order n, n odd. Suppose that the circulant matrices generated by the first rows of X and Y respectively are symmetric, then $XY^t = YX^t$.*

We recall that there is an orthogonal design of order 8 and type $(1, 1, 1, 1, 1, 1, 1, 1)$, (see [5] for this classical design derived from the Cayley Numbers) on the variables y_1, \dots, y_8 . We shall substitute a circulant matrix for y_1 and back-circulants for y_2, \dots, y_8 ; where if A_i is the matrix being substituted for y_i we have:

A_1	with first row	x_1	0	0	0	0
A_2	with first row	x_2	0	0	0	0
A_3	with first row	0	x_2	x_2	x_2	x_2
A_4	with first row	$-x_2$	x_2	x_2	x_2	x_2
$A_5 = A_6$	with first row	x_2	$-x_2$	x_2	x_2	$-x_2$
$A_7 = A_8$	with first row	x_2	x_2	$-x_2$	$-x_2$	x_2 .

If $X, Y \in \{A_1, \dots, A_8\}$ then $XY^t = YX^t$ by Lemma 11 or by the fact that if X is circulant and Y is back-circulant then $XY^t = YX^t$.

This then gives an orthogonal design of order 40 and type $(1, 30)$ since

$$\sum_{i=1}^8 A_i A_i^t = (x_1^2 + 30x_2^2) I_n.$$

We use the same procedure to obtain a design of order 40 and type $(1, 31)$. This time, let

A_1	have first row	x_1	0	x_2	$-x_2$	0
A_2	have first row	0	0	x_2	x_2	0
$A_3 = A_4 = A_5$	have first row	$-x_2$	x_2	x_2	x_2	x_2
$A_6 = A_7 = A_8$	have first row	0	x_2	$-x_2$	$-x_2$	x_2 .

PROPOSITION 12. (a) *Conjecture IV is true for $n = 2 \cdot 7$.*

(b) *For $n = 4 \cdot 7$ conjectures III and I are true.*

(c) For $n=8 \cdot 7$, there is an orthogonal design of type $(1, k)$ for $1 \leq k \leq 55$, except possibly for $k=46, 47$.

Proof. (a) was established in [2].

(b) We first establish conjecture III. This will then yield a proof of conjecture I for 28 (although this was already done in [4]). As in the proof of Theorem 10 (b) we have orthogonal designs of type $(1, k)$, $1 \leq k \leq 12, k \neq 7$. From Proposition 9 we obtain orthogonal designs in order 28 of type $(1, k)$ for $k=16, 17, 18, 20, 21, 24, 25, 26, 27$.

Now in [2] we found circulant matrices

i) A_1, A_2 of order 7 such that

$$\sum_{i=1}^2 A_i A_i^t = (x_1^2 + 4x_2^2) I_7$$

ii) B_1, B_2 of order 7 such that

$$\sum_{i=1}^2 B_i B_i^t = (y_1^2 + 9y_2^2) I_7$$

iii) C_1, C_2 of order 7 such that

$$\sum_{i=1}^2 C_i C_i^t = 13I_7.$$

Hence, if we use A_1, A_2, B_1, B_2 in the Goethals-Seidel array we obtain a design of type $(1, 1, 4, 9)$ in order 28, Using B_1, B_2, B_1, B_2 in the Goethals-Seidel array gives a design of type $(1, 1, 9, 9)$ in order 28, while if we use $B_1, B_2, y_3 C_1, y_3 C_2$ we obtain an orthogonal design of type $(1, 9, 13)$. These last three yield designs of type $(1, k)$ (not already listed) for $k=13, 14, 19, 22$. Since 7, 15 and 23 are not the sum of \leq three squares this proves conjecture III for $n=28$.

(c) Corollary 3 to Theorem 4 gives designs of type $(1, k)$ in order 56 $k \in \{1, \dots, 23, 27, \dots, 30, 36, \dots, 39\}$. Using Proposition 1 and the designs we found in (b) above we can fill in $k=24, 25, 32, \dots, 35, 40, \dots, 45, 48, \dots, 55$. Using Proposition 2 and the design of type $(1, 26)$ in order 28 we have the result for $k=26$ in order 56. The only gaps left are for $k=31, 46, 47$. The design of type $(1, 9, 13)$ in order 28 yields (by Proposition 2) a design of type $(1, 18, 13)$ in order 56. This gives us $(1, 31)$ in order 56. We have been unable to find designs of type $(1, k)$ in order 56 for $k=46, 47$.

With the results of these last two sections we can now sharpen the corollaries to Theorem 4.

PROPOSITION 13. *There are orthogonal designs of order $4n$, n odd and type $(1, k)$ when*

(a) $n \geq 3, k \in \{1, \dots, 6, 8, \dots, 11\}$.

(b) $n \geq 5, k \in \{1, \dots, 6, 8, \dots, 14, 16, 17\}$.

- (c) $n \geq 7$, $k \in \{1, \dots, 6, 8, \dots, 14, 16, 17, 18, 20, 24\}$
 (d) $n \geq 9$, $k \in \{1, \dots, 6, 8, \dots, 14, 16, 17, 18, 20, 21, 24, 32, 33\}$.
 (e) $n \geq 11$, $k \in \{1, \dots, 6, 8, \dots, 14, 16, \dots, 21, 24, 32, 33\}$.

The other corollaries admit of generalization in the same fashion. For the reader's benefit and for future reference we give, in an appendix, the status of our conjectures for small n .

Appendix

In the table given below we shall use the following ideas. If the conjecture is not applicable to that order we write N.A.; if the conjecture is verified for that order we shall put T in the table. If there are still values to test we shall list them. We shall not deal with conjecture IV in this table.

The first few numbers which are not the sum of \leq three squares are:

7, 15, 23, 28, 31, 39, 47, 55, 60, 92, 112.

Order	I	II	III
4	T	N.A.	T
8	T	T	N.A.
12	T	N.A.	T
16	T	T	N.A.
20	T	N.A.	T
24	T	T	N.A.
28	T	N.A.	T
32	T	T	N.A.
36	23, 31	N.A.	19, 22, 25, 26, 27, 29, 30, 34
40	T	T	N.A.
44	23, 29, 31, 35, 41	N.A.	22, 25, 27, 29, 30, 34, 35, 37, 38, 40, 41, 42
48	T	T	N.A.
52	26–31, 35–38, 40–47, 49, 50	N.A.	25, 26, 27, 30, 34–38, 40–46, 48–50
56	47	46, 47	N.A.
60	29–31, 35–58	N.A.	29, 30, 34–38, 40–46, 48–54, 56–58
64	T	T	N.A.

Order	IV	Order	IV
2	T	26	16
6	T	30	16,25
10	T	34	16,25
14	T	38	25,36
18	9,16	42	25,36
22	9		

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