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Some results on weighing matrices

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It is shown that if q is a prime power then there exists a circulant weighing matrix of order $q^2 + q + 1$ with q^2 non-zero elements per row and column.

This result allows the bound N to be lowered in the theorem of Geramita and Wallis that "given a square integer k there exists an integer N dependent on k such that weighing matrices of weight k and order n and orthogonal designs $(1, k)$ of order $2n$ exist for every $n > N$ ".

1. Introduction

An *orthogonal design of order* n *and type* $(s_1, s_2, \ldots, s_l)$ $(s_i > 0)$ on the commuting variables $x_1, x_2, \ldots, x_l$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_l\}$ such that

$$AA^t = \left(\sum_{i=1}^{l} s_i x_i^2\right) I_n .$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$.

In [2], where this was first defined and many examples and properties of such designs were investigated, it is mentioned that

$$A^t A = \left(\sum_{i=1}^{l} s_i x_i^2\right) I_n$$

and so the alternative description of A applies equally well to the

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columns of A. It is also shown in [2] that $l \le \rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

when

$$n = 2^a \cdot b \ , \quad b \text{ odd}, \quad a = 4c + d \ , \quad 0 \le d < 4 \ .$$

Also in [2] it is shown that if there is an orthogonal design of order n and type (a^2, b), then

(i) $n \equiv 2 \pmod 4 \Rightarrow b = c^2$ for some integer c,

(ii) $n = 4t$, t odd $\Rightarrow b$ is the sum of three integer squares;

while in [5] it is shown that if $n \equiv 4 \pmod 8$ and if there exists an orthogonal design of order n and type

(i) (a, a, a, b), then $\frac{b}{a}$ is a rational square;

(ii) (a, a, b), then $\frac{b}{a}$ is the sum of two rational squares;

(iii) (a, b), then $\frac{b}{a}$ is the sum of three rational squares.

A *weighing matrix of weight* k and *order* n is a square $\{0, 1, -1\}$ matrix, $W = W(n, k)$, of order n satisfying

$$WW^t = kI_n \ .$$

In [2] it is shown that the existence of an orthogonal design of order n and type $(s_1, \ldots, s_l)$ is equivalent to the existence of weighing matrices $A_1, \ldots, A_l$, of order n, where A_i has weight s_i and the matrices, $\{A_i\}_{i=1}^{l}$, satisfy the matrix equation

$$XY^t + YX^t = 0$$

in pairs. In particular, the existence of an orthogonal design of order n and type $(1, k)$ is equivalent to the existence of a skew-symmetric weighing matrix of weight k and order n.

It is conjectured that:

(i) for $n \equiv 2 \pmod 4$ there is a weighing matrix of weight k and order n for every $k < n - 1$ which is the sum of two integer squares;

(ii) for $n \equiv 0 \pmod 4$ there is a weighing matrix of weight k and order n for every $k \leq n$;

(iii) for $n \equiv 4 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$, where k is the sum of at most three squares of integers (equivalently, there is an orthogonal design of type $(1, k)$ in order n for every $k < n$ which is the sum of at most three squares of integers. In other words, the necessary condition for the existence of an orthogonal design of type $(1, k)$ in order n , $n \equiv 4 \pmod 8$ is also sufficient);

(iv) for $n \equiv 0 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$ (equivalently there is an orthogonal design of type $(1, k)$ in order n for every $k < n$);

(v) for $n \equiv 2 \pmod 4$ there is an orthogonal design of type $(1, k)$ in order n for every $k < n - 1$ such that k is an integer square.

Conjecture (ii) above is an extension of the Hadamard conjecture (that is, for every $n \equiv 0 \pmod 4$ there is a $\{1, -1\}$ matrix, H , of order n satisfying $HH^t = nI_n$) while (iv) and (iii) generalize the conjecture that for every $n \equiv 0 \pmod 4$ there is a Hadamard matrix, H , of order n , with the property that $H = I_n + S$ where $S = -S^t$.

Conjecture (ii) was established in [*10*] for $n \in \{4, 8, 12, \ldots, 32, 40\}$ and in [*6*] for $n = 2^t$. Conjecture (iii) was established in [*3*, Theorem 17] for $n = 2^t$ $(t \geq 3)$.

Conjectures (iv) and (iii) (and as a consequence conjecture (ii)) were established for $n = 2^{t+1} \cdot 3$, $n = 2^{t+1} \cdot 5$, t a positive integer, in [*4*] and in [*11*] for $n = 2^{t+1} \cdot 9$. Also in [*3*] it was shown that only

$k = 46, 47$ in order 56 remain to be found and the conjectures will be settled for $n = 2^{t+1}\cdot 7$.

It has been established [5] that given a square k there exists an $N(k)$ such that $W(n, k)$ exists for every $n > N$. Consequently an orthogonal design $(1, k)$ exists in every order $2n$, $n > N$.

Here we give some results which allow $N(k)$ to be lowered when k has a factor of 4 .

Let R be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to be *constructed from two circulant matrices* A and B if it is of the form

$$\begin{bmatrix} A & BR \\ BR & -A \end{bmatrix}$$

and to be of *Goethals-Seidel type* if it is of the form

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^tR & -C^tR \\ -CR & -D^tR & A & B^tR \\ -DR & C^tR & -B^tR & A \end{bmatrix}$$

where A, B, C, D are circulant matrices.

Let $S_1, S_2, \ldots, S_n$ be subsets of V , a finite abelian group, containing $k_1, k_2, \ldots, k_n$ elements respectively. Write T_i for the totality of all differences between elements of S_i (with repetitions), and T for the totality of elements of all the T_i . If T contains each non-zero element of V a fixed number of times, λ say, then the sets $S_1, S_2, \ldots, S_n$ will be called $n - \{v; k_1, k_2, \ldots, k_n; \lambda\}$ supplementary difference sets.

2. Weighing matrices of odd order

If A is a $W(n, k)$, then $(\det A)^2 = k^n$. Thus if n is odd and a $W(n, k)$ exists, then k must be a perfect square.

In [2] where they are first discussed it is shown that

$$(n-k)^2 - (n-k) + 2 > n$$

must also hold. It is noted there that the "boundary" values of this condition are of special interest; that is, if

$$(n-k)^2 - (n-k) + 1 = n ,$$

for in this case the zeros of A occur such that the incidence between any pair of rows is exactly one. So if we let $B = J - A*A$, B satisfies

$$BB^t = (n-k-1)I_n + J_n , \quad BJ = (n-k)J_n ;$$

that is, B is the incidence matrix of the projective plane of order $n - k - 1$.

Thus, the Bruck-Ryser Theorem on the non-existence of projective planes of various orders implied the non-existence of the appropriate $W(n, k)$.

We shall prove in this paper that if q is a prime power, then a circulant weighing matrix of the form

$$W\left(q^2+q+1, q^2\right)$$

can be constructed. Our method makes use of near difference sets.

In [8] Ryser has given the following definition of a near difference set.

Let $m \geq 4$ be an even integer, and let k and λ be positive integers. A *near difference set*

$$D = \{d_1, d_2, \ldots, d_k\}$$

is a set of k residues modulo m with the property that, for any residue $a \not\equiv 0$, $\frac{m}{2} \pmod m$, the congruence

$$d_i - d_j \equiv a \pmod m$$

has exactly λ solution pairs $\left(d_i, d_j\right)$ with d_i and d_j in D and no solution pairs for $a \equiv \frac{m}{2} \pmod m$.

A necessary condition for the existence of a near difference set with parameters m, k, λ is that

$$k(k-1) = \lambda(m-2) .$$

Let us put

$$m = 2v .$$

Then the necessary condition becomes

$$k(k-1) = 2\lambda(v-1) .$$

Examples of near difference sets are:-

(i) $v = 7$, $k = 4$, $\lambda = 1$, $m = 14$,

0	1	4	6

;

(ii) $v = 13$, $k = 9$, $\lambda = 3$, $m = 26$,

0	1	6	8	10	11	12	15	18

;

(iii) $v = 21$, $k = 16$, $\lambda = 6$, $m = 42$,

0	1	10	11	18	20	23	25	26	29	30	34	36	37	38	40

.

In [1] Elliott and Butson proved that if q is an odd prime power, then we can construct a near difference set with parameters

$$m = 2\left(1+q+q^2\right) , \quad k = q^2 , \quad \lambda = \tfrac{1}{2}q(q-1) .$$

Spence [9] showed that the construction of Elliott and Butson is also valid when q is a power of 2 .

The three examples of near difference sets that we have given illustrate the cases $q = 2, 3, 4$ of the Elliott-Butson-Spence result.

Suppose that we are given a near difference set

$$D = \{d_1, d_2, \ldots, d_k\}$$

with parameters m, k, λ . Then the polynomial

$$\alpha(x) = \sum_{d \in D} x^d$$

is the Hall polynomial associated with D. Since D is a near difference set we have

$$\alpha(x)\alpha(x^{-1}) \equiv k + \lambda(x + x^2 + \ldots + x^{v-1} + x^{v+1} + \ldots + x^{2v-1}) \pmod{x^{2v}-1} .$$

If we write $T_r(x) = 1 + x + x^2 + \ldots + x^{r-1}$ this takes the form

$$\alpha(x)\alpha(x^{-1}) \equiv k + \lambda\left(T_{2v}(x) - T_2(x^v)\right) \pmod{x^{2v}-1} :$$

In the rest of this discussion let D denote the near difference set of Elliott-Butson-Spence. The parameters of D are given by

$$m = 2(q^2+q+1) , \quad k = q^2 , \quad \lambda = \frac{q(q-1)}{2} .$$

If $\alpha(x) = \sum_{d \in D} x^d$, then we have

$$\alpha(x)\alpha(x^{-1}) \equiv q^2 + \frac{q(q-1)}{2}(x + x^2 + \ldots + x^{v-1} + x^{v+1} + \ldots + x^{2v-1}) \pmod{x^{2v}-1} ,$$

where $v = 1 + q + q^2$. Let k_1 be the number of odd integers in D, and k_2 the number of even integers in D. Since a translate of D is also a near difference set with the same parameters we may assume without loss of generality that

$$k_2 \geq k_1 .$$

For $x = -1$ we have

$$\alpha(-1) = k_2 - k_1 , \quad \alpha^2(-1) = q^2 .$$

Hence

$$\alpha(-1) = q .$$

The two equations

$$-k_1 + k_2 = q ,$$

$$k_1 + k_2 = q^2 ,$$

yield

$$k_1 = \frac{q^2-q}{2} , \quad k_2 = \frac{q^2+q}{2} .$$

Let us now put

$$F(x) = \sum_{\substack{d\in D \\ d \text{ odd}}} x^d , \quad G(x) = \sum_{\substack{d\in D \\ d \text{ even}}} x^d .$$

Then we have

$$\alpha(x) = F(x) + G(x) ,$$

$$\alpha(x^{-1}) = F(x^{-1}) + G(x^{-1}) ,$$

so that

$$\alpha(x)\alpha(x^{-1}) = F(x)F(x^{-1}) + G(x)G(x^{-1}) + F(x)G(x^{-1}) + F(x^{-1})G(x) .$$

It is clear that

(1) $$F(x)F(x^{-1}) + G(x)G(x^{-1}) \equiv$$
$$\equiv q^2 + \frac{q(q-1)}{2} (x^2 + x^4 + \ldots + x^{2v-2}) \pmod{x^{2v}-1} ,$$

(2) $$F(x)G(x^{-1}) + F(x^{-1})G(x) \equiv$$
$$\equiv \frac{q(q-1)}{2} (x + x^3 + \ldots + x^{v-2} + x^{v+2} + \ldots + x^{2v-1}) \pmod{x^{2v}-1} .$$

We next put

$$\alpha_1(x) = \sum_{\substack{d\in D \\ d \text{ odd}}} x^{(d+v)/2} , \quad \alpha_2(x) = \sum_{\substack{d\in D \\ d \text{ even}}} x^{d/2} .$$

Then the reduction of (1) mod x^v-1 yields

(3) $$\alpha_1(x)\alpha_1(x^{-1}) + \alpha_2(x)\alpha_2(x^{-1}) \equiv q^2 + \frac{q(q-1)}{2} (x + x^2 + \ldots + x^{v-1}) \pmod{x^v-1} .$$

The reduction of (2) mod x^v-1 yields

(4) $\alpha_1(x)\alpha_2(x^{-1}) + \alpha_2(x)\alpha_1(x^{-1}) \equiv \frac{q(q-1)}{2}\left(x + x^2 + \ldots + x^{v-1}\right)$

$$\left(\text{mod } x^v-1\right) .$$

We shall prove the following theorem.

THEOREM 1. *Let* q *be a prime power. Then a circulant weighing matrix of the form*

$$W\left(q^2+q+1,\ q^2\right)$$

can be constructed.

Proof. Let $D = \{d_1, d_2, \ldots, d_k\}$ be an Elliott-Butson-Spence near difference set with parameters

$$m = 2\left(q^2+q+1\right) , \quad k = q^2 , \quad \lambda = \frac{q(q-1)}{2} .$$

We again put $v = q^2 + q + 1$. Let S be the set of v integers: 0, 1, 2, ..., v-1 . We partition S into three subsets as follows:

$$S = T_1 \cup T_2 \cup T_3$$

where

$$T_1 = \left\{\frac{d+v}{2} \ (\text{mod } v),\ d \in D,\ d \text{ odd}\right\} ,$$

$$T_2 = \left\{\frac{d}{2} \ (\text{mod } v),\ d \in D,\ d \text{ even}\right\} ,$$

$$T_3 = \left\{s \in S,\ s \notin T_1,\ s \notin T_2\right\} .$$

There are k_1 integers in T_1 , k_2 integers in T_2 , and $v - k_1 - k_2$ integers in T_3 .

The sets T_1 and T_2 are disjoint. For if

$$\frac{d_i+v}{2} \equiv \frac{d_j}{2} \ (\text{mod } v)$$

then

$$d_i - d_j \equiv v \ (\text{mod } 2v) , \quad \left(d_i,\ d_j \in D\right) ,$$

in violation of the definition of a near difference set.

The initial row

$$a_0, a_1, \ldots, a_{v-1}$$

of the circulant $W(q^2+q+1, q^2)$ is now constructed as follows:

$$a_i = \begin{cases} -1 & \text{if } i \in T_1 , \\ 1 & \text{if } i \in T_2 , \\ 0 & \text{if } i \in T_3 . \end{cases}$$

Define $\psi(x) = \sum_{i=0}^{v-1} a_i x^i$. Then we have

$$\psi(x) = \alpha_2(x) - \alpha_1(x) ,$$

$$\psi(x^{-1}) = \alpha_2(x^{-1}) - \alpha_1(x^{-1}) ,$$

so that

$$\begin{aligned}\psi(x)\psi(x^{-1}) &= \alpha_1(x)\alpha_1(x^{-1}) + \alpha_2(x)\alpha_2(x^{-1}) - \alpha_1(x)\alpha_2(x^{-1}) - \alpha_1(x^{-1})\alpha_2(x) \\ &\equiv q^2 + \frac{q(q-1)}{2}(x + x^2 + \ldots + x^{v-1}) - \frac{q(q-1)}{2}(x + x^2 + \ldots + x^{v-1}) \\ &\qquad\qquad\qquad\qquad \pmod{x^v-1} \\ &\equiv q^2 \pmod{x^v-1} .\end{aligned}$$

Replacing x by ζ (where $\zeta^v = 1$) we obtain

$$\psi(\zeta)\psi(\zeta^{-1}) = q^2 .$$

The last relation is valid for each vth root of unity ζ including $\zeta = 1$. For $\zeta = 1$ we have

$$\psi(1) = k_2 - k_1 = \frac{q(q+1)}{2} - \frac{q(q-1)}{2} = q .$$

We next apply the finite Parseval relation:

$$\sum_{i=0}^{v-1} a_i a_{i+r} = \frac{1}{v} \sum_{j=0}^{v-1} |\psi(\zeta^j)|^2 \zeta^{jr} .$$

For $r = 0$ we have

$$\sum_{i=0}^{v-1} a_i^2 = \frac{1}{v} vq^2 = q^2 .$$

For $1 \le r \le v-1$ we get

$$\sum_{i=0}^{v-1} a_i a_{i+r} = \frac{1}{v} \cdot q^2 \cdot 0 = 0 .$$

This completes the proof of the orthogonality of the circulant $W(q^2+q+1,\ q^2)$.

3. Other observations

We next note that the sets T_1, T_2 constitute

$$2 - \left\{v;\ k_1,\ k_2;\ k_1 + k_2 - \frac{v-1}{2}\right\}$$

supplementary difference sets. Since $k_1 = \frac{q(q-1)}{2}$, $k_2 = \frac{q(q+1)}{2}$, we have

$$\lambda = k_1 + k_2 - \frac{v-1}{2} = k_1 .$$

The result follows at once from

$$\alpha_1(x)\alpha_1(x^{-1}) + \alpha_2(x)\alpha_2(x^{-1}) \equiv q^2 + \frac{q(q-1)}{2}\left(x + x^2 + \ldots + x^{v-1}\right) \pmod{x^v-1} .$$

We are now in the position to construct the Hadamard matrix, H_{292} , of Spence. We use the following well-known result.

Let $p = 2n + 1$ be a prime. Let U be the set of quadratic residues of p , and V the set of quadratic non-residues of p . Then U and V constitute

$$2 - \left\{v;\ k_3,\ k_4;\ k_3 + k_4 - \frac{v+1}{2}\right\}$$

supplementary difference sets. Here we have

$$v = p = 2n + 1 \ ; \quad k_3 = k_4 = n \ ; \quad \lambda = n - 1 .$$

Combining our results we find that if $v = q^2 + q + 1$ is a prime, then we construct

$$2 - \left\{v;\ k_1,\ k_2;\ k_1 + k_2 - \frac{v-1}{2}\right\}$$

supplementary difference sets, and also

$$2 - \left\{v;\ k_3,\ k_4;\ k_3 + k_4 - \frac{v+1}{2}\right\}$$

supplementary difference sets. It follows that we have

$$4 - \{v;\ k_1,\ k_2,\ k_3,\ k_4;\ k_1 + k_2 + k_3 + k_4 - v\}$$

supplementary difference sets, which may be used to construct an Hadamard matrix H_{4v} of Williamson type.

In particular for $q = 8$ we have $v = 73$. Therefore we can construct H_{292}.

Our next objective is to show that the $k_1 + k_2$ numbers in $T_1 \cup T_2$ constitute an ordinary difference set with parameters

$$v = q^2 + q + 1\ ,\quad k = q^2\ ,\quad \lambda = q^2 - q\ .$$

For this purpose we form the polynomial

$$A(x) = \alpha_1(x) + \alpha_2(x)$$

so that

$$A\left(x^{-1}\right) = \alpha_1\left(x^{-1}\right) + \alpha_2\left(x^{-1}\right)\ .$$

Then we have

$$\begin{aligned} A(x)A\left(x^{-1}\right) &= \alpha_1(x)\alpha_1\left(x^{-1}\right) + \alpha_2(x)\alpha_2\left(x^{-1}\right) + \alpha_1(x)\alpha_2\left(x^{-1}\right) + \alpha_1\left(x^{-1}\right)\alpha_2(x) \\ &\equiv q^2 + \frac{q(q-1)}{2}\left(x + x^2 + \ldots + x^{v-1}\right) + \frac{q(q-1)}{2}\left(x + x^2 + \ldots + x^{v-1}\right) \\ &\qquad\qquad \left(\mathrm{mod}\ x^v - 1\right) \\ &\equiv q^2 + q(q-1)\left(x + x^2 + \ldots + x^{v-1}\right) \ \left(\mathrm{mod}\ x^v - 1\right)\ . \end{aligned}$$

The set T_3 is the complement of $T_1 \cup T_2$. Therefore the integers in T_3 constitute a difference set with parameters

$$v^* = v\ ,\quad k^* = v - k = q + 1\ ,\quad \lambda^* = v - 2k + \lambda = 1\ .$$

4. Applications to weighing matrices and orthogonal designs

The existence of the $W(21, 16)$ allows us to make the following statements.

THEOREM 2. *There exists a* $W(n, 16)$ *for every* $n \in \{16, 18, 20, 21, 22, 24, 26, \ldots, 36$ *, and all orders* $\geq 36\}$.

Proof. In [5] it was noted that a $W(n, 16)$ exists for $n \in \{16, 18, 20, \ldots, 64$, and all orders $\geq 64\}$. Thus the existence of a $W(21, 16)$ allows this set to be replaced by that of the enunciation.

THEOREM 3. *There exist orthogonal designs* $(1, 9)$ *and* $(1, 16)$ *in every order* $2n$ *,* $n \geq 21$.

Proof. These results follow using the $W(21, 16)$ to obtain a $(1, 16)$ in order 42 and then noting from Tables 1 and 2 of [4] that each order $2n$, $n \geq 21$ can be written as $2m_1 + 2m_2$ where $(1, 9)$ and $(1, 16)$ exist for both orders $2m_1$ and $2m_2$.

THEOREM 4. *There exists a* $W\left(42, a^2+b^2\right)$ *for integers* a, b *except possibly for* $a^2 + b^2 \in \{18, 25, 29, 36, 37\}$.

Proof. Since a $W(22, k)$ and $W(20, k)$ exist for $k \in \left\{a^2+b^2 : a^2+b^2 \leq 20,\ a^2+b^2 \neq 18\right\}$ [4; Table 2] we have $W(42, k) = W(22, k) \oplus W(20, k)$ for the same k .

There is a $W(42, k)$ for $k \in \{26, 32, 40\}$ by [4; Proposition 13]. Writing $A = W(21, 16)$ we see

$$\begin{bmatrix} A+I & A-I \\ A^t-I & -A^t-I \end{bmatrix}$$

is a $W(42, 34)$. Finally since 41 is a prime the construction of Goethals and Seidel [7] gives a $W(42, 41)$ and we have the result.

THEOREM 5. *Since there exists a* $W = W\left(q^2+q+1, q^2\right)$ *for every prime power* q *there exist orthogonal designs*

(i) $\left(1, q^2\right)$ *and* $\left(q^2, q^2\right)$ *in order* $2\left(q^2+q+1\right)$ *;*

(ii) $(1, 1, 1, q^2)$, $(1, 1, q^2, q^2)$, $(1, q^2, q^2, q^2)$, (q^2, q^2, q^2, q^2), $(1, 4, q^2)$, $(1, 1, 2(q^2+1))$, $(1, q^2, 2(q^2+1))$, $(q^2, q^2, 2(q^2+1))$, $(2(q^2+1), 2(q^2+1))$ *in every order* $4(q^2+q+1)$ *;*

(iii) $(1, 1, 2, q^2, q^2, q^4)$ *(at least) in every order* $8(q^2+q+1)$ *;*

(iv) $(2q^2, 2(q^2+2q+2))$ *in order* $4(q^2+q+1)$ *with* $q^2 + q + 1$ *a prime.*

Proof. Use I, W in various combinations in the Geothals-Seidel array for *(i)*, *(ii)*, *(iii)*.

For *(iv)* we note that $W*A = 0$ where A is the incidence matrix of the $(q^2+q+1, q+1, 1)$ configuration satisfying

$$AA^t = qI + J$$

and $*$ is the Hadamard product. For every prime order, p, there exist circulant matrices X, Y satisfying

$$XX^t + YY^t = 2(p+1)I - 2J .$$

Then

$$aW+bA,\ aW-bA,\ bX,\ bY$$

may be used in the Goethals-Seidel array to give the required result.

THEOREM 6. *Since there exists a* $W(q^2+q+1, q^2)$ *for every prime power* q *there exist*

(i) $W(2(q^2+q+1), 2(q^2+1))$ *;*

(ii) $W(4(q^2+q+1), 4(q^2+2))$.

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