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Constructions for amicable orthogonal designs

Jennifer Seberry Wallis

Infinite families of amicable orthogonal designs are constructed with

(i) both of type $(1, q)$ in order $q + 1$ when $q \equiv 3 \pmod 4$ is a prime power,

(ii) both of type $(1, q)$ in order $2(q+1)$ where $q \equiv 1 \pmod 4$ is a prime power or $q + 1$ is the order of a conference matrix,

(iii) both of type $(2, 2q)$ in order $2(q+1)$ when $q \equiv 1 \pmod 4$ is a prime power or $q + 1$ is the order of a conference matrix.

Introduction

The concept of an orthogonal design was first introduced in [1]. An $n \times n$ matrix, X, is an orthogonal design of type $(u_1, u_2, \ldots, u_s)$ on the variables $x_1, x_2, \ldots, x_s$ in order n if X has entries from the set $\{0, \pm x_1, \ldots, \pm x_s\}$ and

$$XX^T = \left(u_1x_1^2 + u_2x_2^2 + \ldots + u_sx_s^2\right)I_n ,$$

where I_n denotes the identity matrix of order n. It was shown in [1] that if there is a pair of orthogonal designs, X, Y, which satisfy the equation $XY^T = YX^T$, then these designs became a powerful tool in the

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construction of new orthogonal designs (for example, see Construction 22 of [1]).

The existence of such designs has been studied further in [3] and limits are given on the number of variables possible in each design. We define

DEFINITION. Two orthogonal designs, X, Y, of the same order, satisfying

$$XY^T = YX^T ,$$

will be called *amicable orthogonal designs.*

In this note we construct infinite families of amicable orthogonal designs.

The constructions

Let $q = p^n$ be a prime power. Then with $a_0, a_1, \ldots, a_{q-1}$ the elements of $GF(q)$ numbered so that

$$a_0 = 0 , \quad a_{q-i} = -a_i , \quad i = 1, \ldots, q-1 ,$$

define $Q = (x_{ij})$ by

$$x_{ij} = \chi(a_j - a_i) ,$$

where χ is the character defined on $GF(q)$ by

$$\chi(x) = \begin{cases} 0 , & x = 0 , \\ 1 , & x = y^2 \text{ for some } y \in GF(q) , \\ -1 , & \text{otherwise.} \end{cases}$$

Then Q is a type 1 matrix (see [2; p. 285-291]) with the properties that

$$(1) \qquad \begin{cases} QQ^T = qI - J , \\ QJ = JQ = 0 , \\ Q^T = \begin{cases} Q & \text{for } q \equiv 1 \pmod 4 , \\ -Q & \text{for } q \equiv 3 \pmod 4 , \end{cases} \end{cases}$$

where I is the identity matrix and J the matrix of all ones.

Now let $U = cI + dQ$ where c, d are commuting variables. Define $R = (r_{ij})$ by

$$r_{ij} = \begin{cases} 1 , & a_i + a_j = 0 , \\ 0 , & \text{otherwise.} \end{cases}$$

Then, as in [2, p. 289] UR is a symmetric type 2 matrix.

We now consider the matrices

$$A = \begin{bmatrix} a & b \dots b \\ -b & \\ \vdots & aI+bQ \\ -b & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -c & d \dots d \\ d & \\ \vdots & (cI+dQ)R \\ d & \end{bmatrix}$$

of order $q + 1$, where a, b, c, d are commuting variables.

We claim that for $q \equiv 3 \pmod 4$,

(i) A and B are orthogonal designs, and

(ii) $AB^T = BA^T$ (this follows since $aI + bQ$ is type 1 and $(cI + dQ)R$ is type 2).

Hence we have

THEOREM 1. *Let $q \equiv 3 \pmod 4$ be a prime power. Then there exists a pair of amicable orthogonal designs of order $q + 1$ and both of type $(1, q)$.*

Further we note that for $q \equiv 1 \pmod 4$ choosing

$$N = \begin{bmatrix} 0 & 1 \dots 1 \\ 1 & \\ \vdots & Q \\ 1 & \end{bmatrix}$$

gives a $(0, 1, -1)$ matrix N satisfying

$$N^T = N , \quad NN^T = qI_{q+1} .$$

Such matrices have been called symmetric conference matrices (see [2, 293, 452]) and we have

THEOREM 2. *Let* $n+1 \equiv 2 \pmod 4$ *be the order of a symmetric conference matrix. Then there exist pairs of amicable orthogonal designs of order* $2(n+1)$ *and both of the pair of type*

(i) $(2, 2n)$,

(ii) $(1, n)$.

Proof. Let N be a symmetric conference matrix and a, b, c, d be commuting variables. Then for *(i)* the required designs are

$$\begin{bmatrix} aI+bN & aI-bN \\ aI-bN & -aI-bN \end{bmatrix} \text{ and } \begin{bmatrix} cI+dN & cI-dN \\ -cI+cN & cI+dN \end{bmatrix},$$

while for *(ii)* they are

$$\begin{bmatrix} aI & bN \\ bN & -aI \end{bmatrix} \text{ and } \begin{bmatrix} cI & dN \\ -dN & cI \end{bmatrix}.$$

COROLLARY. *Let* $q \equiv 1 \pmod 4$ *be a prime power. Then there exist pairs of amicable Hadamard designs of order* $2(q+1)$ *where both of the pair are of type* $(2, 2q)$ *or of type* $(1, q)$.

References

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[3] Warren W. Wolfe, "Clifford algebras and amicable orthogonal designs", Queen's Mathematical Preprint No. 1974-22.

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.