A SURVEY OF ORTHOGONAL DESIGNS

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Abstract

This paper surveys orthogonal designs which are an overview of Baumert-Hall arrays, Hadamard matrices and weighing matrices.

The known results are given and unsolved problems indicated.

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§1 INTRODUCTION AND DEFINITIONS

This paper is intended to survey orthogonal designs and highlight the unsolved problems in the area. No proofs are given but the source of the result is indicated.

Definition. An *orthogonal design of order* n *and type* $(s_1,\ldots,s_\ell)$ on the commuting variable $x_1,\ldots,x_\ell$ is an $n\times n$ matrix, A, with entries chosen from $\{0,\pm x_1,\pm x_2,\ldots,\pm x_\ell\}$ such that

$$AA^t = (s_1x_1^2 + \ldots + s_\ell x_\ell^2)I_n \quad .$$

Alternatively, the rows (and hence columns) of A are formally orthogonal and every row (column) contains s_i entries of the type $\pm x_i$.

If A is as above, we may write

$$A = x_1A_1 + x_2A_2 + \ldots + x_\ell A_\ell$$

where

i) $A_iA_i^t = s_iI_n$

ii) $A_iA_j^t + A_jA_i^t = 0 \quad , 1\le i\ne j\le \ell$

iii) the A_i are $\{0,1,-1\}$ matrices,

iv) $A_i*A_j = 0$ for $i \ne j$ (* is the Hadamard product).

It was shown in [7] that $\ell \leq \rho(n)$, where $\rho(n)$ *(Radon's function)* is defined by

$$\rho(n) = 8c + 2^d$$

when

$$n = 2^a \cdot b \ , \quad b \text{ odd}, \quad a = 4c + d \quad 0 \leq d < 4 \ .$$

<u>Definition</u> A *weighing matrix of weight* k *and order* n, is a square $\{0,1,-1\}$ matrix, A, of order n satisfying

$$AA^t = kI_n$$

In [7] it was shown that the existence of an orthogonal design of order n and type $(s_1,\ldots,s_\ell)$ is equivalent to the existence of weighing matrices $A_1,\ldots,A_\ell$, of order n, where A_i has weight s_i and the matrices, $\{A_i\}_{i=1}^{\ell}$, satisfy the matrix equations

$$XY^t + YX^t = 0 \quad \text{and} \quad A_i * A_j = 0, \ i \neq j.$$

in pairs. In particular, the existence of an orthogonal design of order n and type $(1,k)$ is equivalent to the existence of a skew-symmetric weighing matrix of weight k and order n.

Weighing matrices are a generalization of *Hadamard matrices*, H, which are (1,-1) -matrices satisfying

$$HH^t = nI_n,$$

where n is the order of the matrix. It is conjectured that Hadamard matrices exist for every order $n \equiv 0 \pmod 4$. Further it is conjected that for these orders there exists an Hadamard matrix (called a *skew-Hadamard* matrix)of the form $H = I_n + S$ where $S^t = -S$.

Let R be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to be *constructed from two circulant matrices* A and B if it is of the form

$$\begin{bmatrix} A & B \\ B^t & -A^t \end{bmatrix}$$

and to be of *Goethals-Seidel type* if it is of the form

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^tR & -C^tR \\ -CR & -D^tR & A & B^tR \\ -DR & C^tR & -B^tR & A \end{bmatrix}$$

where A,B,C,D are circulant matrices.

Two weighing matrices W and N of order n and weights w_1 and w_2 will be called *amicable weighing matrices* if

$$W^t = -W,\ N^t = N$$

$$WN^t = NW^t$$

If I + W and N are Hadamard matrices satisfying these equations then they are called *amicable Hadamard matrices* (see [28]).

§2 THE CONJECTURES

It is conjectured that:

(I) for $n \equiv 2 \pmod 4$ there is a weighing matrix of weight k and order n for every $k \leq n-1$ which is the sum of two integer squares;

(II) for $n \equiv 2 \pmod 4$ there is a skew-symmetric weighing matrix of order n for every $k < n-1$ such that k is an integer square; or equivalently

(IIA) for $n \equiv 2 \pmod 4$ there is an orthogonal design of type $(1,k)$ in order n for every $k < n-1$ such that k is an integer square;

(III) for $n \equiv 0 \pmod 4$ there is a weighing matrix of weight k and order n for every $k \leq n$;

(IV) for $n \equiv 4 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$, where k is the sum of $\leq$ three squares of integers;

or equivalently

(IVA) there is an orthogonal design of type $(1,k)$ in order $n \equiv 4 \pmod 8$ for every $k < n$ which is the sum of $\leq$ three squares of integers;

(V) for $n \equiv 0 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$;

or equivalently

(VA) there is an orthogonal design of type $(1,k)$ in order $n \equiv 0 \pmod 8$ for every $k < n$

Conjecture III above is an extension of the Hadamard conjecture (i.e. for every $n \equiv 0 \pmod 4$ there is a $\{1,-1\}$ matrix, H, of order n satisfying $HH^t = nI_n$) while (IV) and (V) generalize the conjecture that for every $n \equiv 0 \pmod 4$ there is a Hadamard, H, matrix of order n, with the property that $H = I_n + S$ where $S = -S^t$.

Conjecture (III) was established in [29] for $n \in \{4,8,12,\ldots,32,40\}$ and in [10] for $n = 2^t$. Conjecture (IV) was established in [7, Theorem 17] for $n = 2^t$ $(t \geq 3)$. In [11] conjectures (IV) and (V) (and as a consequence conjecture (III) were established for $n = 2^{t+1}\cdot 3$, $n = 2^{t+1}\cdot 5$, t a positive integer. Also it is only necessary to find a design $(1,47)$ in order 56 and the conjectures (IV) and (IVA) (and hence (III) will be validated for $n = 2^t.7$ $(t \geq 3)$.

Some conjectures that come to mind on orthogonal designs are:

(i) A necessary and sufficient condition that there exist a design of type (a,b) in order $2t$, t odd, is that $\frac{b}{a}$ is a rational square and that $a+b < 2t$. (We doubt this is true.)

(ii) A necessary and sufficient condition that there exist a design of type (a,a,b) in order $n=4t$, t odd, is that $\frac{b}{a}$ be a sum of two rational squares;

(iii) A necessary and sufficient condition that there exist a design of type (a,a,a,b) in order $n=4t$, t odd, is that $\frac{b}{a}$ be a rational square;

(iv) A necessary and sufficient condition that there exist a design of type (a,b) in order n=4t , t odd, is that $\frac{b}{a}$ be a sum of three rational squares.

§3 EXISTENCE THEOREMS : NECESSARY CONDITIONS

Lemma 1 [7]. *The maximum number of variables in an orthogonal design of order* n *is* $\rho(n)$ *where* $\rho(n)$ *is the Radon number.*

Lemma 2. *(Raghavarao-van Lint-Seidel). Let* $n \equiv 2 \pmod 4$ *and let* A *be a matrix of order* n *with entries in* Q *satisfying* $AA^t = kI_n$ *. Then* $k = q_1^2 + q_2^2$ *where* $q_1 q_2 \in Q$ *. Moreover, if* $k \in Z$ *then* $k = a^2 + b^2$, $a,b \in Z$.

Corollary [7]. *If* $n \equiv 2 \pmod 4$ *and there is an orthogonal design of order* n *and type* (a,b) *, then* a,b *and* $a+b$ *must be the sum of* ≤ 2 *integer squares and* $a + b < n$.

Lemma 3 [13]. *Let* X *be a matrix of order* $n \equiv 2 \pmod 4$ *with entries in the field* $Q(i)$ $(i^2=-1)$ *. Suppose*

$$i) \quad X = -X^t$$

and

$$ii) \quad XX^* = kI_n \quad .$$

Then $k = q_1^2 + q_2^2$ *where* $q_1, q_2 \in Q$ *. If, in addition* $k \in Z$ *then* q_1 *and* q_2 *may be chosen in* Z .

Lemma 4 [13]. *Let* X, Y *be matrices of order* $n = 2s$ *(s odd) with entries in* $Q(i)$*. Suppose*

$$(i) \quad X = -X^t, \quad Y = -Y^t,$$

$$\begin{array}{ll} (ii) & XX^* = I_n, \quad YY^* = kI_n, \\ (iii) & XY^* + YX^* = 0, \end{array} \left\{ \begin{array}{l} \textit{where } * \textit{ is the} \\ \textit{conjugate transpose.} \end{array} \right.$$

then $k = a^2$ *for some* $a \in Q$. *If, in addition,* $k \in Z$ *then "a" may be chosen in* Z.

<u>Lemma 5 [13]</u>. *If* $n \equiv 2 \pmod 4$ *and there is an orthogonal design of order* n *and type* (a,b) *then* $\frac{b}{a}$ *is a rational square.*

<u>Lemma 6 [7]</u>. *Let* $n \equiv 4 \pmod 8$ *and let* X *be an orthogonal design of order* n *and type* (a,b) . *Then* $\frac{b}{a}$ *must be a sum of* ≤ 3 *rational squares.*

<u>Lemma 7 [13]</u>. *Let* $n \equiv 4 \pmod 8$ *and let* X *be an orthogonal design of order* n *and type* (a,a,b). *Then* $\frac{b}{a}$ *must be a sum of* $\leq$ *2 rational squares.*

<u>Lemma 8 [13]</u>. *Let* $n \equiv 4 \pmod 8$ *and let* X *be an orthogonal design of order* n *and type* (a,a,a,b) *then* $\frac{b}{a} = t^2$, $t \in Q$.

§4 THE MOST POWERFUL CONSTRUCTIONS

<u>Baumert-Hall Arrays</u>:

Let $n = 4t$, t odd, then $\rho(n) = 4$. The *Baumert-Hall Arrays* are orthogonal designs of order n and type (t,t,t,t). These exist for a large number of odd t (see [1], [3], [5], [18], [25], [27]). and it has been conjectured they exist for all odd t.

For a discussion of these array see [33; Part 4, chapter VII].

Designs of type $(1,1,\ldots,1)$:

Let n be any integer. In [9] it was shown that there is an orthogonal design of order n and type $(1,1,\ldots,1)$ on the variables $x_1,\ldots,x_{\rho(n)}$.

Plotkin's Array:

Let $n = 8t$, t odd, then $\rho(n) = 8$. A *Plotkin array* is an orthogonal design of order n and type (t,t,t,t,t,t,t,t). If $t = 1$ such an array is a classical one derived from the multiplication of the Cayley numbers.

In his paper Plotkin shows:

Lemma 1 [21]. Let $n = 4t$ be the order of an Hadamard matrix then there are orthogonal designs of type

(i) $(2t,2t)$ in order $4t$;

(ii) $(2t,2t,2t,2t)$ in order $8t$;

(iii) $(2t,2t,2t,2t,2t,2t,2t,2t)$ in order $16t$.

We now give some of the most powerful constructions for orthogonal designs:

Lemma 2 [7]. *If A is an orthogonal design of order n and type $(s_1,s_2,\ldots,s_\ell)$ on the variables $x_1,x_2,\ldots,x_\ell$ then there is an orthogonal design of order n and type $(s_1,\ldots,s_i + s_j,\ldots,s_\ell)$ on the $\ell-1$ variables $x_1,x_2,\ldots,\hat{x}_j,\ldots,x_\ell$.*

Example. An orthogonal design of type $(1,2,3)$ on the variables x_1,x_2,x_3 in order n means there are also orthogonal designs $(1,5)$, $(2,4)$, $(3,3)$ on the variables $\hat{x}_1,\hat{x}_2$ in order n.

Lemma 3 [7]. *If there exists an orthogonal design of type* $(s_1,\ldots,s_\ell)$ *on the variables* $x_1,\ldots,x_\ell$ *in order* n *then there exists an orthogonal design of type* $(s_1,\ldots s_\ell)$ *on the variables* $x_1,\ldots,x_\ell$ *in order* mn *for any integer* $m \geq 1$.

Lemma 4 [7]. *If there exists an orthogonal design of type* $(s_1,\ldots,s_\ell)$ *in order* n *then there exists an orthogonal design of type* $(\varepsilon_1 s_1,\varepsilon_2 s_2,\ldots,\varepsilon_\ell s_\ell)$ *in order* $2n$, $\varepsilon_i = 1$ *or* 2.

Lemma 5 [7]. *Suppose there exist amicable weighing matrices of weights* k *and* m *and order* p *then there exists an orthogonal designs of order* np *and type* $(1,k,m,\ldots.,m)$ *on the variables* $x_0,x_1,x_2,\ldots,x_{\rho(n)}$.

Lemma 6 [7]. *Suppose there exist two circulant matrices* A, B *of order* n *satisfying*

$$AA^t + BB^t = fI_n .$$

Further suppose the R *is the back diagonal matrix. Then*

$$G = \begin{bmatrix} A & BR \\ -BR & A \end{bmatrix}$$

is a $W(2n,f)$ *when* A,B *are* $(0,1,-1)$*-matrices and an orthogonal design of order* $2n$ *and type* $(s_1,s_2,\ldots,s_\ell)$ *on* $x_1,\ldots x_\ell$ *when*

$$f = \sum_{i=1}^{\ell} s_i x_i^2 .$$

Further G *is skew or skew-type if* A *is skew or skew-type.*

Lemma 7 (Goethals and Seidel [15]).

Suppose there exist four circulant matrices A, B, C, D *of order* n *satisfying*

$$AA^t + BB^t + CC^t + DD^t = fI_n.$$

Let R *be the back-diagonal matrix. Then*

$$GS = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^tR & -C^tR \\ -CR & -D^tR & A & B^tR \\ -DR & C^tR & -B^tR & A \end{bmatrix}$$

is a $W(4n,f)$ *when* A, B, C, D *are* $(0,1,-1)$ *matrices, and an orthogonal design of order* $4n$ *and type* $(s_1, s_2, \ldots, s_\ell)$ *on* $x_1, x_2, \ldots x_\ell$ *when*

$$f = \sum_{j=1}^{\ell} s_j x_j^2 .$$

Further GS *is skew or skew-type if* A *is skew or skew-type.*

Lemma 8 [7]. *If there exists an orthogonal design of type* (a,b) *in order* n *there exists an orthogonal design of type* (a,a,b,b) *in order* $2n$.

Lemma 9 [7]. *If there exists an orthogonal design of type* (a,b) *in order* n *there exists an orthogonal design of type* $(a,a,2a,b,b,2b)$ *in order* $4n$.

Lemma 10 [7]. *If there exists an orthogonal design of type* $(s_1, s_2, \ldots, s_i)$ *in order* n_1 *and* n_2 *then there exists an orthogonal design of type* $(s_1, s_2, \ldots, s_i)$ *in order* $n_1 + n_2$.

Lemma 11 [11], [14]. *There exist orthogonal designs* $(1,k)$ *in orders* $2^t m$, $t \geq 3$ *an integer and* $m=1,3,5$ *or* 9.

Part (i) of the following lemma was also discovered by Joan Murphy Geramita.

Lemma 12 [7], [14]. *If there exists an orthogonal design of type* $(s_1,s_2,\ldots,s_\ell)$ *in order* n *there exists orthogonal designs*

(i) $(s_1,s_1,s_2,s_3,\ldots,s_\ell)$ *and*

(ii) $(s_1,s_1,2s_2,2s_3,\ldots,2s_\ell)$

in order $2n$.

§5 APPENDICES

APPENDIX A: The weighing matrix problem : known results.

Lemma 1 [7]. *If a* $W(n,s)$ *exists and*

(i) n *is odd then* s *is a square;*

(ii) $n \equiv 2 \pmod 4$ *then* s *is the sum of two squares;*

(iii) $(n-s)^2 - (n-s) + 2 > n$ *for odd* n;

(iv) $W(m^2 + m + 1, m^2)$ *exists only if* m *is the order of a projective plane.*

Lemma 2 [7]. *If there exists a* $W(n,k)$ *and* $n \equiv 2 \pmod 4$ *then*

(i) k *is the sum of two integer squares;*

(ii) $k \leq n - 1$.

Given these two results we have the conjectures:

(I) *for* $n \equiv 2 \pmod 4$ *there is a weighing matrix of weight* k *and order* n *for every* $k \leq n - 1$ *which is the sum of two integer squares;*

(III) for $n \equiv 0 \pmod 4$ there is a weighing matrix of weight k and order n for every $k \le n$.

As noted before conjecture III is an extension of the Hadamard conjecture.

Lemma 3 [29] [10]. *Conjecture III is true for*

(i) $n \in N = \{4, 8, 12, \ldots, 32, 40\}$;

(ii) $n = 2^t.m$ if $t = 2$ for $m = 1$, 3, or 5, and if $t \ge 3$ for $m = 1$, 3, 5, or 9.

We note that the existence of an orthogonal design $(1,k)$ of order n implies the existence of a $W(n,k)$ and a $W(n,k+1)$. Thus the appendix on designs $(1,k)$ should be consulted for numerical results.

APPENDIX B: The skew weighing matrix problem : known results.

Lemma 1 [7]. *Let $n \equiv 2 \pmod 4$. If X is a skew-symmetric rational matrix of order n satisfying*

$$XX^t = kI_n$$

then k is a rational square. If k is an integer it is an integer square.

Lemma 2 [7]. *Let $n \equiv 4 \pmod 8$. If X is a skew-symmetric rational matrix of order n satisfying*

$$XX^t = kI_n$$

then k is the sum of three rational squares. If k is integer it is the sum of three integer squares.

Lemma 3 [11]. *Let t be an integer ≥ 3.*

(i) Conjecture IV is true for 4m, m=1,3,5, or 7;

(ii) Conjecture V is true for $2^t m$, m=1,3,5 or 9.

Now the existence of a skew-weighing matrix W(n,k) is equivalent to the existence of an orthogonal design (1,k) of order n so the appendix on the designs (1,k) should **be consulted for numerical** results on this conjecture.

APPENDIX C: The design (1,k) problem : known results.

Most of these results are cited from [11], [12], and [14]. We are concerned with conjectures

(II) for $n \equiv 2 \pmod 4$ there is an orthogonal design of type (1,k) in order n for every $k < n-1$ such that k is an integer square;

(IVA) for $n \equiv 4 \pmod 8$ there is an orthogonal design of type (1,k) in order n for every $k < n$ which is the sum of $\leq$ three squares of integers.

(VA) for $n \equiv 0 \pmod 8$ there is an orthogonal design of type (1,k) in order n for every $k < n$.

We now cite the theorems relevant to this problem

Lemma 1 [7, Corollary to Construction 22]. *If there is an orthogonal design of type $(1,\ell)$ in order n then there is an orthogonal design of type $(1,1,\ell,\ell)$ in order $2n$ and of type $(1,1,2,\ell,\ell,2\ell)$ in order $4n$.*

Corollary 1. *If there are orthogonal designs of type* $(1,k)$ $1 \le k \le \ell$ *in order* n *then there are orthogonal designs of type* $(1,m)$, $1 \le m \le 2\ell + 1$ *in order* $2n$.

Corollary 2. *If there are orthogonal designs of type* $(1,k)$, $1 \le k \le n-1$ *in order* n *then there are orthogonal designs of type* $(1,m)$, $1 \le m \le 2^t n - 1$ *in order* $2^t n$, t *a positive integer.*

Lemma 2. *Let* n *be any number of the form* $2^t \cdot 3$, $2^t \cdot 5$ *or* $2^t \cdot 9$, *with* t *a positive integer.*

(a) *If* $t = 1$, *then conjecture I is true (except for* 2.9*).*

(b) *If* $t = 2$ *then conjecture IVA is true (except for* $2^2.9$*).*

(c) *If* $t \ge 3$ *then conjecture VA (and consequently conjecture III) is true.*

Lemma 3.

(a) *Conjecture I is true for* $n=2\cdot 7$.

(b) *For* $n=4\cdot 7$ *conjecture IVA is true.*

(c) *For* $n=8\cdot 7$, *there is an orthogonal design of type* $(1,k)$ *for* $1 \le k \le 55$, *except possibly for* $k = 47$.

Lemma 4. *Let* r *be any number of the form* $2^a \cdot 10^b \cdot 26^c$, a,b,c *non-negative integers, and let* n *be any integer* $> r$. *Then*

(i) *There are orthogonal designs of order* $4n$ *and types* $(1,1,2r)$ *and* $(1,1,r)$.

If, in addition n *is odd, then*

(ii) *there are orthogonal designs of order* $4n$ *and types* $(1,4,r)$ *and* $(1,4,2r)$.

<u>Lemma 5.</u> *The following orthogonal designs exist in the orders $2n$ as indicated:*

(i) $(1,1)$ *in orders* $2n \geq 2$;

(ii) $(1,4)$ *in orders* $2n \geq 6$;

(iii) $(1,9)$ *in orders* $12,14,16,20$, $2n \geq 24$;

<u>Lemma 6.</u> *The following orthogonal designs exist in the orders $4n$ as indicated:*

(i) $(1,1,13)$ *in orders* $4n$, $n(odd) \geq 11$;

(ii) $(1,1,16)$ *in orders* $4n$, $n(odd) \geq 5$;

(iii) $(1,1,18)$ *in orders* $20,24,28,32$ *and all* $4n \geq 40$;

(iv) $(1,1,20)$ *in orders* $24,28,32$ *and all* $4n \geq 40$;

(v) $(1,1,26)$ *in orders* $4n$, $n(odd) \geq 15$;

(vi) $(1,1,32)$ *in orders* $4n$, $n(odd) \geq 9$;

(vii) $(1,1,34)$ *in order* 36 *(from [30])*;

(viii) $(1,1,40)$ *in orders* $4n$, $n(odd) \geq 11$;

(ix) $(1,1,1,4)$ *in orders* $4n \geq 8$;

(x) $(1,1,1,9)$ *in orders* $4n \geq 12$;

(xi) $(1,1,1,16)$ *in orders* $4n$, $n(odd) \geq 7$;

(xii) $(1,1,1,25)$ *in order* 28;

(xiii) $(1,1,2,8)$ *in orders* $4n \geq 12$;

(xiv) $(1,1,5,5)$ *in orders* $4n \geq 12$;

(xv) $(1,2,2,4)$ *in orders* $4n \geq 12$;

(xvi) $(1,2,3,6)$ *in orders* $4n \geq 12$.

Using all means available we have (at least)·

Lemma 7. *If* n *is odd there are orthogonal designs of order* $4n$ *and type* $(1,k)$ *when*

(i) $n \geq 3$, $k \in \{1,\ldots,6,8,\ldots,11\}$;

(ii) $n \geq 5$, $k \in \{1,\ldots,6,8,\ldots,14,16,17,18\}$;

(iii) $n \geq 7$, $k \in \{1,\ldots,6,8,\ldots,14,16,17,18,20,21,22,24,\ldots,27\}$;

(iv) $n \geq 9$, $k \in \{1,\ldots,6,8,\ldots,14,16,17,18,20,21,22,24,\ldots,27,29,32,33,34\}$;

(v) $n \geq 11$, $k \in \{1,\ldots,6,8,\ldots,14,16,\ldots,22,24,\ldots,27,29,32,33,34,40,41\}$;

(vi) $n \geq 13$, $k \in \{1,\ldots,6,8,\ldots,14,16,\ldots,22,24,\ldots,27,29,30,32,33,34,40,41\}$;

Lemma 8. *There are orthogonal designs of order* $8n$ *and type* $(1,k)$ *where*

(i) $n = 3,4,5$ or 6, $k \in \{1,\ldots,8n-1\}$;

(ii) $n \geq 7$, $k \in \{1,\ldots,46\}$.

Lemma 9 [14]. *Given a square integer* k *there exists an integer* $N(k)$ *such that*

(i) $(1,k)$ *exists in every order* $2n$ *for* $n > N(k)$,

(ii) $(1,1,k)$ *exists in every order* $4n$ *for* $n > N(k)$.

Finally we tabulate the unresolved cases in the conjectures.

Order	Unresolved Cases on the conjecture that weighing matrices exist for all k < n	Unresolved Cases of appropriate conjecture on the existence of (1,k)
4	true	true
8	true	true
12	true	true
16	true	true
20	true	true
24	true	true
28	true	true
32	true	true
36	31	19,30
40	true	true
44	31	30,42
48	true	true
52	37,46,47,49	35-38,42-46, 48-50
56	true	47
60	51,53	38,42,43,44, 48,50-54, 56-58
64	true	true
72	true	true
80	true	true

Table 1

True signifies the conjecture is verified.

Order	Unresolved Cases of conjecture the weighing matrices exist for all $k < n$ which are the sum of two squares	Unresolved Cases of conjecture that (1,k) exists for all $k < n$ which are squares
2	true	true
6	true	true
10	true	true
14	true	true
18	9	9,16
22	18	9
26	17	16
30	18	16,25

Table 2

True signifies the conjecture is verified.

APPENDIX D: The two variables problem.

The Radon number indicates that designs in two variables can exist only in even orders. The appropriate conjectures are:

IVA there is an orthogonal design of type (1,k) in order $n \equiv 4 \pmod 8$ for every $k < n$ which is the sum of $\leq$ three squares of integers;

VA there is an orthogonal design of type (1,k) in order $n \equiv 0 \pmod 8$ for every $k < n$;

(i) for $n \equiv 2 \pmod 4$ a necessary and sufficient condition that there exist a design of type (a,b) is that $\frac{b}{a}$ is a rational square and that $a + b < n$;

(iv) for $n \equiv 4 \pmod 8$ a necessary and sufficient condition that there exist a design of type (a,b) is that $\frac{b}{a}$ is a sum of three rational squares.

Some relevant results are Lemmas 2, 3, 4, 5 and 6 of §3. We note that Lemmas 5 and 6 of §3 establish the necessary conditions of conjectures (i) and (iv).

For the known results on designs of order $n \equiv 2 \pmod 4$ we refer the reader to table 2 of Appendix C.

Lemma 1. *If there exist all orthogonal designs $(1,k)$, $k < n$, in order n then there exist all orthogonal designs $(1,\ell)$, $\ell < 2n$, in order $2n$.*

Corollary. *There exist all orthogonal designs $(1,k)$ in order 2^t, $2^t.3$, $2^t.5$ and $2^t.9$, $t \geq 3$ and integer.*

Lemma 2. *The existence of all designs $(1, i, j)$ in order n implies the existence of all designs (k,ℓ) in order $2n$.*

Lemma 3. *Let $n=12$. Then $\frac{b}{a}$ is the sum of three rational squares is a necessary and sufficient condition for the existence of designs (a,b) in order 12.*

We give the results on two variables for a few other orders.

order 16: all two variable designs exist.

order 20: the cases (3,16) , (6,13) , (7,10) and (7,12) remain unsolved.

order 24: all two variable designs exist.

order 28: the status of the two variable problem is that (j,k) exists for each j except for those k explicitly excluded in the following list.

(2,k) , $k \in \{x : 0 \le x \le 26,\ x \ne 22, 23, 24\}$

$k \in \{14\}$ is ruled out by Lemma (iv) of §3.

(3,k) , $k \in \{x : 0 \le x \le 25,\ x \ne 7,10,12,14,15,22,23,24\}$

$k \in \{5,13,20,21\}$ are not possible.

(4,k) , $k \in \{x : 0 \le x \le 24,\ x \ne 19,20,21,22,24\}$

$k \in \{7,15,23\}$ are not possible.

(5,k) , $k \in \{x : 0 \le x \le 23,\ x \ne 20,21,22,23\}$

$k \in \{11,12,19\}$ are not possible.

(6,k) , $k \in \{x : 0 \le x \le 22,\ x \ne 7,11,13,15,17,18,19,20,21,22\}$

$k \in \{10\}$ is not possible.

(7,k) , $k \in \{x : 0 \le x \le 21,\ x \ne 10,11,12,14,15,16,18,19,20,21\}$

$k \in \{9,17\}$ are not possible.

(8,k) , $k \in \{x : 0 \le x \le 20,\ x \ne 17\}$

$k \in \{14\}$ is not possible.

(9,k) , $k \in \{x : 0 \le x \le 19,\ x \ne 12,16,17\}$

$k \in \{15\}$ is not possible.

(10,k) , $k \in \{x : 0 \le x \le 18,\ x \ne 14,15,16,17\}$

(11,k) , $k \in \{x : 0 \le x \le 17,\ x \ne 12,14,15,16,17\}$

$k \in \{13\}$ is not possible.

(12,k) , $k \in \{x : 0 \le x \le 16,\ x \ne 12,13,14,15,16\}$

(13,k) all exist

(14,k) all exist

order 32: all two variable designs exist.

order 36: all (1,k) designs except (1,19) and (1,30) are known. The other two variable designs have not yet been studied.

order 40: the following two variable designs are not known:

(5,32)	(7,25)	(8,31)
(6,29)	(7,32)	(9,30)
(6,31)	(8,29)	(14,23)
(6,33)		

order 44: all (1,k) designs except (1,30) and (1,42) are known. The other two variable designs have not yet been studied.

order 48: all two variable designs are known.

orders 52, 56, 60, 64, 72, 80: the status of the (1,k) designs is given in table 1 of Appendix C. Other two variable designs have not been studied.

APPENDIX E: The four variables problem.

The Radon number indicates that designs in four variables can only exist in orders $n \equiv 0 \pmod 4$. As the number of variables increase the solving of the existence problem becomes correspondingly more difficult. The Goethals-Seidel method is extremely useful but not all designs can be obtained by this method.

The appropriate conjectures are:

(iii) A necessary and sufficient condition that there exist a design of type (a,a,a,b) in orders $n \equiv 4 \pmod 8$ is that $\frac{b}{a}$ be a rational square;

(v) All four variable designs not ruled out by conjectures (ii), (iii), (iv) exist in orders $n \equiv 4 \pmod 8$.

(vi) All four variable designs exist in orders $n \equiv 0 \pmod 8$.

Lemmas 6, 7, 8 of §3 apply and we recall the essence of these

Lemma 1. *Let $n \equiv 4 \pmod 8$ and let X be an orthogonal design of order n and type*

(i) (a,b). Then $\frac{b}{a}$ must be a sum of ≤ 3 rational squares.

(ii) (a,a,b). The $\frac{b}{a}$ must be a sum of two rational squares.

(iii) (a,a,a,b). Then $\frac{b}{a}$ must be a rational square.

order 8: all four variable designs exist.

order 12: all possible four variable designs exist i.e. Conjecture (V) is true in order 12.

order 16: all four variable designs exist except (1,5,5,5).

order 20: the following designs in order 20 are not eliminated by the necessary conditions but are not known either.

(1,1,1,16)	(1,2,2,9)	(1,5,5,8)
(1,1,2,16)	(1,2,3,9)	(1,5,5,9)
(1,1,5,8)	(1,2,6,11)	(2,2,5,5)
(1,1,5,13)	(1,2,8,9)	(2,3,7,8)
(1,1,8,9)	(1,3,6,8)	(3,3,6,6)
(1,1,8,10)	(1,4,4,9)	
(1,2,2,8)	(1,4,5,5)	

order 24: There are 369 four variable designs which have not yet been found.

higher orders The situation deteriorates rapidly with little being known.

The most useful constructions are given in the following results:

Lemma 2 [7]. *If there exists an orthogonal design of type* (a,b) *in order* n *then there exists an orthogonal design of type*

(i) (a,a,b,b) *in order* $2n$;

(ii) $(a,a,2a,b,b,2b)$ *in order* $4n$.

Lemma 3. *If there exists an orthogonal design of type* $(s_1,s_2,\ldots,s_\ell)$ *in order* n *then there exists an orthogonal design of type*

(i) $(s_1,s_2,\ldots,s_i+s_{i+1},s_{i+2},\ldots,s_\ell)$ *in order* n;

(ii) $(s_1,s_1,es_2,es_3,\ldots,es_\ell)$ *with* $e = 1$ *or* 2 *in order* $2n$;

(iii) $(e_1s_1,e_2s_2,e_3s_3,\ldots,e_\ell s_\ell)$ *with* $e_i = 1$ *or* 2 *in order* $2n$;

(iv) $(s_1,s_2,\ldots,s_\ell)$ *in order* $n+m$ *whenever a design of type* $(s_1,s_2,\ldots,s_\ell)$ *also exists in order* m.

The known results on a few orders follow.

APPENDIX F: Unsolved Questions and Known Results for n odd.

Lemma 1 [7]. *An orthogonal design of odd order,* n *, has the following properties*

(i) there is only one variable so we have a weighing matrix;

(ii) the weight of the variable, k *, is a square;*

(iii) $(n - k)^2 + (n - k) + 2 > n$*;*

(iv) a $W(m^2 + m + 1, m^2)$ *exists only if a projective plane of order* m *exists.*

Lemma 2 [13]. *Given a square* k *there exists an integer* $N(k)$ *such that an orthogonal design of weight* k *exists for every order* $n > N(k)$*.*

Lemma 3[13] [32] [37] *With* $N(k)$ *as in the previous lemma* $N(4) = 10$, $N(9) \le 22$, $N(16) \le 36$, $N(25) \le 82$, $N(36) \le 198$.

In the table we give the known results for some orders. These results are taken from [7], [13], and [32].

Order	Weights not eliminated by Lemma 1(iii)	Non-existent weights	Constructed weights	Unsolved weights
5	nil			
7	4		4	
9	4	4,9		
11	4	9	4	
13	4,9		4,9	
15	4,9		4	9
17	4,9	16	4	9
19	4,9	16	4	9
21	4,9,16		4,16	9
23	4,9,16		4,9	16
25	4,9,16	25	4,9	16
27	4,9,16	25	4	9,16
29	4,9,16	25	4,9	16
31	4,9,16,25		4,25	9,16
33	4,9,16,25		4,9	16,25
35	4,9,16,25		4,9,16	25
37	4,9,16,25	36	4,9,16	25
> 11			4	
> 21			4,9	
> 37			4,9,16	
> 83			4,9,16,25	
> 199			4,9,16,25, 36	

Table

Problems: (i) Find new relations between n and k.

(ii) Lower the bounds N(k) by finding the unsolved weights.

APPENDIX G : Unsolved Questions and Known Results for $n \equiv 2 \pmod 4$.

We recall that for $n \equiv 2 \pmod 4$ there can be orthogonal designs on only one or two variables. Further

Lemma 1 [7]. *If there exists an orthogonal design of order* n *and type* (a,b) *then*

(i) a,b *and* $a + b$ *must all be the sum of two integer squares;*

(ii) $\frac{b}{a} = \frac{ab}{a^2}$ *is a rational square (ie* ab *is an integer square);*

(iii) $a + b \le n - 1$.

Corollary. *If there exists a* $W(n,k)$ *then*

(i) k *is the sum of two integer squares;*

(ii) $k \le n - 1$.

Lemma 2 [12]. *There exists a* $W(n,k)$, *constructed using two circulant matrices for* $k \in \{0,1,2,4,5\}$ *in every order* $n \ge 6$.

Lemma 3. *There exists a* $W(n,9)$ *for every* $n \ge 22$.

Lemma 4 [32]. *There exists a* $W(n, 16)$ *for every* $n \ge 18$.

Lemma 5. *There exists a* $(1,16)$ *in every order* $n \ge 26$.

Lemma 6 [32]. *There exists a* $(1,9)$ *in every order* $n \ge 42$.

Lemma 7. *Given a square* k *there exists an integer* $M(k)$ *such that an orthogonal design* $(1,k)$ *exists for every* $n > M(k)$.

Lemma 8 [13], [32], [37] *With* $M(k)$ *as is the previous lemma, for* $n \equiv 2 \pmod 4$, $M(4) = 6$, $M(9) \le 18$, $M(16) = 18$, $M(25) \le 54$, $M(36) \le 106$.

We now consider the different orders with respect to

The weighing matrix conjecture: there exists a $W(n,k)$ for every $k < n$ which is the sum of two squares.

The skew weighing matrix conjecture: there exists an orthogonal design $(1,k)$ for every $k < n - 1$ which is an integer square.

The orthogonal design conjecture: Lemma 1 gives necessary and sufficient conditions for the existence of orthogonal designs of type (a,b).

The known results are summarized in the table. The entry "nil" signifies the conjecture is true for this order. See [7], [12], [29], [32], [37], [39] for details.

Integers which are the sums of two squares: 0,1,2,4,5,8,9, 10,13,16,17,18,20,26,29,32,34,36,37,40,41,45,49,50,....

Order	Unsettled Cases for weighing matrix conjecture	Unsettled Cases for skew weighing matrix conjecture	Unsettled Cases for orthogonal design conjecture
2	nil	nil	nil
6	nil	nil	nil
10	nil	nil	nil
14	nil	nil	nil
18	nil	9,16	
22	18	9	
26	17	16	
30	18	16,25	
34		16,25	
38		16,25,36	
42	18,25,29,36,37	25,36	

Problems: (i) study the existence of orthogonal designs in orders ≥ 14.

(ii) look for weighing (skew-weighing) matrices in the unsettled cases.

(iii) lower the bounds on $M(9)$, $M(25)$, $M(36)$ and find $M(k)$ for other k.

APPENDIX H: Unsolved questions and known results for $n \equiv 4 \pmod 8$.

Here we are concerned with the following conjectures:

(III) for $n \equiv 4 \pmod 8$ there is a weighing matrix of weight k and order n for every $k \leq n$;

(IV) for $n \equiv 4 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$, where k is the sum of $\leq$ three integer squares;

(IVA) there is an orthogonal design of type $(1,k)$ in order $n \equiv 4 \pmod 8$ for every $k < n$ which is the sum of $\leq$ three integer squares;

(ii) a necessary and sufficient condition that there exist a design of type (a,a,b) in order $n \equiv 4 \pmod 8$ is that $\frac{b}{a}$ be a sum of two rational squares;

(iii) a necessary and sufficient condition that there exist a design of type (a,a,a,b) in order $n \equiv 4 \pmod 8$ is that $\frac{b}{a}$ be a rational square;

(iv) a necessary and sufficient condition that there exist a design of type (a,b) in order $n \equiv 4 \pmod 8$ is that $\frac{b}{a}$ be a sum of three rational squares.

The following theorems apply

Lemma 1 [7]. *Let $n \equiv 4 \pmod 8$. The maximum number of variables in an orthogonal design of order n is 4.*

Lemma 2. [7]. *Let $n \equiv 4 \pmod 8$ and let X be an orthogonal design of order n and type (a,b). Then $\frac{b}{a}$ must be a sum of ≤ 3 rational squares.*

Lemma 3 [13]. *Let $n \equiv 4 \pmod 8$ and let X be an orthogonal design of order n and type (a,a,b). Then $\frac{b}{a}$ must be a sum of two rational squares.*

Lemma 4 [13]. *Let $n \equiv 4 \pmod 8$ and let X be an orthogonal design of order n and type (a,a,a,b) then $\frac{b}{a}$ must be a rational square.*

Order 12: In this case $\rho(n) = 4$, so we need only consider designs on ≤ 4 variables. We note that of all the numbers ≤ 12, seven is the only one which is not the sum of ≤ 3 squares.

one variable: From [29] all designs exist.

two variables: Conjecture (iv) is true.

three variables: Conjecture (ii) is true. All designs not ruled out by Lemmas 1,2 or 3 exist.

four variables: Conjecture (iii) is true. All designs not ruled out by Lemmas 1,2,3 or 4 exist.

Order 20: The following designs in order 20 are not eliminated by the necessary conditions but are not known either.

4 variables

(1,1,1,16)	(1,2,2,8)	(1,4,5,5)
(1,1,2,16)	(1,2,2,9)	(1,5,5,8)
(1,1,5,8)	(1,2,6,11)	(1,5,5,9)
(1,1,5,13)	(1,2,8,9)	(2,2,5,5)
(1,1,8,9)	(1,3,6,8)	(2,3,7,8)
(1,1,8,10)	(1,4,4,9)	(3,3,6,6)

3 variables

(1,1,17)	(1,5,13)	(2,7,11)
(1,2,10)	(1,6,12)	(2,8,9)
(1,2,16)	(1,6,13)	(3,3,12)
(1,3,9)	(1,8,10)	(3,4,10)
(1,3,10)	(2,3,8)	(3,4,11)
(1,3,11)	(2,3,13)	(3,6,8)
(1,3,14)	(2,5,6)	(3,7,8)
(1,3,16)	(2,5,7)	(3,7,10)
(1,4,6)	(2,6,11)	(5,5,9)
(1,4,13)	(2,7,8)	(5,6,7)
(1,5,8)	(2,7,10)	(5,6,8)

2 variables Conjecture (IVA) is true.

(3,16)
(6,13)
(7,10)
(7,12)

} these designs are not ruled out by any lemma but are not yet known.

1 variable

all exist and Conjecture (III) is true.

order 28: *1 variable*

all exist and Conjecture (III) is true.

2 variables

all (1,k) where k is the sum of ≤ 3 squares exist so Conjecture (IV)A is true.

See Appendix D for a list of the unknown designs.

orders 36,44,52 and 60:

the known results are summarized in table 1 of Appendix C.

APPENDIX I: Unsolved questions and known results for orders $n \equiv 8 \pmod{16}$.

No necessary conditions, other than there is a maximum of 8 variables, are known for these orders. Part (i) of the following lemma was also noted by Joan Murphy Geramita.

Lemma 1 [14]. *If there exists an orthogonal design of type* $(s_1, s_2, \ldots, s_\ell)$ *in order* n *there exist orthogonal designs*

(i) $(s_1, s_1, s_2, s_3, \ldots, s_\ell)$

(ii) $(s_1, s_1, 2s_2, 2s_3, \ldots, 2s_\ell)$

in order $2n$.

Lemma 2 [14]. *Suppose there exist 8 circulant matrices of order* n, $A_1, A_2, \ldots, A_8$, *of variables* $0 \pm x_1, \ldots, \pm x_\ell$, *satisfying*

$$\sum_{i=1}^{8} A_i A_i^t = \sum_{j=1}^{\ell} s_j x_j^2 I_n .$$

Further suppose

(i) $A_1, A_2, \ldots, A_8$ *are all symmetric; or*

(ii) $A_2, A_3, \ldots, A_8$ *are all skew; or*

(iii) $A_1 = A_2 = \ldots = A_i$ *and* $A_{i+1} = \ldots = A_8$; *or*

(iv) $A_1 = A_2 = \ldots = A_i$ *and* $A_{i+1}, \ldots, A_8$ *are all symmetric or all skew; or*

(v) $A_2 = A_3 = A_4$ *are all skew (symmetric) and* A_5, A_6 A_7, A_8 *are all symmetric (skew); or*

(vi) $A_1 = A_2$, $A_3 = A_4$, *and* A_5, A_6, A_7, A_8 *are all symmetric;*

(vii) A_1 *is zero,* $A_3 = A_4$ *and* A_5, A_6, A_7, A_8 *are all symmetric.*

Then there exists an orthogonal design of order $8n$ *and type* $(s_1, s_2, \ldots, s_\ell)$.

Lemma 3 [14]. *If there exist all designs* $(1,i,j)$ *in order* n *then there exist all designs* (g,h) *in order* $2n$.

Lemma 4 [11],[14]. *Orthogonal designs* $(1,k)$ *exist for* $1 \le k \le n-1$ *when* $n = 2^t, 2^t.3, 2^t.5, 2^t.9$, $t \ge 3$ *a positive integer.*

order 24: No necessary conditions are known ruling out any designs except that the number of variables is limited to $\rho(24) = 8$.

1 variable

all one variable designs exist.

2 variables

all two variable designs exist.

3 variables

the following 3 variable designs have not yet been found

(1,1,21)	(2,8,11)	(4,4,15)
(1,3,17)	(3,3,11)	(4,5,14)
(1,3,19)	(3,3,17)	(4,8,11)
(1,5,17)	(3,6,11)	(5,7,11)
(1,7,15)	(3,7,11)	(5,8,8)
(1,8,14)	(3,8,10)	(6,7,8)
(1,9,13)	(3,8,12)	(7,7,7)
(2,7,11)	(4,4,13)	(7,7,9)
(2,7,14)		(7,8,8)

4 variables

there are 369 four variable designs which have not yet been found

5 and more variables

these cases have virtually been untouched. Most of those known have been found using Lemmas 1, 2 and 3.

order 40: As for order 24 no necessary conditions are known except that the number of variables is limited to $\rho(40) = 8$.

1 variable

all one variable designs exist.

2 variables

The following 2 variable designs have not yet been found:

(5,32)	(7,25)	(8,31)
(6,29)	(7,32)	(9,30)
(6,31)	(8,29)	(14,23)
(6,33)		

3 and more variables

virtually untouched.

order 56: Again $\rho(56) = 8$ so we have at most 8 variables.

1 variable

All one variable designs are known.

2 variables

All (1,k) for $k \in \{x: 0 \le x \le 55, \ x \ne 47\}$ are known but no other results.

order 72: Again at most 8 variables.

1 variable

All one variable designs are known.

2 variables

All designs (1,k) in order 72 are known.

Other designs have not yet been studied.

APPENDIX J: Unsolved Questions and Known Results for orders n a power of 2.

Lemma 1. *For orders $n = 1,2,4,8$ every possible orthogonal design exists.*

Lemma 2 [10], [11]. *For every order n a power of 2*

(i) *$W(n,k)$ exists for every integer k, $0 \le k \le n$;*

(ii) *an orthogonal design $(1,k)$ exists for every integer k, $0 \le k \le n - 1$.*

order 16: $\rho(16) = 9$ so orthogonal designs must have ≤ 9 variables.

By Lemma 2 all 1 variable designs exist and all designs $(1,k)$, $1 \le k \le 15$.

In [7] it is shown all 2 variable designs exist

Also in [7] it is shown all 3 variable designs exist.

The only unresolved case for 4 variables is (1,5,5,5). Unresolved cases for 5 variables:

(1,1,1,1,11)	(1,1,1,5,8)	(1,1,4,5,5)
(1,1,1,1,12)	(1,1,2,2,9)	(1,2,2,5,5)
(1,1,1,2,11)	(1,1,2,5,7)	(1,2,3,5,5)

There are 37 cases which are unresolved for the designs on 6 variables:

1,1,1,1,1,8	1,1,1,1,5,5	1,1,2,2,2,7
1,1,1,1,1,9	1,1,1,1,5,6	1,1,2,2,3,6
1,1,1,1,1,10	1,1,1,1,5,7	1,1,2,2,3,7
1,1,1,1,1,11	1,1,1,1,6,6	1,1,2,2,4,5
1,1,1,1,2,7	1,1,1,2,2,8	1,1,2,2,5,5
1,1,1,1,2,8	1,1,1,2,2,9	1,1,2,3,4,5
1,1,1,1,2,9	1,1,1,2,3,8	1,2,2,2,3,5
1,1,1,1,2,10	1,1,1,2,4,7	1,2,2,3,3,5
1,1,1,1,3,8	1,1,1,2,5,5	1,3,3,3,3,3
1,1,1,1,3,9	1,1,1,2,5,6	2,2,2,3,3,3
1,1,1,1,4,5	1,1,1,3,5,5	2,2,3,3,3,3
1,1,1,1,4,7	1,1,1,4,4,4	
1,1,1,1,4,8	1,1,1,4,4,5	

There are 57 unresolved cases for the designs on 7 variables, they are:

1,1,1,1,1,1,5	1,1,1,1,1,5,5	1,1,1,1,4,4,4
1,1,1,1,1,1,6	1,1,1,1,1,5,6	1,1,1,2,2,2,5
1,1,1,1,1,1,7	1,1,1,1,2,2,5	1,1,1,2,2,2,6
1,1,1,1,1,1,8	1,1,1,1,2,2,6	1,1,1,2,2,2,7
1,1,1,1,1,1,9	1,1,1,1,2,2,7	1,1,1,2,2,3,5
1,1,1,1,1,1,10	1,1,1,1,2,2,8	1,1,1,2,2,3,6
1,1,1,1,1,2,5	1,1,1,1,2,3,3	1,1,1,2,2,4,4
1,1,1,1,1,2,6	1,1,1,1,2,3,4	1,1,1,2,2,4,5
1,1,1,1,1,2,7	1,1,1,1,2,3,5	1,1,1,2,3,3,5
1,1,1,1,1,2,8	1,1,1,1,2,3,6	1,1,1,2,3,4,4
1,1,1,1,1,2,9	1,1,1,1,2,3,7	1,1,2,2,2,2,5
1,1,1,1,1,3,5	1,1,1,1,2,4,4	1,1,2,2,2,3,3
1,1,1,1,1,3,6	1,1,1,1,2,4,5	1,1,2,2,2,3,4
1,1,1,1,1,3,7	1,1,1,1,2,4,6	1,1,2,2,2,3,5
1,1,1,1,1,3,8	1,1,1,1,2,5,5	1,1,2,2,3,3,3
1,1,1,1,1,4,4	1,1,1,1,3,3,5	1,1,2,2,3,3,4
1,1,1,1,1,4,5	1,1,1,1,3,3,6	1,1,2,3,3,3,3
1,1,1,1,1,4,6	1,1,1,1,3,4,4	1,2,2,2,2,3,3
1,1,1,1,1,4,7	1,1,1,1,3,4,5	1,2,2,2,3,3,3

Of 67 possible 8-tuples only 11 are known, viz:

1,1,1,1,1,1,1,1	1,1,2,2,2,2,2,2
1,1,1,1,1,1,1,2	1,1,2,2,2,2,2,4
1,1,1,1,1,1,2,2	1,2,2,2,2,2,2,2
1,1,1,1,1,2,2,2	1,2,2,2,2,2,2,3
1,1,1,1,2,2,2,2	2,2,2,2,2,2,2,2
1,1,1,2,2,2,2,2	

We can only verify that 2 of the possible 9-tuples exist! They are

1,1,1,1,1,1,1,1,1

1,1,2,2,2,2,2,2,2

order 32: Except for the results of Lemma 2 this case has not yet been studied.

Lemma 3 [7, Theorem 17]. *If* $n = 2^t$ *then there is an L-family of order* n *having* n *members; or equivalently If* $n = 2^t$ *then there exist amicable weighing matrices of weights* i *and* n *for every* $1 \le i \le n-1$.

§6 UNSOLVED PROBLEMS

1. Two weighing matrices W and N of order n and weights i and j are called *amicable weighing matrices* if

   $$W^t = -W, \; N^t = N$$
   $$WN^t = NW^t$$

   Their existence is known in many cases for $i = n - 1$, $j = n \equiv 0 \pmod 4$. They have also been studied for $n = 2$ and 4.

   Find such matrices for $i, j < n$.

2. Prove that "If all weighing matrices exist in order n then all weighing matrices exist in order 2n (or 4n)".

3. Find new Baumert-Hall arrays ie orthogonal designs (t,t,t,t) in orders 4t (see §4).

4. Find new Plotkin arrays ie orthogonal designs (t,t,t,t,t,t,t,t) in orders 4t (see §4).

5. Find a construction, similar to the Goethals-Seidel array, using 8 circulant matrices.

6. Find a design $(1,47)$ in order 56 and hence prove there exist all orthogonal designs $(1,k)$ in orders $2^t.7$, $t \geq 3$ an integer.

7. Find more results of the type indicated in Lemmas 5, 6, 7, 8 of Appendix C.

8. Fill in the unknown cases of Tables 1 and 2 of Appendix C.

9. Find the two variable designs indicated as unknown in Appendix D.

10. Find the four variable designs indicated as unknown in Appendix E.

11. Find new relations between the order n and weight k of a weighing matrix (see Appendix F).

12. Lower the bounds $N(k)$ by finding the unsolved weights as indicated in Appendix F.

13. Find more results of the type indicated in Lemmas 2,3, 4,5,6,7 of Appendix G.

14. Study the existence of orthogonal designs in orders $n \equiv 2 \pmod 4$, $n \geq 14$.

15. Find weighing (skew-weighing) matrices in the table in Appendix G.

16. Lower the bounds on $M(9)$, $M(25)$, $M(36)$, and find $M(k)$ for other k (see Appendix G).

17. Find the unknown designs in order 20 listed in Appendix H.

18. Study the case of 3 and 4 variable designs in orders 28, 36, 44, 52, 60.

19. Find the unknown design in order 24 listed in Appendix I.

20. Study the case of 3 and 4 variable designs in orders 40 (and thus 80) and 48.

21. Find the unknown 2, 3 and 4 variable designs in orders 56 and 72.

22. Find more results in orders 16,32.

23. There is a theorem: "The existence of all designs (1, i, j) in order n implies the existence of all designs (k,ℓ) in order 2n." Do there exist similar theorems of the type "the knowledge of all t variable designs in order n implies the knowledge of all t - 1 variable designs in 2n."

24. Show that if $n \equiv 0 \pmod 8$ all orthogonal designs on 1,2,3,4 or 5 variables must exist.

25. Verify the following three conjectures for $n \equiv 0 \pmod 8$.

    i) A necessary sufficient condition that there exist an orthogonal design of type (a,a,a,a,a,b) in order n is that $\frac{b}{a}$ be a sum of ≤ 3 rational squares.

    ii) A necessary and sufficient condition that there exist an orthogonal design of type (a,a,a,a,a,a,b) is that $\frac{b}{a}$ be a sum of ≤ 2 rational squares.

    iii) A necessary and sufficient condition that there exist an orthogonal design of type (a,a,a,a,a,a,a,b) is that $\frac{b}{a}$ be a rational square.

26. If $n \equiv 0 \pmod{16}$ all orthogonal designs exist! (We have no evidence for this statement to be true. If this statement is false then there might be some exceedingly interesting number theoretic conditions involved.)

27. The Baumert-Hall arrays have been much used to construct Hadamard matrices. We have found several designs in order n of type $(s_1,\ldots s_\ell)$ where $\sum_{i=1}^{\ell} s_i = n$. Find matrices to use in these designs to construct new Hadamard matrices.

28. Find the missing cases in order 36.

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