ORTHOGONAL DESIGNS III: WEIGHING MATRICES

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ABSTRACT. A *weighing matrix* $W = W(n,k)$ of order n and *weight* k is a square $(0,1,-1)$-matrix satisfying

$$WW^t = kI_n$$

An orthogonal design of order n on a single variable is a weighing matrix and consequently the study of orthogonal designs is intimately connected with the study of weighing matrices.

This paper reviews and updates the current status of the conjectures:

I. Let $n \equiv 2 \pmod 4$. Then there exists a $W(n,k)$ if and only if $k < n - 1$ is the sum of two integer squares;

II. Let $n \equiv 0 \pmod 4$. Then there exists a $W(n,k)$ for each $k \le n$. This conjecture has been verified for $n = 28$, 2^{t+1}, $2^{t+1}\cdot 3$ and $2^{t+1}\cdot 5$, where t is any positive integer;

III. Let $n \equiv 4 \pmod 8$. Then there exists an orthogonal design $(1,1)$ for all $k < n$ where k is the sum of *three* integer squares;

IV. Let $n \equiv 0 \pmod 8$. Then there exists an orthogonal design $(1,k)$ for all $k < n$. This conjecture has been verified for $n = 2^{t+2}$, $2^{t+2}\cdot 3$ and $2^{t+2}\cdot 5$, where t is any positive integer.

V. Let $n \equiv 2 \pmod 4$. Then there exists an orthogonal design of type $(1,k)$ in order n for all $k < n - 1$ such that $k = a^2$, a an integer.

1. *Introduction.*

An orthogonal design of order n *and type* $(s_1,s_2,\ldots,s_\ell)$ $(s_i > 0)$ on the commuting variables $x_1,x_2,\ldots,x_\ell$ is an $n\times n$

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matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_\ell\}$ such that

$$AA^t = \sum_{i=1}^{\ell} \left(s_i x_i^2\right) I_n .$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$.

In [2], where this was first defined and many examples and properties of such designs were investigated, we mentioned that

$$A^t A = \sum_{i=1}^{\ell} \left(s_i x_i^2\right) I_n$$

and so our alternative description of A applies equally well to the columns of A. We also showed in [2] that $\ell \le \rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

when

$$n = 2^a \cdot b, \ b \ \text{odd}, \ a = 4c + d, \quad 0 \le d < 4 .$$

In [2] we also showed that if there is an orthogonal design of order n and type (a,b) then

(i) $n \equiv 2 \pmod 4 \Rightarrow \frac{b}{a} = c^2$ for some rational integer c.

(ii) $n = 4t$, t odd $\Rightarrow \frac{b}{a}$ is a sum of three rational integer squares.

A *weighing matrix of weight* k *and order* n, is a square $\{0,1,-1\}$ matrix, A, of order n satisfying

$$AA^t = kI_n .$$

In [2] we showed that the existence of an orthogonal design of order n and type $(s_1, \ldots, s_\ell)$ is equivalent to the existence of weighing matrices $A_1, \ldots, A_\ell$, of order n, where A_i has weight s_i and the matrices, $\{A_i\}_{i=1}^{\ell}$, satisfy the matrix equation

$$XY^t + YX^t = 0$$

in pairs. In particular, the existence of an orthogonal design of order n and type $(1,k)$ is equivalent to the existence of a skew-symmetric weighing matrix of weight k and order n.

It is conjectured that:

(i) for $n \equiv 2 \pmod 4$ there is a weighing matrix of weight k and order n for every $k < n - 1$ which is the sum of two integer squares;

(ii) for $n \equiv 0 \pmod 4$ there is a weighing matrix of weight k and order n for every $k \leq n$;

(iii) for $n \equiv 4 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$, where k is the sum of $\leq$ three squares of integers (equivalently, there is an orthogonal design of type $(1,k)$ in order n for every $k < n$ which is the sum of $\leq$ three squares of integers. In other words, the necessary condition, given below in (iv), for the existence of an orthogonal design of type $(1,k)$ in order n, $n \equiv 4 \pmod 8$ is also sufficient).

(iv) for $n \equiv 0 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k < n$ (equivalently there is an orthogonal design of type $(1,k)$ in order n for every $k < n$) .

(v) for $n \equiv 2 \pmod 4$ there is an orthogonal design of type $(1,k)$ in order n for every $k < n - 1$ such that k is an integer square.

Conjecture (ii) above is an extension of the Hadamard conjecture (i.e. for every $n \equiv 0 \pmod 4$ there is a $\{1,-1\}$ matrix, H, of order n satisfying $HH^t = nI_n$) while (iv) and (iii) generalize the conjecture that for every $n \equiv 0 \pmod 4$ there is a Hadamard matrix, H, of order n, with the property that $H = I_n + S$ where $S = -S^t$.

Conjecture (ii) was established in [4] for $n \in \{4,8,12,\ldots,32,40\}$ and in [1] for $n = 2^t$. Conjecture (iii) was established in [2, Theorem 17] for $n = 2^t$ $(t \geq 3)$. In [3] conjectures (iv) and (iii) (and as a consequence conjecture (ii)) were established for $n = 2^{t+1}\cdot 3$, $n = 2^{t+1}\cdot 5$, t a positive integer. Also in [3] it was shown that only k = ~~46,47~~ in order 56 remain to be found and the conjectures will be settled for $n = 2^{t+1}\cdot 7$. done

Let R be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to be *constructed from two circulant matrices* A and B if it is of the form

$$\begin{bmatrix} A & BR \\ BR & -A \end{bmatrix}$$

and to be of Goethals-Seidel type if it is of the form

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^tR & -C^tR \\ -CR & -D^tR & A & B^tR \\ -DR & C^tR & -B^tR & A \end{bmatrix}$$

where A,B,C,D are circulant matrices.

2. *Known Results.*

PROPOSITION 1. [2, Corollary to Construction 22]. *If there is an orthogonal design of type* $(1,\ell)$ *in order* n *then there is an orthogonal design of type* $(1,1,\ell,\ell)$ *in order* 2n *and of type* $(1,1,2,\ell,\ell,2\ell)$ *in order* 4n .

COROLLARY 1. *If there are orthogonal designs of type* $(1,k)$ $1 \leq k \leq \ell$ *in order* n *then there are orthogonal designs of type* $(1,m)$, $1 \leq m \leq 2\ell + 1$ *in order* 2n .

COROLLARY 2. *If there are orthogonal designs of type* $(1,k)$, $1 \le k \le n-1$ *in order* n *then there are orthogonal designs of type* $(1,m)$, $1 \le m \le 2^t n - 1$ *in order* $2^t n$, t *a positive integer.*

PROPOSITION 2. *There exist orthogonal designs of type* $(1,k)$ *in order* $8n$ *where*

(i) $n \ge 3$, $k \in \{1,\dots,19\}$;
(ii) $n = 4$, $k \in \{1,\dots,31\}$;
(iii) $n = 5$, $k \in \{1,\dots,39\}$;
(iv) $n \ge 6$, $k \in \{1,\dots,45\}$.

PROPOSITION 3. *Let* $n = 2^{t+2} \cdot s$, $t \ge 0$, $s \in \{1,3,5\}$. *Then*

(i) *there is a* $W(n,k)$ *for all* $k \le n$;
(ii) *if* $t \ge 1$ *there is a skew-symmetric* $W(n,k)$ *for all* $k < n$;
(iii) *if* $t \ge 1$ *there is an orthogonal design* $(1,k)$ *for all* $k < n$.

PROPOSITION 4. *There exists an orthogonal design* $(1,k)$ *in order* n *where*

(i) $n = 28$, $k \in \{x: x \ne 7,15,23,\ 0 \le x \le 27\}$;
(ii) $n = 56$, $k \in \{x: x \ne 46,47,\ 0 \le x \le 55\}$.

PROPOSITION 5. *If* n *is odd there are orthogonal designs of order* $4n$ *and type* $(1,k)$ *when*

(i) $n \ge 3$, $k \in \{1,\dots 6,8,\dots,11\}$;
(ii) $n \ge 5$, $k \in \{1,\dots 6,8,\dots,14,16,17\}$;
(iii) $n \ge 7$, $k \in \{1,\dots 6,8,\dots,14,16,17,18,20,24\}$;
(iv) $n \ge 9$, $k \in \{1,\dots 6,8,\dots,14,16,17,18,20,21,24,32,33\}$;
(v) $n \ge 11$, $k \in \{1,\dots 6,8,\dots,14,16,\dots 21,24,32,33\}$.

Note: These sets will be amended in this paper.

PROPOSITION 6. [5, p. 294]. *A* $W(n,k)$ *can only exist for* $n \equiv 2 \pmod 4$ *where* k *is the sum of two integer squares.*

COROLLARY. *For* $n \equiv 2 \pmod 4$ *a* $W(n,k)$ *can only exist for* $k \in \{0,1,2,4,5,8,9,10,13,16,17,18,20,25,26,29,32,34,36,37,40,41$ $45,49,50,\dots\}$.

PROPOSITION 7. *If* n *is odd and there exist* $W(2n,k)$ *and* $W(2n,\ell)$ *constructed from circulant matrices then there exists a* $W(4n,k+\ell)$ *constructed using four circulant matrices in the Goethals-Seidel array.*

PROPOSITION 8. *If there exist a* $W(n,k)$ *and a* $W(m,k)$ *there exists a* $W(m+n,k)$.

PROPOSITION 9. *There exists a* $W(2n,k)$, *constructed using two circulant matrices, for* $k \in \{0,1,2,4,5\}$ *in every order* $2n$, $n(\text{odd}) \geq 3$.

3. *Golay Sequences and Orthogonal Designs.*

Let $X = \{[a_{11},\ldots,a_{1n}],\ [a_{21},\ldots,a_{2n}],\ldots,[a_{m1},\ldots,a_{mn}]\}$ be m sequences of integers of length n .

Definition. (1) The *non-periodic auto-correlation function of the family of sequences* X (denoted N_X) is a function from the set of integers $\{1,2,\ldots,n-1\}$ to Z (the integers) where

$$N_X(j) = \sum_{i=1}^{n-j} (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \cdots + a_{m,i}a_{m,i+j}) \ .$$

Note that if the following collection of m matrices of order n is formed

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{11} & & a_{1,n-1} \\ & & \ddots & \\ 0 & & & a_{11} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ & a_{21} & & a_{2,n-1} \\ & & \ddots & \\ 0 & & & a_{21} \end{bmatrix}, \ldots \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \\ & \ddots & & a_{m,n-1} \\ & & \ddots & \\ 0 & & & a_{m1} \end{bmatrix},$$

then $N_X(j)$ is simply the sum of the inner products of rows 1 and $j+1$ of these matrices.

(2) The *periodic auto-correlation function of the family of sequences* X (denoted P_X) is a function from the set of integers $\{1,2,\ldots,n-1\}$ to Z where

$$P_X(j) = \sum_{i=1}^{n} (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j})$$

where we assume the second subscript is actually chosen from the complete set of residues mod(n), $\{1,2,\ldots,n\}$.

We can interpret the function P_X in the following way: Form the m circulant matrices which have first rows respectively, $[a_{11}\ a_{12} \ldots a_{1n}]$, $[a_{21}\ a_{22} \ldots a_{2n}], \ldots, [a_{m1}\ a_{m2} \ldots a_{mn}]$, then $P_X(j)$ is the sum of the inner products of rows 1 and j + 1 of these matrices.

PROPOSITION 10. *Let* X *be a family of sequences as above, then*

$$P_X(j) = N_X(j) + N_X(n-j), \quad j = 1,\ldots,n-1$$

COROLLARY. *If* $N_X(j) = 0$ *for all* $j = 1,\ldots,n-1$ *then* $P_X(j) = 0$ *for all* $j = 1,\ldots,n-1$.

Note: $P_X(j)$ may equal 0 for all $j = 1,\ldots,n-1$ even though the $N_X(j)$ are not.

Definition. If $X = \{\{a_1,\ldots,a_n\}, \{b_1,\ldots,b_n\}\}$ are two sequences where $a_i, b_j \in \{1,-1\}$ and $N_X(j) = 0$ for $j = 1,\ldots,n-1$ then the sequences in X are called *Golay complementary sequences of length* n . name ?

We note that if X is as above and A is the circulant formed by $\{a_1,\ldots,a_n\}$ and B the circulant formed by $\{b_1,\ldots,b_n\}$ then

$$AA^t + BB^t = \sum\left(a_i^2 + b_i^2\right)I_n .$$

Consequently, such matrices may be used in the Goethals-Seidel array to obtain Hadamard matrices.

Example. $X = \{\{1,-1\}, \{1,1\}\}$ are Golay complementary sequences of length 2.

By results of R. J. Turyn, Golay complementary sequences exist having length r for

$$r = 2^a \cdot 10^b \cdot 26^c \ , \ a,b,c$$

non-negative integers.

Since our interest is in orthogonal designs we shall not be restricted to sequences with entries only ± 1, but shall allow 0's also. One very simple remark is in order. If we have a collection of sequences, X, (each having length n) such that $N_X(j) = 0$, $j = 1,\ldots,n-1$, then we may augment each sequence at the beginning with k zeros and at the end with ℓ zeros so that the resulting collection, (say $\bar{X}$), of sequences having length $k + n + \ell$ still has $N_{\bar{X}}(j) = 0$, $j = 1,\ldots,k+n+\ell-1$. More interesting is the following result of Turyn.

PROPOSITION 11. *Let* $X = \{\{a_1,\ldots,a_n\}, \{b_1,\ldots,b_n\}\}$ *be Golay complementary sequences. Then the sequences in*

$$X' = \{\{\tfrac{1}{2}(a_1+b_1),\ldots,\{\tfrac{1}{2}(a_n+b_n)\}, \{\tfrac{1}{2}(a_1-b_1),\ldots,\tfrac{1}{2}(a_n-b_n)\}\}$$

satisfy

(i) $N_{X'}(j) = 0$, $1 \le j \le n-1$.

(ii) exactly half of the $a_i + b_i$ are equal to 0 and exactly half of the $a_i - b_i$ equal 0.

Thus, if we let $X = \{g_r, h_r\}$ represent Golay complementary sequences of length r, we obtain a new pair of sequences of length r, which we donote g'_r, h'_r each having exactly $r/2$ non-zero members which can be chosen from $\{1,-1\}$ and such that if $X' = \{g'_r, h'_r\}$ then $N_{X'}(j) = 0$, $1 \le j \le r-1$.

One more piece of notation is in order. If g_r denotes a sequence of integers of length r then by xg_r we mean the sequence of length r obtained from g_r by multiplying each member of g_r by x.

PROPOSITION 12. *Let* r *be any number of the form* $2^a \cdot 10^b \cdot 26^c$ (a,b,c) *non-negative integers and let* n *be any odd integer* $\geq$ r. *Then there exist orthogonal designs of order* 4n *and types*

(i) (1,1,2r) *and* (1,1,r)

(ii) (1,4,r) *and* (1,4,2r) .

PROPOSITION 13. *Let* r *be as in Proposition 12 and* n *be any integer* $\geq$ r . *Then there exist* W(2n,r) *and* W(2n,2r) *constructed from circulant matrices.*

4. Some More Results.

LEMMA 1. *There exist orthogonal designs in every order* 4n , n odd, *as follows*

(i)	of type	(1,25)	for	n (odd) $\geq$ 9 ;
(ii)	of type	(1,1,40)	for	n (odd) $\geq$ 11 ;
(iii)	of type	(1,22)	for	n (odd) $\geq$ 7 ;
(iv)	of type	(1,1,32)	for	n (odd) $\geq$ 9 ;
(v)	of type	(1,24)	for	n (odd) $\geq$ 7 ;
(vi)	of type	(1,1,1,16)	for	n (odd) $\geq$ 7 ;
(vii)	of type	(1,1,13)	for	n (odd) $\geq$ 11 ;
(viii)	of type	(1,1,26)	for	n (odd) $\geq$ 15 ;
(ix)	of type	(1,34)	for	n (odd) $\geq$ 11 .

Proof. Use the following four first rows to form circulant matrices which are then used in the Goethals-Seidel array. Use $\underline{0}$ for a sequence of all zeros (its length is given in brackets). Use

(i) $(\frac{1}{2}(n-9)),\ \{x_1\underline{0}0x_20x_2 - x_20 - x_20\underline{0},\ x_20x_2x_20x_2x_2x_2 - x_2\underline{00},$

$x_20x_2 - x_2 - x_2x_2 - x_2 - x_2x_2\underline{00},$

$x_2x_2 - x_2 - x_2 - x_2x_2000\underline{00}\}$;

(ii) $(\frac{1}{2}(n-11)),\ \{x_1\underline{0} - x_3x_3x_3x_3 - x_3x_3 - x_3 - x_3 - x_3x_3\underline{0},$

$x_2\underline{0}x_3 - x_3 - x_3x_3 - x_3x_3 - x_3x_3x_3 - x_3\underline{0}$,

$0\underline{0}x_3 - x_3 - x_3x_3x_3x_3x_3 - x_3 - x_3x_3\underline{0},$

$0\underline{0} - x_3x_3x_3x_3x_3x_3x_3x_3x_3 - x_3\underline{0}\}$;

(iii) $(\frac{1}{2}(n-7)),\ \{x_1\underline{0}x_2x_2 - x_2x_2 - x_2x_2\underline{0},\ 0\underline{0}x_2 - x_2 - x_2x_2x_2x_2\underline{0}$,

$00\underline{0}x_2x_2x_2 - x_2x_2\underline{0},\ 00\underline{0}x_2x_2x_2 - x_2x_2\underline{0}\}$;

The proof of (iv), (v), (vi) is given in [3].

(vii) $(\frac{1}{2}(n-11)),\ \{x_1\underline{0}00000000000\underline{0},\ x_2\underline{0}00000000000\underline{0}$,

$x_3x_3x_30 - x_3x_3x_30 - x_3x_3 - x_3\underline{00},$

$x_30x_3000 - x_3000x_3\underline{00}\}$;

(viii) $(\frac{1}{2}(n-15)),\ \{x_1\underline{0}000000000000000\underline{0},\ x_2\underline{0}000000000000000\underline{0},$

$x_3x_3x_3 - x_3 - x_3x_3x_3x_3 - x_3x_3 - x_3 - x_30 - x_30\underline{00},$

$x_3x_3x_3x_3 - x_3x_3x_3 - x_3 - x_3x_3 - x_3x_30x_30\underline{00}\}$

(ix) $(\frac{1}{2}(n-11)),\ \{\underline{0}x_20 - x_20 - x_2x_1x_20x_20x_2\underline{0},$

$\underline{0}x_20x_20 - x_20 - x_20x_20x_2\underline{0}$,

$\underline{0}x_2x_2x_2x_2 - x_2x_2x_2x_2 - x_2x_2 - x_2\underline{0}$,

$\underline{0}x_2x_2x_2 - x_2 - x_2x_2x_2 - x_2 - x_2x_2 - x_2\underline{0}\}$.

LEMMA 2. *Let* $X = \{\{a_1,\dots a_n\}, \{b_1,\dots,b_n\}\}$ *be two sequences of* 0,1,-1, *with total weight* r *(i.e. non-zero elements), such that* $N_X(j) = 0$. *Then*

$$Y = \{\{a_1,\dots,a_n,-a_n,\dots,-a_1\}, \{b_1,\dots,b_n,-b_n,\dots,-b_1\},$$
$$\{a_1,\dots,a_n,a_n,\dots,a_1\}, \{b_1,\dots,b_n,b_n,\dots,b_1\}\}$$
$$= \{A,B,C,D\}$$

have $N_Y(j) = 0$. *Furthermore with* $\underline{0}$ *the* $\frac{1}{2}(t - 2n - 1)$ *sequence of all zeros* $(t(\text{odd}) \geq 2n)$, *the following four sequences may be used to generate circulant matrices which when used in the Goethals-Seidel array give an orthogonal design* $(1,1,4r)$ *in every order* $4t$:

$$\{\{x_1,\underline{0},x_3A,\underline{0}\}, \quad \{x_2,\underline{0},x_3B,\underline{0}\}, \quad \{0,\underline{0},x_3C,\underline{0}\}, \quad \{0,\underline{0},x_3D,\underline{0}\}\} .$$

LEMMA 3. *There exists a* $W(2n, 10)$ *constructed from circulant matrices for every* $n \geq 6$. *It uses the sequences*

$$X = \{\{101\text{--}1\underline{0}\}, \{10111\text{-}\underline{0}\}\}$$

which have $N_X(j) = 0$ ($\underline{0}$ *is a sequence of* $n - 6$ *zeros*).

LEMMA 4. *Suppose* A *and* B *are circulant matrices from which a* $W(2n,k)$ *may be constructed. Then if the number of zeros in the first row of*

(i) A *is* > 0 *there exists a* $W(4n,2k + 2)$;

(ii) A *is* > 0 *and* B *is* > 0 *there exists a* $W(4n,2k + 4)$;

(iii) A *is* ≥ 2 *and* B *is* ≥ 2 *in the* (i,i) *and* $(i,i + j)$ *positions (possibly after permuting) then there exists a* $W(4n,2k + 8)$;

(iv) A *is* > 2 *and* B *is* > 2 *in the* (i,i), $(i,i + j)$, $(i,i + 2j)$ *positions (possibly after permuting) then there exists a* $W(4n,2k + 10)$.

Proof. Circulate the first row of A and B (if possible) to form matrices A' and B' with zero diagonal. Now using the following matrices in the Goethals-Seidel array we get the result.

(i) $A' + I,\ A' - I,\ B', B'$;

(ii) $A' + I,\ A' - I,\ B' + I,\ B' - I$;

(iii) $A + T^{i} + T^{i+j},\ A - T^{i} - T^{i+j},\ B + T^{i} - T^{i+j},\ B - T^{i} + T^{i+j}$;

(iv) $A + T^{i} + T^{i+2j},\ A - T^{i} - T^{i+2j},\ B + T^{i} + T^{i+j} - T^{i+2j},$
$B - T^{i} - T^{i+j} + T^{i+2j}$

where T^{i+k} is the circulant matrix with 1's in the positions $(i, i + k)$, $1 \le i \le n$, and zeros elsewhere.

LEMMA 5. *If there exists a circulant* $W(n,k)$ [*and necessarily* $(n - k)^2 - (n - k) + 2 > n$] *then there exist*

(i) $W(2n,\ell)$ *for* $\ell \in \{2k, 2k+2\}$;

(ii) $W(4n,\ell)$ *for* $\ell \in \{2k, 2k+1, 2k+2, 2k+3, 2k+4, 2k+6\}$;

(iii) $W(4n,\ell)$ *for* $\ell \in \{3k, 3k+1, 3k+2, 3k+3\}$;

(iv) $W(4n,\ell)$ *for* $\ell \in \{4k, 4k+2, 4k+4, 4k+8\}$;

(among others), all constructed using four circulant matrices in the Goethals-Seidel array.

Proof. Along the lines of the proof of Lemma 4.

LEMMA 6. *Suppose* A, B, C, D *are circulant matrices from which a* $W(4n,k)$ *may be constructed. Further suppose* $A + C$, $A - C$, $B + D$, $B - D$ *are also* $(0,1,-1)$-*matrices. Then* $A + C$, $A - C$, $B + D$, $B - D$ *may be used in the Goethals-Seidel array to form a* $W(4n,2k)$.

5. *Updating the Weighing Matrix Results.*

LEMMA 7. *There exist a* $W(18,k)$ *for* $k \in \{0,1,2,4,5,8,10,13,16,17\}$. *All but* $W(18,13)$ *are constructed from circulant matrices. The only undecided case is* $W(18,9)$.

Proof. The existence of a W(18,k) for $k \in \{0,1,2,4,5,8,10,16\}$ constructed from circulant matrices is established by Propositions 9 and 10, and Lemma 3.

Let $C = \begin{bmatrix} - & 1 & 0 \\ 1 & 0 & - \\ 0 & - & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $A = I \times J + (J - I) \times C$,

$B = C \times C + R \times (J - I)$, then

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

is a W(18,13) .

A W(18, 17) constructed from circulant matrices may be obtained by circulating the following first rows:

1--1111--, 0-1----1-

LEMMA 8. *There exists a* W(36,k) *for* $k \in \{x: x \neq 31, 0 \le x \le 36\}$. *All but* $W(36,\ell), \ell \in \{29\}$ *are constructed using four circulant matrices in the Goethals-Seidel array.*

Proof. The existence of W(36,k) for $k \in \{0,1,2,\ldots,21,22,24,25,26,27,32,33,34\}$ constructed using four circulant matrices in the Goethal-Seidel array follows from Proposition 8 and Lemma 7.

The $W(36,\ell)$ for $\ell \in \{23,28,30,35,36\}$ may be constructed from the Goethals-Seidel array by circulating the following first four rows:

23: 1011-1--0, 0001--111, 0000111-1, 0000111-1 ;

28: 101101--1, -0-101--1, 10-10111-, -0110111- ;

30: 101101--1, -0-101--1, 11-10111-, --110111- ;

35: 0111-1---, 11-1--1-1, 1---11---, 111-11-11 ;

36: 1111-1---, 11-1--1-1, 1---11---, 111-11-11 .

There is an orthogonal design (1,5,6) in order 12. If the variables are replaced by the matrices

$$\begin{bmatrix} 0 & 1 & - \\ 1 & - & 0 \\ - & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} - & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

respectively, we get a W(36,29) .

LEMMA 9. *There exists a* W(22,k) *for* $k \in \{0,1,2,4,5,8,9,10,13,16,17,20\}$. *All but* W(22,9) *are constructed from circulant matrices. The only undecided case is* W(22,18) .

Proof. The existence of a W(22,k) for $k \in \{0,1,2,4,5,8,10,16,20\}$ constructed from circulant matrices is established by Propositions 9 and 13.

A W(22,13) and W(22,17) constructed from circulant matrices may be obtained by circulating the following first rows:

1110-110-1-, 101000-0001
-1011100010, 11-111---1- .

$W(22,9) = W(12,9) \oplus W(10,9)$ and hence W(22,9) exists.

LEMMA 10. *There exists a* W(44,k) *for* $k \in \{x: x \neq 31,\ 0 \le x \le 44\}$. *All are constructed using four circulant matrices in the Goethals-Seidel array.*

Proof. The existence of W(44,k) for k ∈ {0,1,2,...,29,30,32, 33,34,36,37,40} constructed using four circulant matrices in the Goethals-Seidel array follows from Proposition 8 and Lemma 9.

We may use the W(22,20) in Lemma 4 to get the four circulant matrices which give W(44,42) and W(44,44).

A W(44,41) and W(44,43) may be obtained from the circulant matrices with the following first rows:

0-111-1---1,11--1-1-11-, 01--1111--1, 0-11111111- ;
01-1--11-1-, 1111----111, 1-111--111-, 1---1--1--- .

Let X,Y,Z,W be the circulant matrices with the following first rows, respectively

01011100010, 01-111---1-, -1011100010, 00100000000 .

Then a W(44,38), W(44,39) and W(44,35) can be constructed using

$$X+I,\ Y+I,\ Y+I,\ Y\ ,$$
$$X+I,\ Y+I,\ Y+I,\ Y+I\ ,$$
$$\text{and}\quad Z+W,\ Z-W,\ Y+I,\ Y$$

respectively, in the Goethals-Seidel array.

LEMMA 11. *There exists a* W(26,k) *for* k ∈ {0,1,2,4,5,8,9,10, 13,16,18,20,25}. *All are constructed from circulant matrices. The only undecided case is* W(26,17) .

Proof. The existence of a W(26,k) for k ∈ {0,1,2,4,5,8,10,16,20} constructed from circulant matrices is established by Propositions 9 and 13.

There exists a circulant W(13,9) = B (see [3]), it has first row

0010-11110-1- .

Hence there exists a W(26,9) and a W(26,18) constructed from circulant matrices.

The W(26,13) and W(26,25) may be obtained by using the circulant matrices with the following first rows:

1110-110-1-00, 101000-000100
01---1--1---1, 1-11------11- .

LEMMA 12. *There exists a* W(52,k) *for* $k \in \{x: x \neq 37,46,47,49, 0 \le x \le 52\}$. *All are constructed from four circulant matrices using the Goethals-Seidel array.*

Proof. The existence of W(52,k) for $k \in \{0,1,2,\ldots,35,36,38, 40,41,43,45,50\}$ constructed using four circulant matrices in the Goethals-Seidel array follows from Proposition 8 and Lemma 11.

Use Lemma 4 and the W(26,20) to obtain the W(52,42) and W(52,44) .

The W(52,39), W(52,48), W(52,51) and W(52,42) may be obtained, respectively, from the circulant matrices with the following first rows:

39: 1--111111-1-1, 100--01-1-000, 1110-1-0010-1, 01111-010--1- ;

48: 01---1--1---1, 0-11------11-, 0---1-11-1---, 01-11----11-1 ;

51: 01-1---111-1-, 1---111111---, 11-1--11--1-1, 1----1--1---- ;

52: 11-1---111-1-, 1---111111---, 11-1--11--1-1, 1----1--1---- .

LEMMA 13. (see [3]). *There exists an orthogonal design* (1,k) *in order* 56 for $k \neq 46,47$, $0 \le k \le 55$.

COROLLARY 14. (see [3]). *There exists a* W(56,k) *for* $k \in \{x: x \neq 47, 0 \le x \le 56\}$. [*The* W(56,48) *may be constructed by circulating the following first rows:* 11-1--11--1100, 11-1----11--00, 111-1-1-111100, 111-1--1----00] .

LEMMA 15. *There exists a* $W(30,k)$ *for* $k \in \{0,1,2,4,5,8,9,10,13,16,20,25,26,29\}$. *All but* $W(30,9)$ *and* $W(30,25)$ *are constructed from circulant matrices. The only undecided case is* $W(30,18)$.

Proof. The existence of a $W(30,k)$ for the k of the enunciation with the exception of $k = 9,13,25,26$ and 29 is given by Propositions 9 and 13.

The $W(30,13)$, $W(30,26)$ and $W(30,29)$ can be constructed using the following first rows

1110-110-1-0000,101000-00010000 ;

111--111-1--0-0, 1111-11--1-1010 ;

011--1----1--11, 1-1----11----1- .

Let B and C be the circulant symmetric matrices with first rows

01--1, -1111 ,

respectively.

Then $D = \begin{bmatrix} B & C \\ C^t & -B^t \end{bmatrix}$ is a $W(10,9)$ and $D \oplus D \oplus D$ is a $W(30,9)$.

Let $A = \begin{bmatrix} 0 & 1\ 1\ 1\ 1\ 1 \\ \begin{matrix}1\\1\\1\\1\\1\end{matrix} & B \end{bmatrix}$, a $W(6,5)$, then $A \times B + I \times J$ is a $W(30,25)$.

LEMMA 16. *There exists a* $W(60,k)$ *for* $k \in \{x: x \neq 51,53,\ 0 \le x \le 60\}$. *All but* $W(60,\ell)$ *for* $\ell \in \{19,35,38,41,43,47,50,57\}$ *are constructed from circulant matrices.*

Proof. The existence of $W(60,k)$ for $k \in \{x: 0 \le x \le 18, 20,21,\ldots,33,34,36,37,39,40,45,46,49,52,55,58\}$ constructed from circulant matrices using the Goethals-Seidel array follows from Proposition 8 and Lemma 15.

The $W(60,\ell)$ for $\ell \in \{48,59,60\}$ may be obtained by constructing circulant matrices with the following first rows:

48: 11-1--11--11000, 11-1----11--000,111-1-1-1111000,
111-1--1----000;

59: 01111-1-1-1----, 1---11-11-11---, 111--1-11-1--11,
111-111--111-11;

60: as for 59 but replace the zeros in the first matrix by 1's

$W(60,\ell) = W(32,\ell) \oplus W(28,\ell)$ so $W(60,\ell)$ exists for $\ell \in \{19\}$. $W(60,42)$, $W(60,44)$, $W(60,54)$ and $W(60,56)$ constructed using the Goethals-Seidel array may be obtained by using the $W(30,26)$ and $W(30,20)$ of Lemma 15 with Lemma 4.

$$W(10,9) \times W(6,5) = W(60,45) \text{ and}$$

$$W(2,2) \times W(30,25) = W(60,50)\ ,$$

so the $W(60,45)$ and $W(60,50)$ exist. See the proof of Lemma 15 for the existence of $W(10,9)$, $W(6,5)$ and

$$W(2,2) = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}.$$

We define the circulant symmetric $(0,1,-1)$-matrices A,B,C,I,J,K, each of order 5, by defining their first rows:

01111, 01--1, 1-11-, 10000, 11111, -1111 .

We also define the back-circulant $(0,1,-1)$-matrix, D, of order 5, with first row

011-- .

Now

$$XY^T = YX^T \quad \text{for} \quad X,Y \in \{A,B,C,D,I,J,K\} .$$

We note the orthogonal designs (2,3,6), (2,4,4), (1,1,8), (1,3,6), (2,5,5), (1,1,5,5), (3,3,6) all exist in order 12. Hence the following weighing matrices may be constructed by replacing the variables of the orthogonal design by the matrices indicated:

W(60,35) use A,I,B in the (2,3,6) orthogonal design;

W(60,38) use I,B,K in the (2,4,4) orthogonal design;

W(60,41) use A,J,B in the (1,1,8) orthogonal design;

W(60,43) use A,K,B in the (1,3,6) orthogonal design;

W(60,47) use I,K,B in the (2,5,5) orthogonal design;

W(60,57) use D,C,K in the (3,3,6) orthogonal design.

6. *Orthogonal Designs* (1,k)

LEMMA 17. *There exists an orthogonal design* (1,k) *in order* 36 *for* $k \in \{x\colon x$ *is the sum of three integer squares,* $x \neq 19,30,34, \quad 0 \le x \le 36\}$.

Proof. In [3] it is shown there exists a (1,k) in every order $4n$, $n \ge 6$ for $k \in \{1,\ldots,6,8,\ldots,12,16,17,20\}$.

It is proved in this paper that there exists a (1,18), (1,22), (1,24), (1,25), (1,32) and (1,33) in every order $4n$, $n(\text{odd}) \ge 9$.

Now if there exists $B = W(2n,k)$ then

$$\begin{bmatrix} x_1 I & x_2 B \\ x_2 B^t & -x_1 I \end{bmatrix}$$

is an orthogonal design (1,k) in order 4n. There exists a W(18,13) so there is a (1,13) in order 36.

Skew-symmetric W(36,k) may be constructed using the circulant matrices with the following first rows in the Goethals-Seidel array:

k = 14: 00001-000, 000011000, 1--101000, 10111-000 ;
k = 21: 00001-000, 000011000, 1--1111--, 0-1----1- ;
k = 35: 0111-1---, 11-1--1-1, 1---11---, 111-11-11 ;

Hence there exist orthogonal designs (1,14), (1,21) and (1,35) in order 36.

Now the following orthogonal designs exist in order 12 (see [2]) (1,2,3,6), (1,1,4,4) and (1,1,5,5) .

Let I,0,A,B,C,J,R be the following matrices respectively:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}, \begin{bmatrix}-&1&1\\1&-&1\\1&1&-\end{bmatrix}, \begin{bmatrix}0&1&-\\-&0&1\\1&-&0\end{bmatrix},$$

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

Then

$$MN^t = NM^t \quad \text{for} \quad M,N \in \{I,0,A,B,CR,(C \pm I)R,J\} .$$

The following orthogonal designs (1,k) may be constructed by replacing the variables of the orthogonal designs specified by the matrices indicated:

for (1,26) use x_1I, x_2I, x_2A, x_2B in the (1,1,5,5) orthogonal design;

for (1,27) use x_1I, x_2J, x_2I, x_2B in the (1,2,3,6) orthogonal design;

for (1,29) use $(x_1I + x_2C)R$, x_2A, x_2A, x_2B in the (1,1,5,5) orthogonal design.

LEMMA 18. *There exists an orthogonal design* $(1,k)$ *in order* 72 *for* $k \in \{x : x \neq 31,46,47,56,60,61,62,63,68, \quad 0 \le x \le 71\}$.

Proof. We know that the existence of a $(1,k)$ in order n means there is a $(1,1,k,k)$ in order $2n$. Hence by the results of the previous lemma we have a design $(1,k)$ in order 72 for $k \in \{1\text{-}14,16\text{-}29,32\text{-}37,40\text{-}45,48\text{-}55,58,59,64\text{-}67,70,71\}$.

We recall (from the previous lemma) that if there exists a $W(2n,k)$ there exists an orthogonal design $(1,k)$ in order $4n$. Now a $W(36,15)$, $W(36,30)$ exist so we have a $(1,15)$ and $(1,30)$ in order 72.

Let L,M be the matrices with first rows 00001-000 and 000011000 .

Now there exists a $W(36,36)$ constructed from four circulant symmetric matrices of order 9 (see [5, p. 388]). Call these matrices A,B,C,D . There exists a $W(36,35)$ (see the previous lemma) constructed of one skew-symmetric and three symmetric circulant matrices of order 9. Call these matrices X,Y,Z,W respectively. Finally there exists a $W(18,17)$ (see Lemma 7) constructed of two circulant symmetric matrices of order 9. Call these matrices P and Q.

The following orthogonal designs may be constructed by replacing the variables of the orthogonal design $(1,1,1,1,1,1,1,1)$ of order 8 by the matrices specified (R, as defined previously, is the back-diagonal matrix):

$(1,1,1,1,36)$ use $x_1I, x_2I, x_3I, x_4I, x_5A, x_5B, x_5C, x_5D$;

$(1,4,17,36)$ use $(x_1I+x_2L)R, x_2M, x_3P, x_3Q, x_4A, x_4B, x_4C, x_4D$;

$(1,17,17,35)$ use $(x_1I+x_4X)R, x_2, P, x_2Q, x_3P, x_3Q, x_4Y, x_4Z, x_4W$.

Hence we have a $(1,38)$, $(1,39)$, $(1,57)$ and $(1,69)$ in order 72.

COROLLARY. *There exists a* $W(72,k)$ *for* $k \in \{x : x \neq 47,61,63, \ 0 \le x \le 72\}$.

Proof. The existence of (1,k) in order 72 as stated in Lemma 20 gives us the result for all k except {47,61,62,63} .

We note that if A is a $W(n,\ell)$ (rearranged so it has zero diagonal) then

$$\begin{bmatrix} A + I & A - I \\ A^t - I & -A^t - I \end{bmatrix}$$

is a $W(2n,2\ell+2)$.

Hence since there exists a W(36,30) we obtain a W(72,62) and the corollary.

LEMMA 19. *There exists an orthogonal design* (1,k) *in order* 44 *for* k $\in$ {x: x *is the sum of three integer squares,* $x \neq 30$, 42, $0 \le x \le 43$} .

Proof. There exist orthogonal designs (1,k) in order 20 for $k \in \{x: x \neq 7,15,\ 0 \le x \le 19\}$ and in order 24 for $k \in \{x: 0 \le x \le 23\}$ so there exist orthogonal designs (1,k) in order 44 for $k \in \{x: x \neq 7,15,\ 0 \le x \le 19\}$ (see [3]) .

It is proved in [3] that a (1,20),(1,21) and (1,24) exist in every order 4n, n(odd) $\ge$ 11 and earlier in this paper that a (1,25), (1,22), (1,40) and (1,41) exist in every order 4n, n (odd) $\ge$ 11 .

Let A,B,I,T be the circulant matrices of order 11 with the following first rows:

01011100010, 01-111---1-, 10000000000, 00100000000 .

Now we can get the orthogonal designs indicated below by using the specified circulant matrices in the Goethals-Seidel array (0 is the zero matrix of order 11, W(22,k) indicates the two circulant matrices which may be used to form this weighing matrix):

Now the following orthogonal designs exist in order 12 (see [2]) (1,2,3,6), (1,1,4,4), (1,1,5,5) .

Let I,L,B,K,Y be the circulant symmetric matrices with first rows

10000, 01111, 01--1, -1111, 00110 ,

and let X be the back circulant matrix with first row

$$x_1 0 x_2 - x_2 0 .$$

Then

$$MN^t = NM^t \quad \text{for} \quad M,N \in \{I,L,B,K,X,Y\} .$$

The following orthogonal designs (1,k) may be constructed by replacing the variables of the orthogonal designs, specified by the matrices indicated (0 is the zero matrix of order 5) :

(1,32) use x_1I, x_2L, 0, x_2B in the (1,2,3,6) orthogonal design;

(1,35) use x_1I, x_2L, x_2I, x_2B in the (1,2,3,6) orthogonal design;

(1,36) use x_1I, 0, x_2K, x_2B in the (1,1,4,4) orthogonal design;

(1,37) use x_1I, x_2I, x_2K, x_2B in the (1,1,4,4) orthogonal design;

(1,45) use x_1I, 0, x_2K, x_2B in the (1,1,5,5) orthogonal design;

(1,46) use x_1I, x_2I, x_2K, x_2B in the (1,1,5,5) orthogonal design;

(1,49) use X, x_2Y, x_2K, x_2B in the (1,1,5,5) orthogonal design.

The design (1,59) may be constructed by using the circulant matrices with the following first rows in the Goethals-Seidel array to give a skew-symmetric W(60,59):

for (1,26) use $x_1I + x_2B,\ x_2A,\ x_2(B+I),\ 0$;

for (1,27) use $x_1I + x_2B,\ x_2A,\ x_2(B+I),\ x_2I$;

for (1,29) use $x_1I + x_2B,\ x_2(A-I),\ x_2W(22,13)$;

for (1,32) use $x_1I + x_2B,\ x_2(A-I),\ x_2B, x_2(A-I)$;

for (1,33) use $x_1I + x_2B,\ x_2(A-I),\ x_2W(22,17)$;

for (1,34) use $x_1I + x_2B,\ x_2(A-I+T),\ x_2(A-I-T),\ x_2B$;

for (1,35) use $x_1I + x_2B,\ x_2(A-I+T),\ x_2(A-I-T),\ x_2(B+I)$;

for (1,36) use $x_1I + x_2B,\ x_2(A+I),\ x_2B,\ x_2B$;

for (1,37) use $x_1I + x_2B,\ x_2(A+I),\ x_2B,\ x_2(B+I)$;

for (1,38) use $x_1I + x_2B,\ x_2(A+I),\ x_2(B+I),\ x_2(B+I)$.

The design (1,43) may be constructed by using the circulant matrices with the following first rows in the Goethals-Seidel array to give a skew-symmetric W(44,43) .

01-1--11-1-, 1111----111, 1-111--111-, 1---1--1--- .

LEMMA 20. *There exists an orthogonal design* (1,k) *in order* 60 *for* $k \in \{1,\dots,6,8,\dots,14,16,\dots,22,24,\dots,27,29,30,32,\dots$ $\dots,37,40,41,45,46,49,59\}$.

Proof. There exists an orthogonal design (1,k) for $k \in \{x\colon x \neq 7,15,23,\ 0 \le x \le 27\}$ in order 28 and $k \in \{x\colon 0 \le x \le 31\}$ in order 32 thus a (1,k) exists for $k \in \{x\colon x \neq 7,15,23,\ 0 \le x \le 27\}$ in order 60.

Now there exists a W(30,29) constructed from two circulant matrices so there exist a (1,1,29) and (1,4,29) in order 60. These give (1,29), (1,30) and (1,33) .

We have shown in this paper that there are designs (1,1,40) and (1,34) in every order 4n, n(odd) ≥ 11, thus we have a (1,34), (1,40) and (1,41) .

01111-1-1-1----, 1---11-11-11---, 111--1-11-1--11,

111-111--111-11 .

7. *Summary.*

Using all means available we have (at least):

Summary 1. If n is odd there are orthogonal designs of order $4n$ and type $(1,k)$ when

(i) $n \geq 3$, $k \in \{1,\ldots,6,8,\ldots,11\}$;

(ii) $n \geq 5$, $k \in \{1,\ldots,6,8,\ldots,14,16,17,18\}$;

(iii) $n \geq 7$, $k \in \{1,\ldots,6,8,\ldots,14,16,17,18,20,21,22,24,\ldots,27\}$;

(iv) $n \geq 9$, $k \in \{1,\ldots,6,8,\ldots,14,16,17,18,20,21,22,24,\ldots,27,29,32,33\}$;

(v) $n \geq 11$, $k \in \{1,\ldots,6,8,\ldots,14,16,\ldots,22,24,\ldots,27,29,32,33,34,40,41\}$;

(vi) $n \geq 13$, $k \in \{1,\ldots,6,8,\ldots,14,16,\ldots,22,24,\ldots,27,29,30,32,33,34,40,41\}$;

Summary 2. There are orthogonal designs of order $8n$ and type $(1,k)$ where

(i) $n = 3,4,5$ or 6, $k \in \{1,\ldots,8n-1\}$;

(ii) $n \geq 7$, $k \in \{1,\ldots,45\}$.

Finally we tabulate the unresolved cases in the conjectures I, II, III, IV, V of the abstract.

Order	Applicable Conjecture	Unresolved Cases	Applicable Conjecture	Unresolved Cases
4	II	true	III	true
8	II	true	IV	true
12	II	true	III	true
16	II	true	IV	true
20	II	true	III	true
24	II	true	IV	true
28	II	true	III	true
32	II	true	IV	true
36	II	31	III	19,30,34
40	II	true	IV	true
44	II	31	III	30,42
48	II	true	IV	true
52	II	37,46,47,49	III	35-38,42-46, 48-50
56	II	47	IV	46,50
60	II	51,53	III	38,42,43, 44,48,50-54, 56-58
64	II	true	IV	true
72	II	52,55,65	IV	30,46,47,56 60-63,68
80	II	true	IV	true

Table 1

True signifies the conjecture is verified.

Order	Applicable Conjecture	Unresolved Cases	Applicable Conjecture	Unresolved Cases
2	I	true	V	true
6	I	true	V	true
10	I	true	V	true
14	I	true	V	true
18	I	9	V	9,16
22	I	16	V	9
26	I	17	V	16
30	I	18	V	16,25

Table 2

True signifies the conjecture is verified.

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