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Families of weighing matrices

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A weighing matrix is an $n \times n$ matrix $W = W(n, k)$ with entries from $\{0, 1, -1\}$, satisfying $WW^t = kI_n$. We shall call k the *degree of* W. It has been conjectured that if $n \equiv 0 \pmod 4$ then there exist $n \times n$ weighing matrices of every degree $k \le n$.

We prove the conjecture when n is a power of 2. If n is not a power of two we find an integer $t < n$ for which there are weighing matrices of every degree $\le t$.

Taussky [1] suggested the following generalization of Hadamard matrices:

A *weighing matrix* is an $n \times n$ matrix $W = W(n, k)$ with entries from $\{0, 1, -1\}$, satisfying $WW^t = kI_n$. We shall call k the *degree of* W. In [3, p. 433], it was conjectured that

(*) *If* $n \equiv 0 \pmod 4$ *then there exist* $n \times n$ *weighing matrices of every degree* $k \le n$.

(Note that an $n \times n$ weighing matrix of degree n is an Hadamard matrix and so (*) is a generalization of the conjecture on the existence of Hadamard matrices of order n for every $n \equiv 0 \pmod 4$.)

In [2] the validity of (*) was established for $n \in \{4, 8, 12, 16, 20, 24, 28, 32, 40\}$ and partial results were obtained

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for $n \in \{36, 44, 52, 56\}$ in that sets of values of k were obtained for which $W(n, k)$ exists.

For all n let $g(n)$ be the maximum degree q for which there exist weighing matrices $W(n, k)$ for all degrees $k \le q$. Thus, conjecture (*) is equivalent to:

(*) $$g(n) = n \text{ for all } n \equiv 0 \pmod 4 .$$

The methods of [2] can be used to show that $g(2^n) \ge 34$ for all $n > 5$. We show [Corollary 2 to our theorem] that in fact $g(2^n) = 2^n$ for all n and hence establish (*) for all powers of 2. As another corollary to the theorem we show that $g(2^k n) \ge 2^k$ for all odd n and all $k \ge 1$. This is better, asymptotically, than results obtained by the methods of [2].

Call $\{M_1, M_2, \ldots, M_m\}$ an *M-family of order* n if for each i, $1 \le i \le m$:

(1) M_i is a weighing matrix of order n and degree i, and

(2) $M_i M_m^t = M_m M_i^t$.

Let $\mu(n)$ be the largest m for which an M-family of order n exists. Evidently $g(n) \ge \mu(n)$.

THEOREM. *If* $\mu(n) \ge m$ *then* $\mu(2n) \ge 2m$.

Proof. Suppose $\{M_1, M_2, \ldots, M_m\}$ is an M-family of order n, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and I_p is the $p \times p$ identity matrix.

Define

(a) $\overline{M}_i = I_2 \otimes M_i$ for each i, $1 \le i \le m$,

(b) $\overline{M}_{m+i} = \overline{M}_i + A \otimes M_m$ for each i, $1 \le i \le m-1$, and

(c) $\overline{M}_{2m} = H \otimes M_m$.

It is easily verified that $\{\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_{2m}\}$ is an M-family of order $2n$. The matrices defined in (a) and (c) satisfy (1) and (2) because the

M_i do. The matrices defined in (b) satisfy (1) because the Hadamard product of A and I_2 being the zero matrix implies they are $(1, -1, 0)$-matrices, and $\overline{M}_{m+i}\overline{M}^t_{m+i} = (m+i)I_{2n}$ because A is skew symmetric; they satisfy (2) because $HA^t = AH$.

COROLLARY 1. *$\mu(2^k) = 2^k$ for all integers $k \geq 1$.*

Proof. $\{I_2, H\}$ is an M-family of order 2 .

COROLLARY 2. *$g(2^k) = 2^k$ for all integers $k \geq 1$.*

COROLLARY 3. *(*) is true for all powers of 2 .*

COROLLARY 4. *$g(2^k n) \geq 2^k$ for all integers n and $k \geq 1$.*

Proof. Each matrix $I_n \otimes M_i$ is a weighing matrix of order nm and degree i if M_i is a weighing matrix of order n and degree i .

Lemma 1 (i), 2 (i) and (iii) of [2] imply immediately that

(†) *If (*) holds for n then $g(2^t n) \geq n + 2t$ for all integers $t \geq 0$.*

But Corollary 4 gives far better estimates of $g(2^t n)$ than does (†) for all sufficiently large t . For example, the results of [2] and (†) give us $g(2^t 24) \geq 24 + 2t$ but Corollary 4 gives us $g(2^t 24) \geq 2^{t+3}$ which is a better estimate for all $t \geq 2$.

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