A NOTE ON BIBD'S

Dedicated to the memory of Hanna Neumann

JENNIFER WALLIS

(Received 21 February 1972; revised 20 March 1972)

Communicated by G. Szekeres

A *balanced incomplete block design or BIBD* is defined as an arrangement of v objects in b blocks, each block containing k objects all different, so that there are r blocks containing a given object and λ blocks containing any two given objects.

In this note we shall extend a method of Sprott [2, 3] to obtain several new families of BIBD's. The method is based on the first Module Theorem of Bose [1] for pure differences.

We shall frequently be concerned with collections in which repeated elements are counted multiply, rather than with sets. If T_1 and T_2 are two such collections then T_1 & T_2 will denote the result of adjoining the elements of T_1 to T_2, with total multiplicities retained. We use the brackets, $\{\ \}$, to denote sets and square brackets, $[\ ]$, to denote collections of elements which may have repetitions. See [5] for results using these concepts.

1. Preliminaries

Let $v = mh + 1 = p^\alpha$, where p is a prime. Let x be a primitive element of $GF(v)$ and write G for the group generated by x. Define H_0 a subgroup of G and H_i, $i \neq 0$, its cosets by

$$H_i = \{x^{hj+i}\colon 0 \leqq j \leqq m-1\} \qquad i = 0, 1, \cdots, h-1,$$

Now consider the collection of differences between elements of H_i

$$\begin{aligned}
&[x^{hj+i} - x^{hl+i}\colon l \neq j,\ 1 \leqq j,\ l \leqq m-1] \\
=\ &[x^{hl+i}(x^{h(j-l)} - 1\colon l \neq j,\ 1 \leqq j,\ l \leqq m-1] \\
=\ &a_0H_0 \mathbin{\&} a_1H_1 \mathbin{\&} \cdots \mathbin{\&} a_{h-1}H_{h-1} \\
=\ &\mathop{\&}_{s=0}^{h-1} a_sH_s.
\end{aligned}$$

This follows because $H_i = \{x^{hl+i}: 1 \leqq l \leqq m-1\}$ is a coset and whenever it is multiplied by some element x^r of the group we have H_{i+r}. Now there are $m(m-1)$ differences between elements of H_i so

$$\sum_{s=0}^{h-1} a_s = m-1,$$

where the a_s are non-negative integers.

The differences from $H_i \cup H_j$ where $i \neq j$ are (differences from H_i) & (differences from H_j) & (elements of $H_i - H_j$) & $-$(elements of $H_i - H_j$)

$$= \left(\mathop{\&}_{s=0}^{h-1} a_s H_s\right) \;\&\; \left(\mathop{\&}_{s=0}^{h-1} b_s H_s\right) \;\&\; \left(\mathop{\&}_{s=0}^{h-1} c_s H_s\right) \;\&\; -\left(\mathop{\&}_{s=0}^{h-1} c_s H_s\right)$$

$$= \mathop{\&}_{s=0}^{h-1} d_s H_s$$

where

$$\sum_{s=0}^{h-1} a_s = \sum_{s=0}^{h-1} b_s = m-1, \; \sum_{s=0}^{h-1} c_s = m, \text{ and } \sum_{s=0}^{h-1} d_s = 2(2m-1).$$

Note that if we had started by considering the differences between elements of H_{i+1} we would have

$$\mathop{\&}_{s=0}^{h-1} a_s H_{s+1},$$

and for $H_{i+1} \cup H_{j+1}$

$$\mathop{\&}_{s=0}^{h-1} d_s H_{s-1}.$$

So we have, by considering, the totality of differences from the sets $H_i, H_{i+1}, \cdots H_{i+h-1}$,

$$\mathop{\&}_{i=0}^{h-1} \left(\sum_{s=0}^{h-1} a_s\right) H_i = (m-1)G,$$

and for the totality of differences from the sets

$$H_i \cup H_j,\, H_{i+1} \cup H_{j+1}, \cdots, H_{i+h-1} \cup H_{j+h-1}$$

we have

$$\mathop{\&}_{i=0}^{h-1} \left(\sum_{s=0}^{h-1} d_s\right) H_i = 2(2m-1)G.$$

Similarly, by considering the totality of differences from the sets $H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_t}$, where $i_1 = 0, 1, \cdots, h-1, i_j = i_1 + s_j$ for positive integers s_j, $0 = s_1 < s_2 < \cdots < s_t < h$, we will have

$$t(mt-1)G.$$

2. Results

It follows from the preceding observation that the blocks formed by the elements of the sets

$$B_{i_1} = B_{i_1}(s_2, \cdots, s_t) = H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_t}$$
$$= \{x^{i_1}, x^{h+i_1}, \cdots, x^{(m-1)h+i_1}, x^{i_2}, x^{h+i_2}, \cdots, x^{(m-1)h+i_2}, \cdots, x^{i_t}, x^{h+i_t}, \cdots, x^{(m-1)h+i_t}\},$$

$i_1 = 0, 1, \cdots, h-1$ can be taken as "initial blocks" in Bose's first Module Theorem [1]. That is, the collection of all blocks $B_{i_1, \theta}$, $\theta \in GF(v)$, obtained from B_{i_1} by adding an arbitrary element θ of $GF(v)$ to each member of B_{i_1}, form a BIBD with parameters

$$v = mh + 1 = p^{\alpha}, b = hv, r = tmh, k = tm, \lambda = t(mt-1).$$

So we obtain

THEOREM 1. (Series Z_1). *If $v = mh + 1 = p^{\alpha}$ where p is a prime, and t is a positive integer $\leqq h$, then a design with parameters*

$$v = mh + 1, b = hv, r = tmh, k = tm, \lambda = t(mt-1)$$

can be constructed via the initial blocks

$$B_{i_1}(s_2, \cdots, s_t) = H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_t},\ i_1 = 0, 1, \cdots, h-1,$$

where $i_j = i_1 + s_j$ for fixed positive integers s_j,

$$0 = s_1 < s_2 < \cdots < s_t < h.$$

If instead of considering the previous sets we consider the differences from

$$0 \cup H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_t},\ i_1 = 0, 1, \cdots, h-1, t \leqq h,$$

then the totality of differences from these sets is

$$t(mt + 1)G,$$

and hence we have

THEOREM 2. (Series Z_2). *If $v = mh + 1 = p^{\alpha}$ where p is a prime, and t is a positive integer $\leqq h$, then the design with parameters*

$$v = mh + 1,\ b = hv,\ r = (tm + 1)h, k = tm + 1,\ \lambda = (tm + 1)t$$

can be constructed via the initial blocks

$$B_{i_1}(s_2, \cdots, s_t) = 0 \cup H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_t},\ i_1 = 0, 1, \cdots, h-1,$$

where $i_j = i + s_j$ for fixed positive integers s_j, $0 = s_1 < s_2 < \cdots < s_t < h$.

THEOREM 3. (Series Z_3). *If* $v = (2\mu + 1)2h + 1 = p^\alpha$, *where* p *is a prime, and* t *is a positive integer* $\leqq h$, *then the design with parameters*

$$v = (2\mu + 1)2h + 1,\ b = vh,\ r = (2\mu + 1)ht,\ k = (2\mu + 1)t,\ \lambda = \tfrac{1}{2}t[t(2\mu + 1) - 1]$$

can be constructed via the initial blocks

$$B_{i_1}(s_2, \cdots, s_t) = H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_t},\ i_1 = 0, 1, \cdots, h-1,$$

$i_j = i_1 + s_j$ *for fixed positive integers* s_j, $0 = s_1 < s_2 < \cdots < s_t < h$.

THEOREM 4. (Series Z_4). *If* $v = (2\mu + 1)2h + 1 = p^\alpha$, *where* p *is a prime, and* t *is a positive integer* $\leqq h$, *then the design with parameters*

$$v = (2\mu + 1)2h + 1,\ b = vh,\ r = h[(2\mu + 1)t + 1],\ k = (2\mu + 1)t + 1,\ \lambda = t[(2\mu + 1)t + 1]$$

can be constructed via the initial blocks

$$B_{i_1}(s_2, \cdots, s_t) = 0 \cup H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_t},\ i_1 = 0, 1, \cdots, h-1,$$

where $i_j = i_1 + s_j$ *for fixed positive integers* s_j, $0 = s_1 < s_2 < \cdots < s_t < h$.

PROOF OF THEOREM 3 AND 4. In our previous discussion we have replaced m by $2\mu + 1$ and h by $2h$. Now $-1 \in H_h$ so the totality of differences from H_1 becomes

$$a_0 H_0 \ \&\ a_1 H_1 \ \&\ \cdots \ \&\ a_{h-1} H_{h-1} \ \&\ a_0 H_h \ \&\ a_1 H_{h+1} \ \&\ \cdots \ \&\ a_{h-1} H_{2h-1}$$

because if $x^{gh+i_s} - x^{rh+i_n} \in H_l$ then $x^{rh+i_n} - x^{gh+i_s} \in H_{l+h}$.

We may then proceed as before while noting the dependence of the coefficients of H_i and H_{i+h} in the collection of sums of differences.

By observing that our series are extensions of those of Sprott we can also show

THEOREM 5. (Series Z_5). *If* $v = (4\mu + 1)4h + 1 = p^\alpha$, *where* p *is a prime and if the collection of differences from the initial block*

$$B_{i_1}(s_2, s_3, \cdots, s_t) = H_{2i_1} \cup H_{2i_2} \cup \cdots \cup H_{2i_t},\ i_1 = 0, 1, \cdots, h-1.$$

are written as

$$\mathop{\&}_{s=0}^{4h-1} a_s\{x^{s+4hj} : 0 \leqq j \leqq 4\mu\}$$

where we may pair the coefficients a_s *such that* $a_{2i} = a_{2t_i+1}$ *for all* $i = 0, 1, \cdots, 2h(4\mu + 1) - 1$, *then the design with parameters*

$$v = 4h(4\mu + 1) + 1,\ b = hv,\ r = ht(4\mu + 1),\ k = (4\mu + 1)t,\ \lambda = \tfrac{1}{4}t[(4\mu + 1)t - 1]$$

can be constructed via these initial blocks where $\frac{1}{4}t[(4\mu+1)t-1]$ *is a positive integer,* $i_j = i_1 + s_j$ *for fixed positive integers* s_j, $0 = s_1 < s_2 < \cdots < s_t < h$.

Acknowledgement

The author wishes to thank Professor G. Szekeres for his suggestions which have substantially improved the readability of this note.

References

[1] R. C. Bose, 'On the construction of balanced incomplete block designs', *Ann. Eugenics*, 9 (1939), 353–399.

[2] D. A. Sprott, 'A note on balanced incomplete block designs', *Can. J. Math.*, 6 (1965), 341–346.

[3] D. A. Sprott, 'Some series of balanced incomplete block designs', *Sankhya, Ser. A*, 17 (1956), 185–192.

[4] E. Nemeth, 'A new family of BIBD'S', *Ann. Math. Stat.*, 42 (1971), 1118–1120.

[5] Jennifer Wallis, 'On supplementary difference sets', *Aeq. Math.* 8 (1972), 242–257.

Institute of Advanced Studies
Australian National University
Canberra, A.C.T.