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Note

Kronecker Products and BIBDs

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Recursive constructions are given which permit, under conditions described in the paper, a (v, b, r, k, λ)-configuration to be used to obtain a (v', b', r', k, λ)-configuration.

Although there are many equivalent definitions we will mean by a (v, b, r, k, λ)-*configuration* or *BIBD* that (0, 1)-matrix A of size $v \times b$ with row sum r and column sum k satisfying

$$AA^T = (r - \lambda)I + \lambda J$$

where, as throughout the remainder of this paper, I is the identity matrix and J the matrix with every element $+1$ whose sizes should be determined from the context or by a subscript (J_n is *square* of order n).

In the case of block matrices, $(X)_{ij}$ and (X_{ij}) mean the matrix whose (i, j)-th block is X; for example, $(T^{i-j})_{ij}$ is the matrix whose (i, j)-th block is T^{i-j}. We define the *Kronecker product* of two matrices $A = (a_{ij})$ of order $m \times n$ and B of any order as the $m \times n$ block matrix

$$A \times B = (a_{ij}B)_{ij} .$$

For more details the reader is referred to Marshall Hall [1].

We will use T for the circulant matrix of order q given by

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

For q a prime we have shown in Jennifer Wallis [3] that

$$Q = (T^{(i-1)(j-1)})_{ij}$$

satisfies

$$QQ^T = qI_q \times I_q + (J_q - I_q) \times J_q ,$$
$$J \cdot Q = qJ.$$

We are concerned with the existence of a (0, 1) matrix Q of size $mv \times v^2$ which satisfies

$$\left.\begin{aligned} QQ^T &= vI_m \times I_v + (J_m - I_m) \times J_v , \\ JQ &= mJ; \end{aligned}\right\} \tag{1}$$

if such a matrix exists we will say $\mathbf{P}(m, v)$ holds. Thus the result cited above shows that $\mathbf{P}(q, q)$ holds for any prime q; we also showed in [3] that $\mathbf{P}(q, q)$ holds for any prime power q. Further, it was proved in [4] that $\mathbf{P}(m, v)$ holds if and only if there exists a set of $m - 2$ mutually orthogonal Latin squares of order v, and that a (0, 1) matrix Q satisfying (1) must have the form

$$Q = \begin{bmatrix} E \\ A \\ \vdots \\ A_{m-1} \end{bmatrix},$$

where E and the A_i are of size $v \times v^2$ and have constant row sums v and column sums 1. From the latter fact it is clear that if Q satisfies (1) then the matrix formed by deleting A_n and subsequent blocks satisfies (1) with m replaced by n, so

$$\mathbf{P}(m, v) \Rightarrow \mathbf{P}(n, v) \qquad \text{when} \quad n < m.$$

If we are referring to $\mathbf{P}(m, v)$, then Q, E and A_i will mean the matrices just mentioned.

MAIN THEOREM

We shall exploit the following theorem, which is a generalization of Lemma 6 of [3]:

THEOREM 1. *Suppose B is a (v, b, r, k, λ)-configuration and suppose R is a* (0, 1) *matrix of size $lv \times tv^2$ satisfying*

$$\left.\begin{aligned} RR^T &= a_1 vI_l \times I_v + a_2(J_l - I_l) \times J_v , \\ JR &= kJ, \end{aligned}\right\} \tag{2}$$

where a_2 divides λ. Then necessarily $la_1 = kt$ and $(l-1)\,a_2 = (k-1)\,a_1$, and

$$[I_l \times B \mid R, R, \ldots, R] \qquad (\lambda/a_2 \text{ copies of } R)$$

is an $(lv, lb + \lambda tv^2/a_2, r + \lambda a_1 v/a_2, k, \lambda)$-configuration.

Proof. By summing the entries of R in two ways we obtain $la_1 = kt$. It is easy to check that the matrix exhibited is the required configuration; one of the standard necessary conditions for a (v, b, r, k, λ)-configuration is $\lambda(v-1) = r(k-1$; substituting in the parameters of the configuration we constructed we have

$$(l-1)\,a_2 = (k-1)\,a_1 .$$

In the particular case where $t = a_1 = a_2 = 1$ and $k = l$, the existence of a suitable R satisfying (2) is simply $\mathbf{P}(k, v)$.

COROLLARY 2. *If there exist a (v, b, r, k, λ)-configuration and a set of $k-2$ mutually orthogonal Latin squares of order v, then there is a $(kv, kb + \lambda v^2, r + \lambda v, k, \lambda)$-configuration.*

EXAMPLE. Suppose $v = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ is a decomposition of v into powers of distinct primes, and suppose

$$k \leqslant \min_i \,(p_i^{\alpha_i}) + 1.$$

Then there is a set of $k-2$ mutually orthogonal Latin squares of order v [1, p. 192], so the existence of a (v, b, r, k, λ)-configuration for this k and v implies the existence of a $(kv, bk + \lambda v^2, r + \lambda v, k, \lambda)$-configuration.

EXAMPLE. Hanani [2] has shown (in terms of Latin squares) that $\mathbf{P}(5, v)$ always holds when $v \geqslant 52$ and $\mathbf{P}(7, v)$ always holds when $v \geqslant 63$, so Corollary 2 can be applied in the corresponding cases.

FIRST APPLICATION

Suppose q is a prime and ω is a primitive q-th root of unity. T is of order q. Define a $q \times q$ matrix P by

$$P = (p_{ij}), \qquad p_{ij} = \omega^{(i-1)(j-1)}.$$

Now define square matrices S_{ij}, $i = 1, 2, \ldots q^s$ and $j = 1, 2, \ldots, q^s$ where

s is any positive integer, as follows: if the (i, j) element of the Kronecker product of s copies of P is ω^a, then $S_{ij} = T^a$.

Assume that $\mathbf{P}(k, v)$ holds for some $k \leqslant q^s$. Write

$$R = \begin{bmatrix} S_{11} \times E & S_{12} \times E & \cdots & S_{1q^s} \times E \\ S_{21} \times A_1 & S_{22} \times A_1 & \cdots & S_{2q^s} \times A_1 \\ \vdots & & & \\ S_{k1} \times A_{k-1} & S_{k2} \times A_{k-1} & \cdots & S_{kq^s} \times A_{k-1} \end{bmatrix}.$$

R is a (0, 1) matrix of size $kvq \times v^2q^{s+1}$, and it is readily shown that

$$RR^T = q^s v I_k \times I_{qv} + q^{s-1}(J_k - I_k) \times J_{qv},$$

so R satisfies (2) with v replaced by vq, $l = k$ and $t = a_1 = a_2 = q^{s-1}$. So we have proved the following:

THEOREM 3. *Suppose there exists a* (qv, b, r, k, λ)*-configuration, where* q *is a prime, and suppose* $\mathbf{P}(k, v)$ *holds. If* s *is a positive integer such that* q^{s-1} *divides* λ *and* $k \leqslant q^s$, *then there exists a* $(kvq, kb + \lambda q^2v^2, r + \lambda qv, k, \lambda)$*-configuration.*

Corollaries can easily be constructed using the examples in the preceding section.

SECOND APPLICATION

Suppose q is a prime and suppose $\mathbf{P}(k, v)$ holds where $k \leqslant q + 1$. The matrices I and T will be of order q.

We consider the (0, 1) block matrix P,

$$P = \begin{bmatrix} I \times A_1 & I \times A_1 & I \times A_1 & \cdots & I \times A_1 \\ I \times A_2 & T \times A_2 & T^2 \times A_2 & \cdots & T^{q-1} \times A_2 \\ I \times A_3 & T^2 \times A_3 & T^4 \times A_3 & \cdots & T^{2(q-1)} \times A_3 \\ \vdots & & & & \\ I \times A_{k-1} & T^{k-2} \times A_{k-1} & T^{2(k-2)} \times A_{k-1} & \cdots & T^{(k-2)(q-1)} \times A_{k-1} \end{bmatrix},$$

which is a $(k - 1) \times q$ array of $qv \times qv^2$ blocks. Write E' for $[I \times E, I \times E, \ldots, I \times E]$, there being q copies of $I \times E$, and denote by $T^i \cdot P$ the result of multiplying the first of the two components of every block entry of P by T^i. Then

$$R = \begin{bmatrix} E' & E' & E' & \cdots & E' \\ P & T \cdot P & T^2 \cdot P & \cdots & T^{q-1} \cdot P \end{bmatrix}$$

is a (0, 1) matrix of suitable size which satisfies (2) with $l = k$, v replaced by qv and $a_1 = a_2 = t = q$. Hence we have

THEOREM 4. *If* $\mathbf{P}(k, v)$ *holds and there is a* (qv, b, r, k, λ)*-configuration, where* q *is a prime not less than* $k - 1$ *and* q *divides* λ, *then there exists a* $(kqv, kb + \lambda q^2v^2, r + \lambda qv, k, \lambda)$*-configuration.*

Again corollaries can be formed at will.

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