ORTHOGONAL (0,1,-1)-MATRICES

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ABSTRACT

We study the conjecture:

There exists a square (0,1,-1)-*matrix* $W = W(w,k)$ *of order* w *satisfying*

$$WW^T = kI_w$$

for all $k = 0, 1, \ldots, w$ *when* $w \equiv 0 \pmod 4$.

We prove the conjecture is true for 4, 8, 12, 16, 20, 24, 28, 32, 40 and give partial results for 36, 44, 52, 56.

One generalization of *Hadamard matrices* is to *weighing matrices* (see Olga Taussky [3]), that is square (0,1,-1)-matrices, W, of order n satisfying

$$WW^T = kI_n, \quad k \leq n, \tag{1}$$

where I_n is the identity matrix of order n, W^T denotes W transposed. Clearly

$$WW^T = W^TW = kI_n. \tag{2}$$

These matrices have application both in design of weighing experiments (see Raghavarao [2]) and in coding theory.

Write $W(w,k)$ for a weighing matrix satisfying (1).

RELEVANT MATRICES

Clearly $(0,1,-1)$-matrices satisfying $WW^T = 0$ and $1I$, always exist. Matrices satisfying

$$WW^T = nI_n, \qquad n \equiv 0 \pmod 4)$$

are *Hadamard matrices* and if $U = I + W$ is a *skew-Hadamard matrix*

$$WW^T = (n-1)I_n.$$

For up-to-date results about these matrices we refer the reader to [1, 4, 5]. These matrices exist for 2 and all $n \equiv 0 \pmod 4$, $n \leq 100$.

If $n \equiv 2 \pmod 4$ a matrix satisfying $U = I + W$ with

$$WW^T = (n-1)I_n, \quad W^T = W$$

is called a *symmetric conference matrix* and these can only exist if

$$n - 1 = a^2 + b^2$$

a,b integer (see Raghavarao [2]). These matrices exist for $n \equiv 2 \pmod 4$, $n - 1$ a prime power for $n < 100$.

Write H_n for the Hadamard matrix of order n, J_n for the matrix of order n of all ones, S_n for the matrix of order n with zero diagonal and other elements ± 1 satisfying

$$S_n S_n^T = (n-1)I_n.$$

The symbol $\times$ denotes the Kronecker product and the orders of all matrices are assumed to be compatible under binary operations.

SOME CONSTRUCTIONS

CONSTRUCTION 1. Provided $AA^T + BB^T + CC^T + DD^T = mI_n$ and for any X, $Y \in \{A, B, C, D\}$, X and Y are (0,1,-1)-matrices and $XY^T = YX^T$, then

$$W = \begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}$$

satisfies

$$WW^T = mI_{4n}$$

CONSTRUCTION 2. Provided $\sum_{i=1}^{8} A_i A_i^T = mI_n$, and each A_i is a (0,1,-1)-matrix and $A_i A_j^T = A_j A_i^T$, then

$$W = \left[\begin{array}{cccc|cccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \\ -A_2 & A_1 & A_4 & -A_3 & A_6 & -A_5 & A_8 & -A_7 \\ -A_3 & -A_4 & A_1 & A_2 & -A_7 & A_8 & A_5 & -A_6 \\ -A_4 & A_3 & -A_2 & A_1 & A_8 & A_7 & -A_6 & -A_5 \\ \hline -A_5 & -A_6 & A_7 & -A_8 & A_1 & A_2 & -A_3 & A_4 \\ -A_6 & A_5 & -A_8 & -A_7 & -A_2 & A_1 & A_4 & A_3 \\ -A_7 & -A_8 & -A_5 & A_6 & A_3 & -A_4 & A_1 & A_2 \\ -A_8 & A_7 & A_6 & A_5 & -A_4 & -A_3 & -A_2 & A_1 \end{array}\right] \qquad (3)$$

satisfies

$$WW^T = mI_{8n}.$$

CONSTRUCTION 3. Let $CC^T = cI_d$ and $SS^T = sI_t$ where $t \equiv 0 \pmod 2$ also let C have zero diagonal and $C^T = C$. Then with $K = \begin{bmatrix} & & & & 1 \\ & & & -1 & \\ & & \cdot & & \\ & \cdot & & & \\ & 1 & & & \\ -1 & & & & \end{bmatrix}$

$$D = I \times SK + C \times S$$

satisfies

$$DD^T = (c + 1)sI_{dt}.$$

CONSTRUCTION 4. Provided $AA^T + BB^T + CC^T + DD^T = mI_n$, and for any $X, Y \in \{A, B, C, D\}$, X and Y are circulant $(0,1,-1)$-matrices and $XY = YX$, then

$$W = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & -D^TR & C^TR \\ -CR & D^TR & A & -B^TR \\ -DR & -C^TR & B^TR & A \end{bmatrix} \qquad (4)$$

satisfies

$$WW^T = mI_{4m}.$$

CONSTRUCTION 5. If there is an Hadamard array (Baumert-Hall array, see [5]) on 4 indeterminates each repeated t times then there exists a $W(4t,t)$, $W(4t,2t)$, $W(4t,3t)$ and a $W(4t,4t)$ by replacing the indeterminates by 0 or 1 as appropriate.

Such arrays exist [1, 5] for $t \in \{1, 3, 5, 7, \ldots, 19, 25\}$.

CONSTRUCTION 6. If there exist two circulant (0,1,-1)-matrices A and B of order n satisfying

$$AA^T + BB^T = kI_n$$

then

$$W = \begin{bmatrix} A & B \\ -B^T & A^T \end{bmatrix} \tag{5}$$

satisfies

$$WW^T = kI_{2n}.$$

CONSTRUCTION 7. If there exist two circulant (0,1,-1)-matrices A and B of order n satisfying

$$AA^T + BB^T = (k + 2)I - 2J$$

$$AJ = 0, \quad BJ = J$$

then

$$W = \left[\begin{array}{cc|cc} 0 & 1 & e & e \\ 1 & 0 & e & -e \\ \hline e^T & -e^T & A & B \\ e^T & e^T & -B^T & A^T \end{array}\right] \tag{6}$$

satisfies

$$WW^T = \left[\begin{array}{cc|c} 2n+1 & 0 & \\ 0 & 2n+1 & 0 \\ \hline 0 & & kI_{2n} \end{array}\right]. \tag{7}$$

CONSTRUCTION 8. Suppose $S^T = S = W(w,w-1)$

$$\begin{bmatrix} S & S \\ S & -S \end{bmatrix}, \left[\begin{array}{cc|cc} S & S+I & S & S-I \\ -S-I & S & -S+I & S \\ \hline -S & S-I & S & -S-I \\ -S+I & -S & S+I & S \end{array}\right], \begin{bmatrix} 0 & S & S & S \\ -S & 0 & S & -S \\ -S & -S & 0 & S \\ -S & S & -S & 0 \end{bmatrix},$$

$$\begin{bmatrix} I & S & S & S \\ -S & 0 & S & -S \\ -S & -S & 0 & S \\ -S & S & -S & 0 \end{bmatrix}, \begin{bmatrix} S & S & S & S \\ -S & S & S & -S \\ -S & -S & S & S \\ -S & S & -S & S \end{bmatrix}, \left[\begin{array}{cc|cc} 0 & S & S+I & S-I \\ -S & 0 & -S+I & S+I \\ \hline -S-I & S-I & 0 & -S \\ -S+I & -S-I & S & 0 \end{array}\right] \text{are}$$

$W(2w,2w-2)$, $W(4w,4w-2)$, $W(4w,3w-3)$, $W(4w,3w-2)$, $W(4w,4w-4)$, $W(4w,3w-1)$ respectively.

CONSTRUCTION 9. Let C be a $(0,1,-1)$-matrix with zero diagonal satisfying

$$CC^T = cI_d$$

and let B be a $(0,1,-1)$-matrix satisfying

$$BB^T = aI_c - J_c, \quad BJ = 0.$$

Consider

$$K = I\times J + C\times B,$$

then

$$KK^T = I\times cJ + cI_d\times(aI - J) + C^T\times JB^T + C\times BJ^T$$

$$= acI_{cd}$$

and hence is a $W(cd, ac)$.

[Since these conditions are always satisfied when $a + 1 = c + 1 = d$ is the order of a conference matrix or a skew-Hadamard matrix we have a $W(d(d - 1), (d - 1)^2)$ for these orders.]

SOME RESULTS ON THE CONJECTURE

First we give a theorem and then some results.

THEOREM. *There can only exist* $W(2n,k)$ *constructed of two circulant matrices* A *and* B *of order* n, *of the form*

$$W = \begin{bmatrix} A & B \\ -B^T & A^T \end{bmatrix},$$

when

$$k = a^2 + b^2.$$

PROOF. Let $T = (t_{ij})$ of order n be given by

$$t_{1j} = \begin{cases} 1 & j = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$t_{ij} = t_{1,j-i+1},$$

then $A = \sum_{i=1}^{n} a_i T^i$, $B = \sum_{i=1}^{n} b_i T^i$ where $a_i, b_i = 0, 1, -1$.

Now

$$AA^T + BB^T = kI = (\sum_{i=1}^{n} a_i T^i)(\sum_{i=1}^{n} a_i T^{n-i}) + (\sum_{i=1}^{n} b_i T^i)(\sum_{i=1}^{n} b_i T^{n-i}).$$

This is the matrix representation of

$$(\sum_{i=1}^{n} a_i \omega^i)(\sum_{i=1}^{n} a_i \omega^{n-i}) + (\sum_{i=1}^{n} b_i \omega^i)(\sum_{i=1}^{n} b_i \omega^{n-i}) = k,$$

where ω is an nth root of unity. This must be true for all nth roots of unity including 1 so

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 = k,$$

and we have the result.

COROLLARY. *There can only exist* W(2n,k) *constructed of two circulant matrices* A *and* B *of order* n *for*

$k < n$, *and* $k = 0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 20, 25, 26, 29, 34, 41, \ldots$

LEMMA 1. *(i) If there exists a* $W(w,k)$ *then* $W(w,k) \oplus W(w,k)$ *is a* $W(2w,k)$ *and* $W(w,k) \times H_2$ *is a* $W(2w,2k)$.

(ii) If there exist $W_1(w_1,k)$ *and* $W_2(w_2,k)$ *then* $W_1(w_1,k) \oplus W_2(w_2,k)$ *is a* $W(w_1 + w_2,k)$.

(iii) If there exist $W_1(w_1,k_1)$ *and* $W_2(w_2,k_2)$ *then* $W_1(w_1,k_1) \times W_2(w_2,k_2)$ *is a* $W(w_1w_2,k_1k_2)$.

LEMMA 2. *If the conjecture is true for* w *then there exist*

(i) $W(2w,k)$, $0 \le k \le w$,

(ii) $W(2w,2k)$, $0 \le k \le w$,

(iii) $W(2w,w+1)$.

PROOF. Use Lemma 1 for (i) and (ii). For (iii) use the matrix

$$\begin{bmatrix} W(w,w) & I_w \\ I_w & -W^T(w,w) \end{bmatrix}.$$

LEMMA 3. *The conjecture is true for* $w = 2, 4, 8, 16$.

PROOF. (i) For $w = 2$, the required matrices are 0, I_2, H_2;

(ii) for $w = 4$, the result follows using part (i), Lemma 2 and S_4;

(iii) for $w = 8$, by part (ii), Lemma 2 and S_8 we have the conjecture;

(iv) for $w = 16$, by part (iii), Lemma 2 and S_{16} we have that $W(8,k)$ exists for $k = 0, 1, 2, \ldots, 8, 9, 10, 12, 14, 15, 16$. Now with $S = S_4$, and $H = J_4 - 2I_4$, in the matrices

$$\begin{bmatrix} 0 & S & S+I & S+I \\ -S & 0 & -S-I & S+I \\ -S+I & S-I & 0 & -S \\ -S+I & -S+I & S & 0 \end{bmatrix} \text{ and } \begin{bmatrix} I & H & H & H \\ -H & I & H & -H \\ -H & -H & I & H \\ -H & H & -H & I \end{bmatrix},$$

we have the result for 16.

LEMMA 4. *If there exists a* $W(w,k) = A$ *then*

$$\left[\begin{array}{cc|cc} A & A & & \\ A^T & -A^T & & \\ \hline & & -A^T & -A \\ & & -A^T & A \end{array}\right], \left[\begin{array}{cc|cc} A & A & I_w & \\ A^T & -A^T & & I_w \\ \hline I_w & & -A^T & -A \\ & I_w & -A^T & A \end{array}\right], \left[\begin{array}{cc|cc} A & A & A & \\ A^T & -A^T & & A \\ \hline A^T & & -A^T & -A \\ & A^T & -A^T & A \end{array}\right], \left[\begin{array}{cc|cc} A & A & I & I \\ A^T & -A^T & I & -I \\ \hline I & I & -A^T & -A^T \\ I & -I & -A & A \end{array}\right]$$

are $W(4w,2k)$, $W(4w,2k + 1)$, $W(4w,3k)$ *and* $W(4k,2k + 2)$ *respectively.*

LEMMA 5. *If there exists a* $W(w, w-1) = S$ *with* $S^T = -S$ *then*

$$\begin{bmatrix} S & S+I \\ S-I & -S \end{bmatrix},$$

$$\left[\begin{array}{cc|cc} S & S+I & S & S+I \\ S-I & -S & S-I & -S \\ \hline S & S+I & -S & -S-I \\ S-I & -S & -S+I & S \end{array}\right], \quad \left[\begin{array}{cc|cc} S & S+I & S+I & S+I \\ S-I & -S & S-I & -S+I \\ \hline S-I & S+I & -S & -S-I \\ S-I & -S-I & -S+I & S \end{array}\right]$$

are $S_{2w} = W(2w, 2w-1)$, $W(4w, 4w-2)$, $S_{4w} = W(4w, 4w-1)$ *respectively, while*

$$\begin{bmatrix} 0 & S & S+I & S+I \\ -S & 0 & -S-I & S+I \\ -S+I & S-I & 0 & -S \\ -S+I & -S+I & S & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & S & S & S \\ -S & 0 & S & -S \\ -S & -S & 0 & S \\ -S & S & -S & 0 \end{bmatrix} \text{ and } \begin{bmatrix} S & S & S & S \\ -S & S & S & -S \\ -S & -S & S & S \\ -S & S & -S & S \end{bmatrix}$$

are $W(4w, 3w-1)$, $W(4w, 3w-3)$ *and* $W(4w, 4w-4)$ *respectively.*

LEMMA 6. *If the conjecture is true for* $w \equiv 0 \pmod 4$ *then there exists a* $(0,1,-1)$ $W = W(4w,k)$ *of order* $4w$ *satisfying*

$$WW^T = kI_{4w}$$

for $k = 0, 1, \ldots, 2w+2, 2w+4, 2w+8, \ldots, 4w$ and $3, 6, 9, \ldots, 3w-3, 3w-1, 3w$.

PROOF. Since $W(w,k)$, $0 \le k \le w$ exists so does $W(4w,k) = W(w,k) \oplus W(w,k) \oplus W(w,k) \oplus W(w,k)$. By Lemma 4, for $k = \frac{1}{2}w, \frac{1}{2}w + 1, \ldots, w$ we get a $W(4w,\ell)$ with $\ell = w+1, w+2, \ldots, 2w+1, \frac{3}{2}w, \frac{3}{2}w + 3, \ldots, 3w$, and $2w+2$.

By Lemma 2 the existence of $W(w,k)$ implies the existence of $W(2w,2k)$ and hence $W(4w,4k)$.

Thus we have the result.

LEMMA 7. *The conjecture is true for* $w = 32$.

PROOF. By Lemmas 3 and 6 there exists a $(0,1,-1)$ W of order 32 satisfying

$$WW^T = kI_{32}$$

for $k = 0, 1, \ldots, 18, 20, 21, 23, 24, 28, 32$.

Since S_8 exists, by Lemma 5, (2) is satisfied for $k = 30, 31$. By Lemmas 2 and 3 (2) is satisfied for $k = 22, 26$.

For $k = 25$, use $A_1 = \ldots = A_6 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} = A$,

$A_7 = I_4$, $A_8 = 0$.

For $k = 29$ use $A_1 = A_2 = \dots = A_7 = A$, $A_8 = I$, in (3) to get $W(32,25)$ and $W(32,29)$.

Now let
$$C = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \quad \text{and} \quad D = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

then $CD^T = DC^T$. Now choosing $A_1 = A_2 = A_3 = A_4 = C$, $A_5 = D$, $A_6 = I_4$, $A_7 = A_8 = 0$ in (3) we get a $W(32,19)$ and choosing $A_1 = A_2 = \dots = A_6 = C$, $A_7 = D$, $A_8 = I$ in (3) we get $W(32,27)$.

Thus we have the conjecture for 32.

LEMMA 8. *There exists a* $(0,1,-1)$ *-matrix* W *of order* 6 *satisfying*

$$WW^T = kI_6$$

for $k = 0, 1, 2, 4, 5$ *i.e., there exists a* $W(6,k)$ *for* $k \in \{0,1,2,4,5\}$.

PROOF. Clearly the Lemma is true for $k = 0, 1$. The symmetric conference matrix of order 6 gives the result for $k = 5$. The required matrices for 2 and 4 are

$$H_2 \times I_3 \quad \text{and} \quad \left[\begin{array}{cc|cc|cc} 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ \hline 1 & -1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \end{array}\right].$$

LEMMA 9. *The conjecture is true for* $w = 12$.

PROOF. By Lemmas 2 and 8 we have a $W(12,k)$ for $k = 0,1,2,4,5,8,10$. The existence of an Hadamard and skew-Hadamard matrix of order 12 gives a $W(12,k)$ for $k = 11, 12$. For

(i) $k = 3$ use $A = B = C = I_3$, $D = 0$,

(ii) $k = 5$ use $A = J_3 - I_3$, $B = J_3 - 2I_3$, $C = D = 0$,

(iii) $k = 6$ use $A = J_3 - I_3$, $B = J_3 - 2I_3$, $C = I_3$, $D = 0$

(iv) $k = 7$ use $A = J_3 - I_3$, $B = J_3 - 2I_3$, $C = D = I_3$,

(v) $k = 9$ use $A = J_3$, $B = C = D = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

respectively, in construction 1.

LEMMA 10. *The conjecture is true for* $w = 24$.

PROOF. From Lemmas 2 and 9 there exists a $W = W(24,k)$ satisfying (2) for $k = 0, 1, \ldots, 14, 16, 18, 20, 22, 24$. Since S_{12} exists, S_{24} exists and (2) is satisfied for $k = 23$.

Write $J = J_3$, $K = J_3 - 2I$, $I = I_3$. Then using for

$k = 15$, $A_1 = J$, $A_2 = A_3 = A_4 = K$, $A_5 = A_6 = A_7 = I$, $A_8 = 0$
$k = 17$, $A_1 = A_2 = A_3 = J - I$, $A_4 = A_5 = A_6 = K$, $A_7 = A_8 = I$,
$k = 19$, $A_1 = J$, $A_2 = A_3 = A_4 = A_5 = K$, $A_6 = J - I$, $A_7 = A_8 = I$,

in construction 2 and using the following first rows in construction 4

10: 0 - 1 1 - -, 0 1 1 1 - 1, 0 0 0 0 0 0, 0 0 0 0 0 0
12: 0 1 1 0 1 -, 0 1 1 0 1 -, 0 1 1 0 1 -, 0 0 0 0 0 0
16: 0 1 1 0 1 -, 0 1 1 0 1 -, 0 1 1 0 1 -, 0 1 1 0 1 -
17: - 1 1 1 1 1, - 1 1 - 1 -, 0 - 1 1 1 -, 0 0 0 0 0 0

20: 0 1 - - - 1, 0 - 1 1 - 1, 0 1 - 1 1 1, 0 1 1 1 1 -

0 - 1 1 - 1, 0 - 1 1 - -, 0 1 1 1 - 1, 0 1 1 1 - 1

22: 1 1 - - - 1, 0 - 1 1 - 1, - 1 1 1 1 1, 0 - 1 - 1 1

23: - 1 1 1 1 1, - 1 1 - 1 -, 0 - 1 1 1 -, 0 1 1 0 1 -

24: - 1 1 1 1 1, - - 1 1 1 1, - 1 - 1 1 1, - 1 1 - 1 -

where, as before, - denotes -1, we have the conjecture for $w = 24$.

LEMMA 11. *There exists a* $(0,1,-1)$*-matrix* $W = W(10,k)$ *satisfying (2) for* $k = 0, 1, 2, 4, 5, 8, 9$.

PROOF. The result is clear for $k = 0, 1, 2$ and the symmetric conference matrix of order 10 gives the result for 9. For $k = 4$ use

$$A = \begin{bmatrix} \cdot & 1 & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 \\ 1 & \cdot & \cdot & 1 & \cdot \end{bmatrix} \text{ and } B = \begin{bmatrix} \cdot & 1 & \cdot & \cdot & -1 \\ -1 & \cdot & 1 & \cdot & \cdot \\ \cdot & -1 & \cdot & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & 1 \\ 1 & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

in

$$\begin{bmatrix} A & B \\ B^T & -A^T \end{bmatrix} . \tag{5}$$

For $k = 5$ use

$$A = \begin{bmatrix} -1 & 1 & \cdot & 1 & \cdot \\ \cdot & -1 & 1 & \cdot & 1 \\ 1 & \cdot & -1 & 1 & \cdot \\ \cdot & 1 & \cdot & -1 & 1 \\ 1 & \cdot & 1 & \cdot & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

in (5).

For k = 8 use (for the first rows of) for A and B

0 1 -1 -1 -1 and 0 1 -1 1 1

in (5).

We note

0 1 -1 -1 1 and -1 1 1 1 1

may also be used to obtain W(10,9).

LEMMA 12. *The conjecture is true for* w = 20.

PROOF. By Lemmas 1 and 11 we have a W satisfying (2) for k = 0, 1, 2, 4, 5, 8, 9, 10, 16, 18. There is an Hadamard matrix and a skew-Hadamard matrix of order 20 so we have a W for k = 19, 20. W(8,6) ⊕ W(12,6) and W(8,7) ⊕ W(12,7) give the result for k = 6 and 7.

The following first rows may be used to generate circulant matrices which can then be used in construction 1:

3:	1 0 0 0 0,	1 0 0 0 0,	1 0 0 0 0,	0 0 0 0 0
8:	0 1 - - 1,	0 0 1 1 0,	0 1 0 0 1,	0 0 0 0 0
9:	- 1 1 1 1,	0 1 - - 1,	0 0 0 0 0,	0 0 0 0 0
10:	- 1 1 1 1,	0 1 - - 1,	1 0 0 0 0,	0 0 0 0 0
11:	- 1 1 1 1,	- 0 1 1 0,	- 1 0 0 1,	0 0 0 0 0
12:	- 1 1 1 1,	- 0 1 1 0,	- 1 0 0 1,	1 0 0 0 0
18:	1 1 - - 1,	0 1 1 1 1,	0 1 - - 1,	- 1 - - 1

If we use the following first rows to generate circulant matrices in construction 4:

13:	1 1 0 0 1,	0 1 0 0 1,	0 1 - - 1,	0 1 - - 1
14:	0 - 1 - 0,	0 1 1 1 0,	0 1 1 1 -,	0 1 - - 1
17:	0 - 1 - 1,	1 1 - - 1,	0 1 1 1 1,	0 1 - - 1.

W(20,15) may be obtained from construction 5. Thus we have the conjecture for $w = 20$.

LEMMA 13. *There exists a* (0,1,-1)*-matrix* W *of order* 14 *satisfying* (2) *for* $k = 0, 1, 2, 4, 5, 8, 9, 10, 13$.

PROOF. The result for $k = 0, 1, 2, 4, 5$ follows using Lemmas 1, 3 and 8. W(14,13) exists because there is a symmetric conference matrix of order 14.

Use the following first rows to generate circulant matrices in (4) to obtain the remainder of the results

8:		- 1 1 0 1 0 0,	- 1 1 0 1 0 0
	or	0 1 1 0 0 - 1,	0 - 1 0 0 1 1
9:		0 1 1 0 1 0 0,	0 1 1 - 1 - -
10:		0 1 1 0 1 0 0,	1 1 1 - 1 - -
	or	- 1 1 1 1 0 0,	- 1 1 - 1 0 0
	or	0 1 1 - 0 - 0,	0 1 1 - 0 1 1
13:		- 1 1 - 1 0 1,	- 1 1 1 1 1 -

where - denotes -1.

LEMMA 14. *The conjecture is true for* $w = 28$.

PROOF. Since W(16,k) and W(12,k) exist for $0 \leq k \leq 12$ we have W(28,k) for $0 \leq k \leq 12$. By construction 5 we have W(28,14), W(28,21) and W(28,28). A W(28,27) exists since there is a skew-Hadamard matrix of order 28. There is a symmetric conference matrix of order 14 so $S = W(14,13)$ exists and $S \oplus S$ and $S \times H_2$ and W(28,13) and W(28,26) respectively.

We use the following first rows in construction 4

15:	1 0 0 0 0 0 0,	0 1 1 0 1 0 0,	- 1 1 0 1 0 0,	1 1 1 - 1 - -
16:	0 0 0 0 0 0 0,	1 1 1 0 1 0 0,	0 1 1 - 1 - -,	0 1 1 - 1 - -
	- 1 1 0 1 0 0,	1 - 1 0 1 0 0,	1 1 - 0 1 0 0,	1 1 1 0 - 0 0
17:	1 0 0 0 0 0 0,	1 1 1 0 1 0 0,	0 1 1 - 1 - -,	0 1 1 - 1 - -
18:	0 1 1 0 1 0 0,	0 1 1 0 1 0 0,	0 1 1 - 1 - -,	0 1 1 - 1 - -
19:	0 1 1 0 1 0 0,	0 1 1 0 1 0 0,	1 1 1 - 1 - -,	0 1 1 - 1 - -
20:	0 1 1 0 1 0 0,	0 1 1 0 1 0 0,	1 1 1 - 1 - -,	1 1 1 - 1 - -
21:	0 1 1 1 - 1 -,	0 - 1 - - - 1,	0 1 1 - - - 1,	0 1 1 - 1 1 1
22:	1 1 1 0 1 0 0,	- 1 1 0 1 0 0,	1 1 1 - 1 - -,	1 1 1 - 1 - -
23:	0 1 1 0 1 0 0,	1 1 1 - 1 - -,	- 1 1 - 1 0 1,	- 1 1 1 1 1 -
24:	0 1 1 1 - 1 -,	0 - 1 - - - 1,	0 1 1 - - - 1,	0 1 1 - 1 1 1
25:	- 1 1 1 1 1 1,	0 1 1 - 1 - -,	0 1 1 - 1 - -,	0 1 1 - 1 - -
26:	- 1 1 1 1 1 1,	1 1 1 - 1 - -,	0 1 1 - 1 - -,	0 1 1 - 1 - -

we have the conjecture for w = 28.

LEMMA 15. *The conjecture is true for* w = 40.

PROOF. Since the conjecture is true for 20 by Lemma 2 we have the results for k = 0, 1, 2, ..., 20, 21, 22, 24, 26, ..., 38, 40.

W(40,39) exists since there is a skew-Hadamard matrix of order 40. By Lemmas 4 and 11, W(40,k) exists for k = 0,3,6,12,15,24,27; and by Lemma 11 and construction 8 we have W(40,k) for k = 38,27,28,36,29.

Let B the matrix generated by the first row

0 1 - - 1.

Then using

$$A_1 = J,\ A_2 = A_3 = A_4 = A_5 = A_6 = B,\ A_7 = A_8 = 0$$

in construction 2 gives W(40,25), while using

$$A_1 = J,\ A_2 = A_3 = B + I,\ A_4 = A_5 = B - I,\ A_6 = B,\ A_7 = A_8 = I$$

gives W(40,31).

Let C and D be the matrices generated by the first rows

$$-\ 1\ 0\ 0\ 1 \quad \text{and} \quad -\ 0\ 1\ 1\ 0$$

respectively then using

$$A_1 = J - I,\ A_2 = A_3 = A_4 = C,\ A_5 = A_6 = A_7 = D,\ A_8 = I$$

in construction 2 gives W(40,23).

Let E and F be the matrices generated by the first rows

$$1\ 0\ 1\ 1\ 0 \quad \text{and} \quad 0\ 0\ 1\ 1\ 0$$

then $A_1 = A_2 = J - 2I$, $A_3 = B + I$, $A_4 = B - I$, $A_5 = A_6 = B$, $A_7 = E$ and $A_8 = F$ used in construction 2 gives W(40,33), while using

$A_1 = A_2 = J - 2I$, $A_3 = A_4 = B + I$, $A_5 = A_6 = B - I$, $A_7 = E$ and $A_8 = F$

gives W(40,35).

Thus we have the result

OTHER RESULTS

LEMMA 16. *There exists a* (0,1,-1)-*matrix* W = W(18,k) *satisfying (2) for* k = 0, 1, 2, 4, 5, 8, 10, 17.

PROOF. Since W(10,k) and W(8,k) exist for k = 0, 1, 2, 4, 5, 8, we have $W(18,k) = W(10,k) \oplus W(8,k)$ exists for these k values.

If we use the following first rows to generate the circulant matrices to use in (4) we get the remaining results.

8:		0 1 - 0 0 1 0 1 0,	0 1 1 0 0 - 0 1 0;
	or	1 1 0 1 - 0 0 0 0,	1 1 0 - 1 0 0 0 0;
10:		1 1 1 0 1 - 0 0 0,	- 1 1 0 - 1 0 0 0;
	or	- 1 0 - 1 0 0 - 0,	1 1 0 1 1 0 0 - 0;
17:		1 - - 1 1 1 1 - -,	0 - 1 - - - - 1 - .

(We note this leaves k = 9, 13, 16 which were not ruled out by the theorem).

LEMMA 17. *There exists a* (0,1,-1)*-matrix* W = W(36,k) *satisfying (2) for* k = 0, 1, 2,..., 22, 25, 27, 32, 33, 34, 35, 36.

PROOF. Since there exists a W(20,k) and W(16,k) for $0 \leq k \leq 16$ there exists a $W(36,k) = W(20,k) \oplus W(16,k)$ for these k. W(36,20), W(36,34) and W(36,17) exist using Lemmas 1 and 15. W(36,36) and W(36,35) exist becuase there is a skew-Hadamard matrix of order 36.

If we use the following first rows in construction 4, we get the other results of the lemma.

18:	0 1 - 0 0 1 0 1 0,	0 1 1 0 0 - 0 1 0
	1 1 1 0 1 - 0 0 0,	- 1 1 0 - 1 0 0 0
19:	1 - - 1 1 1 1 - -,	0 - 1 - - - - 1 -,
	1 0 0 0 0 0 0 0 0,	1 0 0 0 0 0 0 0 0
21:	1 - - 1 1 1 1 - -,	0 - 1 - - - - 1 -,
	1 1 0 0 0 0 0 0 0,	1 - 0 0 0 0 0 0 0
22:	1 1 1 1 1 - 0 0 0,	1 1 1 - 1 - 0 0 0,
	- 1 1 0 - 1 0 0 0,	- 1 1 0 - 1 0 0 0

25:   0 1 - 0 0 1 0 1 0,   0 1 1 0 0 - 0 1 0,

1 - - 1 1 1 1 - -,   0 - 1 - - - - 1 -

27:   1 1 1 0 1 - 0 0 0,   - 1 1 0 - 1 0 0 0,

1 - - 1 1 1 1 - -,   0 - 1 - - - - 1 -

32:   0 - 1 - - 1 1 - 1,   0 - 1 1 1 1 1 1 -,

0 1 - - - 1 1 1 -,   0 1 - 1 1 1 1 - 1

33:   1 - 1 - - 1 1 - 1,   0 - 1 1 1 1 1 1 -,

0 1 - - - 1 1 1 -,   0 1 - 1 1 1 1 - 1

LEMMA 18. *There exists a* W(22,k) *for* k ε {0,1,2,4,5,8,9,10,16,17,20}.

PROOF. There exists a W(10,ℓ) and a W(12,ℓ) by Lemmas 9 and 11 for ℓ ε {0, 1, 2, 4, 5, 8, 9} so $W_1 \oplus W_2$ is the required matrix for ℓ. To obtain the other matrices use the following first rows to generate A and B in (5).

10:   0 0 1 1 - 0 - 0 0 0 1,   0 1 0 0 0 1 0 1 1 - 0   (a)

16:   - 1 0 1 1 1 0 0 0 1 0,   0 1 - 1 1 1 - - - 1 -   (b)

17:   - 1 0 1 1 1 0 0 0 1 0,   1 1 - 1 1 1 - - - 1 -   (c)

20:   0 1 1 1 - 1 - 1 1 - 1   0 1 - - 1 1 1 1 1 - -   (d)

LEMMA 19. *There exists a* W(44,k) *for* k ε {i : 0 ≤ i ≤ 20,21,25 ≤ i ≤ 28, 30, 32, 33, 34, 36 ≤ i ≤ 40, 42 ≤ i ≤ 44}.

PROOF. Since the conjecture is true for 24 and 20, there exists a $W_1(20,m)$ and a $W_2(24,m)$ for m ε {i : 0 ≤ i ≤ 20} and $W_1 \oplus W_2$ is the required matrix for m.

W(44,44) and W(44,43) exist as there is a skew-Hadamard matrix of order 44.

If we use the matrices used to form W(22,k) with the indicated matrices in construction 4 we get W(44,n) for

$n \in \{21, 22, 26, 27, 30, 32, 33, 34, 36, 37, 40\}$

k = 21: (d) 0 and I_{11};

k = 22: (d) I_{11}, I_{11};

k = 26: (a) and (b);

k = 27: (a) and (c);

k = 30: (a) and (d);

k = 32: (b) and (b);

k = 33: (b) and (c);

k = 34: (c) and (c);

k = 36: (b) and (d);

k = 37: (c) and (d);

k = 40: (d) and (d);

Let A be the circulant incidence matrix of an (11, 6, 3) configuration and B = J - I - 2A. Then

$$AA^T = 3I + 3J \quad \text{and} \quad BB^T = 11I - J,$$

and have 6 and 10 non-zero elements respectively.

Further A - I and B + I satisfy

$$(A - I)(A - I)^T = 5I + 2J \text{ and } (B + I)(B + I)^T = 11I - J,$$

and have 7 and 11 non-zero elements respectively.

So we may use the following matrices in construction 4 to get W(44, k):

k = 25:	J - A	B	B	0
k = 26:	J - A	B	B + I	
k = 27:	A - I	B	B	0
k = 28:	A - I	B	B	I
k = 28:	A - I	B + I	B	0
k = 29:	A - I	B + I	B + I	0
k = 30:	A - I	B + I	B + I	I
k = 36:	A	B	B	B
k = 37:	A	B + I	B	B
k = 38:	A	B + I	B + I	B
k = 39:	A	B + I	B + I	B + I

LEMMA 20. *There exists a* W(26,k) *for* k ε {0, 1, 2, 4, 5, 8, 9, 10, 25}.

PROOF. This follows from the existence of W(12,k) and W(14,k) for k ε {0, 1, 2, 4, 5, 8, 9, 10}.

The following first rows generate matrices which can be used in (5) to form a W(26,25):

0 1 - - - 1 - - 1 - - - 1, 1 - 1 1 - - - - - - - 1 1 - .

LEMMA 21. *There exists a* W(52,k) *for* k ε {i : 0 ≤ i ≤ 24, 32, 34, 39, 48, 51, 52}.

PROOF. The existence of a W(52,k) for k ε {i: 0 ≤ i ≤ 24} follows from the existence of Hadamard matrices of orders 24 and 28. The W(52,52) and W(52,51) exist because there is a skew-Hadamard matrix of order 52.

W(52,48) may be obtained by using the following first rows to generate matrices which are then used in (4):

0 1 - - - 1 - - 1 - - - 1, 0 - 1 1 - - - - - - 1 1 -

0 - - - 1 - 1 1 - 1 - - -, 0 1 - 1 1 - - - - 1 1 - 1 .

Write B for the last of these four matrices.

Let Q be the circulant incidence matrix of a (13, 4, 1) configuration then

$$BB^T = 13I - J \quad \text{and} \quad QQ^T = 12I + J.$$

So if we use the following four matrices in (4) we get W(52,k) for $k \in \{32, 34\}$:

k = 32: Q, Q, B, B;

k = 34: Q, Q, B+I, B-I.

We get a W(52,39) by putting A = B = C = 1, D = 0 in the 52 × 52 Hadamard array.

LEMMA 22. *There exists a* W(56,k) *for*
$k \in \{i: 0 \le i \le 30, 32, 33, 34, 36, 37, 38, 40, 42, 44, 46, 48, 50, 52, 54, 55, 56\}$.

PROOF. This follows from Lemmas 2 and 14, the existence of a skew-Hadamard matrix of order 56 and from construction 8, since there exists a W(14,13).

REFERENCES

1. JOAN COOPER AND JENNIFER WALLIS, A construction for Hadamard matrices, *Bull. Austral. Math. Soc.* 7 (1972), 269-277.

2. DAMARAJU RAGHAVARAO, "Constructions and Combinatorial Problems in Design of Experiments", Wiley Series in Probability and Mathematical Statistics, Wiley, New York - London - Sydney - Toronto, 1971.

3. OLGA TAUSSKY, (1,2,4,8)-sums of squares and Hadamard matrices, "Combinatorics, Proceedings of Symposia in Pure Mathematics", American Mathematical Society, Providence, R.I., 1971.

4. JENNIFER WALLIS, (v,k,λ)-configurations and Hadamard matrices, *J. Austral. Math. Soc.* 11 (1970), 297-309.

5. W.D. WALLIS, ANNE PENFOLD STREET, JENNIFER SEBERRY WALLIS, "Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices" (to appear).