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On Integer Matrices Obeying Certain Matrix Equations

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We discuss integer matrices B of odd order v which satisfy

$$B^T = \pm B, \ BB^T = vI - J, \ BJ = 0.$$

Matrices of this kind which have zero diagonal and other elements ± 1 give rise to skew-Hadamard and n-type matrices; we show that the existence of a skew-Hadamard (n-type) matrix of order h implies the existence of skew-Hadamard (n-type) matrices of orders $(h-1)^5 + 1$ and $(h-1)^7 + 1$.

Finally we show that, although there are matrices B with elements other than ± 1 and 0, the equations force considerable restrictions on the elements of B.

1. INTRODUCTION

Some interesting theorems have been discovered by H. J. Ryser [6–8], K. Majindar [5] and more recently by W. G. Bridges and H. J. Ryser [1] on integer matrices. Other specialized matrices have been studied and some of these results may be found in Marshall Hall Jr.'s book [4], G. Szekeres [9], J. M. Goethals and J. J. Seidel [2] and Jennifer Wallis [10, 11].

In this paper I propose to look at some integer matrices satisfying very restrictive matrix equations.

2. PRELIMINARIES

An *Hadamard matrix* H, of order $h = 2$ or $\phi \equiv 0 \pmod 4$, has every element $+1$ or -1 and satisfies $HH^T = hI_h$, where I denotes the identity matrix. An Hadamard matrix $H = I + R$ of order h will be called *skew-Hadamard* if R has zero diagonal, $R^T = -R$ and $RR^T = (h-1)I_h$. Skew-Hadamard matrices are discussed in [2], [3], [4], and [10].

A matrix $N = I + P$ of order $n \equiv 2 \pmod 4$ is called *n-type* if it has every element $+1$ or -1, P has zero diagonal, $P^T = P$ and $PP^T = (n-1) I_n$. These matrices are discussed in [2] and [11].

K. Goldberg has proved in [3] that, if there is a skew-Hadamard matrix of order h (Goldberg refers to "type 1" matrices—we use the more recent nomenclature "skew-Hadamard"), there is a skew-Hadamard matrix of order $(h-1)^3 + 1$. We examine the fifth and seventh powers.

$M \times N$ is the Kronecker product of M and N, and we use J to denote the matrix with every element $+1$.

The *core* of a skew-Hadmard of n-type matrix is found by altering the rows and columns of the matrix unitil it can be written in the form

$$\begin{bmatrix} 0 & e \\ \pm e^T & W \end{bmatrix} + I,$$

where W, the *core*, has zero diagonal and $+1$ or -1 elsewhere, and $e = (1, 1, \ldots, 1)$. If W is of order h then $WJ = 0$, $WW^T = hI - J$, $W^T = -W$ if $h \equiv 3 \pmod 4$ and $W^T = W$ if $h \equiv 1 \pmod 4$.

We will mean by the notation $\sum' A \times B \times C \times \cdots D$, where $\times$ is the Kronecker product, the sum obtained by circulating the letters formally; thus

$$\sum' A \times B \times C = A \times B \times C + B \times C \times A + C \times A \times B.$$

The sum $\sum'$ over x letters has x terms in the sum.

For convenience we restate some results from Jennifer Wallis [11] and K. Goldberg [3]:

LEMMA 2.1 [11, class II]. *If h is the order of a skew-Hadamard matrix there is a n-type matrix of order $(h-1)^2 + 1$.*

LEMMA 2.2 [11, class II]. *If h is the order of an n-type matrix there is an n-type matrix of order $(h-1)^2 + 1$.*

LEMMA 2.3 [11, class III]. *If h is the order of an n-type matrix there is an n-type matrix of order $(h-1)^3 + 1$.*

LEMMA 2.4 [3]. *If h is the order of a skew-Hadamard there is a skew-Hadamard matrix of order $(h-1)^3 + 1$.*

My thanks go to W. D. Wallis who suggested the line of proof of the next lemma and theorem:

DEFINITION. Let $A = [a_{ij}]$ and $C = [c_{ij}]$ be two matrices of order n. The *Hadamard product* $A * C$ if A and C is given by

$$A * C = [a_{ij}c_{ij}].$$

LEMMA 2.5. *When A, B, C, D are matrices of the same order*

$$(A \times B) * (C \times D) = (A * C) \times (B * D).$$

THEOREM 2.6. *Suppose I, J, and W are of order h, where W is a matrix with zero diagonal and $+1$ or -1 elsewhere. Suppose A is a matrix of order $q = h^p$ which is of the form*

$$A = B_1 + B_2 + \cdots + B_k$$

where each B_i is a Kronecker product of p terms I, J or W in some order, such that

(a) *each B_i contains at least one term W,*

(b) *for any two summands B_i and B_j there is a position r such that one of the summands has I in the r-th position and the other has W in that position,*

and suppose A satisfies

$$AA^T = qI_q - J_q.$$

Then A has zero diagonal and $+1$ or -1 in every other place.

Proof. Clearly

$$W * I = I * W = 0;$$

so, by part (b) of the hypothesis and Lemma 2.5,

$$B_i * B_j = B_j * B_i = 0$$

whenever $i \neq j$.

Hence no two B_i have non-zero elements in the same position, so each non-zero element of A comes from exactly one of the B_i; since each B_i is a $(0, 1, -1)$ matrix it follows that A is a $(0, 1, -1)$ matrix. But

$$AA^T = qI_q - J_q.$$

So, if $A = [a_{ij}]$,

$$\sum_j a_{ij}^2 = q - 1$$

for any i; therefore at most one element in each row of A is zero. W appears in each B_i, and W has zero diagonal, so each B_i—and consequently A—has zero diagonal. So we have the result.

3. n-Type and Skew-Hadamard Matrices of Order $(h-1)^n+1$ Where $n=5$ and 7

THEOREM 3.1. *If h is the order of a skew-Hadamard (n-type) matrix then there is a skew-Hadamard (n-type) matrix of order $(h-1)^5+1$.*

Proof. Let W be the core of the skew-Hadamard (n-type) matrix. Then $WJ=0$, $WW^T=(h-1)I_{h-1}-J_{h-1}$ and $W^T=-W$ for $h\equiv 0 \pmod 4$ and $W^T=W$ for $h\equiv 2 \pmod 4$.

Now if I, W, and J are all of order $h-1$ write

$$\begin{aligned} B = {} & W\times W\times W\times W\times W+\textstyle\sum' I\times J\times I\times J\times W \\ & +\textstyle\sum' I\times J\times W\times W\times W. \end{aligned}$$

Then, since $WJ=0$, $B(J\times J\times J\times J\times J)=0$, and if $W^T=-W$ then $B^T=-B$ but if $W^T=W$ then $B^T=B$.

It is easy to see that, if we multiply one summand of B by the transpose of a different summand, then either WJ or JW^T appears as one term of the Kronecker product expansion, so the product is zero; thus

$$\begin{aligned} BB^T = {} & WW^T\times WW^T\times WW^T\times WW^T\times WW^T \\ & +\textstyle\sum' I\times J^2\times I\times J^2\times WW^T \\ & +\textstyle\sum' I\times J^2\times WW^T\times WW^T\times WW^T. \end{aligned}$$

substituting for WW^T, and using the distributive law for $+$ over $\times$, we find

$$BB^T=(h-1)^5 I_{(h-1)^5}-J_{(h-1)^5}\,.$$

B satisfies Theorem 2.6, so it has zero diagonal and 1 and -1 elsewhere.

So B is the core of the required matrix of order $(h-1)^5+1$.

THEOREM 3.2. *If h is the order of a skew-Hadamard (n-type) matrix then there is a skew-Hadamard (n-type) matrix of order $(h-1)^7+1$.*

Proof. Proceed as in the previous theorem but use instead

$$\begin{aligned} B = {} & W\times W\times W\times W\times W\times W\times W \\ & \textstyle\sum' I\times J\times [W\times W+I\times J]\times[W\times W+I\times J]\times W. \end{aligned}$$

COROLLARY 3.3. *If h is the order of a skew-Hadamard matrix then there is a skew-Hadamard matrix of order $(h-1)^c+1$ where $c=3^j5^k7^e$, j, k, e are non-negative integers.*

This follows from Theorems 2.4, 3.1, and 3.2.

COROLLARY 3.4. *If h is the order of a skew-Hadamard matrix then there is an n-type matrix of order $(h-1)^c+1$ where $c = 2^i 3^j 5^k 7^e$, i a positive integer, j, k, e non-negative integers.*

This follows from Theorems 2.1, 2.2, 2.3, 3.1, and 3.2.

So we see, writing W^n to mean $W \times W \times \cdots \times W$ the Kronecker product of n W's, that the formulae for the third, fifth, and seventh powers may be written as

$$W^n + \sum{}' IJ[W^2 + IJ]^{(n-3)/2}\, W,$$

where juxtaposition denotes Kronecker product. This formula breaks down for 9 and 11.

4. THE NATURE OF MATRICES SATISFYING THE EQUATION FOR THE CORE OF SKEW-HADAMARD MATRICES

In the preceding sections we have seen that solutions to the equations

$$AA^T = vI - J, \qquad AJ = 0 = JA, \qquad \text{and} \qquad A^T = -A \tag{1}$$

exist for an infinitude of matrices A of order v. In these cases it is clear that A is a $(0, 1, -1)$ matrix with zero diagonal and 1 or -1 elsewhere. We now show that it is possible to have a matrix A satisfying (1) but which is not a $(0, 1, -1)$ matrix. Consider

$$A = \begin{bmatrix} 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 \end{bmatrix};$$

then $AA^T = 7I - J$, $AJ = 0 = JA$ and $A^T = -A$.

However there are considerable restrictions on the elements of A as Theorem 2.6 and the following theorem show:

THEOREM 4.1. *Let A be an integer matrix of order v, where $v \equiv 3 \pmod 4$, satisfying*

$$AA^T = vI - J, \qquad AJ = 0 = JA, \qquad A^T = -A.$$

Suppose $-a$ is the least element of A; define

$$B = A + a(J - I);$$

write e for the g.c.d. of non-zero elements of B. Then either $e = 1$ or A has zero diagonal and $+1$ or -1 elsewhere.

Proof. $A^T = -A$, so A has zero diagonal, and hence B has zero diagonal. B satisfies

$$BB^T = (v + a^2)I + [a^2(v - 2) - 1]J,$$
$$BJ = a(v - 1)J.$$

Consider the matrix $C = (1/e)B$. C is an integer matrix and

$$CC^T = \frac{v + a^2}{e^2} I + \frac{a^2(v - 2) - 1}{e^2} J,$$
$$CJ = \frac{a(v - 1)}{e} J.$$

The element $-a$ occurs in A; and since $A^T = -A$ the element a must occur in A and must be the greatest element of A. Then $2a$ is the greatest element of B and $2a/e$, which is therefore an integer, the greatest element of C. $e \mid 2a$, so there are two possibilities:

(i) e is odd, $e \mid a$;

(ii) e is even.

If (i) is true, since

$$\frac{a^2(v - 2) - 1}{e^2}$$

is an integer we have $e^2 \mid 1$ and so $e = 1$.

Now consider (ii). $v \equiv 3 \pmod 4$, so since $e^2 \mid v + a^2$ and e is even we must have a odd. Since e is even, every element of B is even; all the non-diagonal elements of A differ from elements of B by a, an odd number, so they are odd.

Now $AA^T = vI - J$, so (evaluating the i-th diagonal element)

$$\sum_{i=1}^{v} a_{ij}^2 = v - 1.$$

Each a_{ij} is an odd integer, unless $j = i$; so $a_{ij}^2 \geqslant 1$, and

$$\sum_{j=1}^{v} a_{ij}^2 \geqslant v - 1,$$

with equality only when each a_{ij} (except a_{ii}) is ± 1. This holds for every i, so we have the result.

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