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Combinatorial matrices

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We investigate the existence of integer matrices B satisfying the equation

(1) $$BB^T = rI + sJ ,$$

where T denotes transpose, r and s are integers, I is the identity matrix and J is the matrix with every element $+1$.

Hadamard matrices are $(1, -1)$ matrices of order $n = 2$ or $4t$ which have $r = n$ and $s = 0$ in (1). We discuss equivalence of Hadamard matrices over the integers and show that all Hadamard matrices of order $4t$, where t is odd and square-free are equivalent over the integers. Further, if t is even and square-free and there is a Hadamard matrix of order $2t$, then there is a Hadamard matrix of order $4t$ which is equivalent over the integers to the diagonal matrix

$$\mathrm{diag}(1, \underbrace{2, \ldots, 2}_{2m-1 \text{ times}}, \underbrace{2m, \ldots, 2m}_{2m-1 \text{ times}}, 4m) .$$

We now develop many methods for constructing Hadamard matrices. Many of these constructions use skew-Hadamard matrices, that is Hadamard matrices $H = I + R$ where $R^T = -R$, or n-type matrices, that is $(1, -1)$ matrices $N = I + P$ of order n where $P^T = P$ and $PP^T = (n-1)I$. We first develop some theory on the Williamson method for constructing skew-Hadamard matrices and show if h is the order of a skew-Hadamard matrix (n-type matrix) then there exists a skew-Hadamard (n-type) matrix of order $(h-1)^u + 1$ where $u = 2^a3^b5^c7^d$, b, c, d non-negative integers while a is a positive (non-negative) integer.

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The concept of supplementary difference sets, that is, a set of subsets such that when we take all the differences in each subset and collect them, each difference occurs a fixed number of times in the totality, is introduced and an example given. Hadamard designs on n distinct letters are shown to exist for $n = 2, 4$ and 8.

(v, k, λ)-configurations are considered, that is, $(0, 1)$-matrices B of order v such that $r = k - \lambda$ and $s = \lambda$ in (1). We show two similar but distinct methods for proving there exists a $\{q^2(q+2), q(q+1), q\}$ configuration whenever q is prime or $q = 2^2, 2^3, 2^4, 3^2, 3^3$ or 7^2. We prove that whenever a (q, k, λ)-configuration exists, q a prime power, then a $\{q(k^2+\lambda), qk, k^2+\lambda, k, \lambda\}$-configuration exists.

We consider integer matrices satisfying

$$BB^T = vI - J\ ,\quad BJ = 0 = JB \quad \text{and} \quad B^T = -B$$

and find that either the greatest common divisor of the elements of B is 1 or B has zero diagonal and $+1$ or -1 elsewhere. Also we show that if B is an integer matrix of order b satisfying

$$BB^T = (p-q)I + qJ$$

$$BJ = dJ$$

where p, q and $d > 0$ are constants then if z, the least element of B, satisfies

$$z \le \frac{d}{b} \quad \text{and} \quad z \le \frac{|w|d+P}{d+|w|b}\ ,$$

where w is the greatest element of B, then

$$B = \frac{d}{b} J\ .$$

We give tables of the orders < 4000 of known Hadamard, skew-Hadamard and n-type matrices at the date of submission as well as lists of known classes of these matrices.