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Reprinted from

THE JOURNAL OF THE AUSTRALIAN
MATHEMATICAL SOCIETY

Volume XII – Part 3 – p.p. 378–384
1971

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(Received 14 May 1969; revised 23 September 1969)

Communicated by B. Mond

A (v, k, λ) configuration is conjectured to exist for every v, k and λ satisfying
$\lambda(v-1) = k(k-1)$
and
$k-\lambda$ is a square if v is even,
$x^2 = (k-\lambda)y^2+(-1)^{(v-1)/2}\lambda z^2$ has a solution in integers x, y and z not all zero for v odd.
See Ryser [5, p. 111] for further discussion.

Necessary conditions for the existence of (b, v, r, k, λ) configurations are that

$$bk = vr$$
$$r(k-1) = \lambda(v-1).$$

We write I for the identity matrix and J for the matrix with every element $+1$. In the case of block matrices, $(X)_{ij}$ means the matrix whose (i, j)th block is X; for example, $(T^{i-j})_{ij}$ is the matrix whose (i, j)th block is T^{i-j}. We define the *Kronecker product* of two matrices $A = (a_{ij})$ of order $m \times n$ and B of any order as the $m \times n$ block matrix

$$A \times B = (a_{ij}B)_{ij}.$$

THEOREM 1. *There exists a $(q(q^2+2), q(q+1), q)$ configuration whenever q is a prime.*

Takeuchi [7] and Ahrens and Szekeres [1] have proven that Theorem 1 holds for all prime powers q. Our method can be extended to $q = 2^2, 2^3, 2^4, 3^2, 3^3$ or 7^3. We include Theorem 1 as our method is entirely different to the others' and closely connected to the proof of Theorem 2.

THEOREM 2. *A $(q(k^2+\lambda), qk, k^2+\lambda, k, \lambda)$ configuration exists whenever a (q, k, λ) configuration exists and q is a prime power.*

THEOREM 3. *If there exists a matrix N of odd order $v-1$ with zero diagonal and every other element $+1$ or -1, such that $NJ = JN = 0$ and*

$$NN^T = (v-1)I_{v-1}-J_{v-1},$$

then there is a $(2(v-1), v, v-1, \frac{1}{2}v, \frac{1}{2}(v-2))$ configuration.

COROLLARY 4: *If v is the order of a skew-Hadamard or n-type matrix (see [8] for definitions) then there is a $(2(v-1), v, v-1, \frac{1}{2}v, \frac{1}{2}(v-2))$ configuration.*

1. Preliminary remark

We require that there exist (0, 1) matrices R_i, $0 \leqq i \leqq q-1$, Q of order q^2 and $\bar{Q}$ which is $kq \times q^2$, k an integer less than q, which together with P (defined in (iv) below) satisfy the following conditions

$$(1) \quad \begin{cases} \text{(i)} & PR_j^T = J \times J \\ \text{(ii)} & R_i R_j^T = J \times J \quad i \neq j \\ \text{(iii)} & \sum_{i=0}^{q-1} R_i R_i^T = q^2 I \times I + q(J-I) \times J \\ \text{(iv)} & P = I \times J, \quad PP^T = qI \times J \\ \text{(v)} & QQ^T = qI \times I + (J-I) \times J \\ \text{(vi)} & \bar{Q}\bar{Q}^T = qI_{kq} + (J_k - I_k) \times J \\ \text{(vii)} & J_{kq}\bar{Q} = k\bar{J} \\ \text{(viii)} & \bar{Q}J_{q^2} = q\bar{J}. \end{cases}$$

In formula (1), unless subscripted otherwise, I and J are of order q and $\bar{J}$ is the $kq \times q^2$ matrix with every element $+1$.

We will show in § 3 some cases where these conditions are satisfied.

2. Constructions

LEMMA 5. *If P, a (0, 1) matrix, is defined as in (1, iv), and if (0, 1) matrices R_i, $0 \leqq i \leqq q-1$ satisfying conditions (1, i, ii, iii) exist then there exists a $(q^2(q+2), q(q+1), q)$ configuration.*

PROOF. It is easily seen that this triplet satisfies the necessary conditions for (v, k, λ) configurations.

Let S be the $q^2(q+2)$ block matrix given by

$$S = \begin{bmatrix} 0 & P & R_0 & R_1 & \cdots & R_{q-3} & R_{q-2} & R_{q-1} \\ R_{q-1} & 0 & P & R_0 & \cdots & R_{q-4} & R_{q-3} & R_{q-2} \\ \vdots & & & & & & \vdots & \\ R_0 & R_1 & R_2 & R_3 & \cdots & R_{q-1} & 0 & P \\ P & R_0 & R_1 & R_2 & \cdots & R_{q-2} & R_{q-1} & 0 \end{bmatrix}$$

then

$$SS^T = I_{q+2} \times \{PP^T + \sum_{i=0}^{q-1} R_i R_i^T\} + (J_{q+2} - I_{q+2}) \times qJ \times J$$
$$= q^2 I_r + qJ_r,$$

where $r = q^2(q+2)$.

Every element of s is 0 or 1 so s is the incidence matrix of a $(q^2(q+2), q(q+1), q)$ configuration.

LEMMA 6. *If there exists a* $(0, 1)$ *matrix* $\bar{Q}$ *satisfying the conditions* (1, *vi*, *vii*, *viii*) *and a* (q, k, λ) *configuration exists then there exists a* $(q(k^2+\lambda), qk, k^2+\lambda, k, \lambda)$ *configuration.*

PROOF. A (q, k, λ) configuration exists, so

$$\lambda(q-1) = k(k-1);$$

hence it is easily verified that the five numbers satisfy the necessary conditions for (b, v, r, k, λ) configurations.

Let V be the incidence matrix of the (q, k, λ) configuration. Then A defined by

$$A^T = [I_k \times V, \bar{Q}, \bar{Q}, \cdots, \bar{Q}]$$

($\bar{Q}$ occuring λ times), has k non-zero elements in every row and $\lambda q + k = k^2 + \lambda$ non-zero elements in each column. Now

$$A^T A = I_k \times VV^T + \lambda \bar{Q}\bar{Q}^T$$
$$= (k - \lambda + \lambda q) I_{kq} + \lambda J_{kq}$$
$$= k^2 I_{qk} + \lambda J_{kq};$$

so A is the incidence matrix of the required configuration.

PROOF OF THEOREM 3. Since N has zero diagonal and every other element $+1$ or -1, C and D defined (with I and J of order $v-1$) by

$$C = \tfrac{1}{2}(N+I+J)$$
$$D = \tfrac{1}{2}(N-I+J)$$

are (0, 1) matrices. Now

$$CC^T + DD^T = \tfrac{1}{2}(NN^T + I + (v-1)J) = \tfrac{1}{2}vI + \tfrac{1}{2}(v-2)J$$

and

$$JC = \tfrac{1}{2}vJ = CJ$$
$$JD = \tfrac{1}{2}(v-2)J = DJ.$$

We define ω_v, ω_b and e to be the vectors of v, b and $(v-1)$ l's respectively and A^T by

$$A^T = \begin{bmatrix} D & C \\ e & 0 \end{bmatrix}.$$

A is $2(v-1)\times v$, and

$$\omega_v A^T = \tfrac{1}{2}v\omega_b, \quad A^T\omega_b^T = (v-1)\omega_v^T,$$

$$A^TA = \begin{bmatrix} D & C \\ e & 0 \end{bmatrix}\begin{bmatrix} D^T & e^T \\ C^T & 0 \end{bmatrix}$$

$$= \begin{bmatrix} DD^T+CC^T & \frac{1}{2}(v-2)e^T \\ \frac{1}{2}(v-2)e & v-1 \end{bmatrix}$$

$$= \frac{v}{2}I_v + \frac{v-2}{2}J_v.$$

So A is the incidence matrix of a $(2(v-1), v, v-1, \frac{1}{2}v, \frac{1}{2}(v-2))$ configuration.

3. Matrices satisfying condition (1)

We shall show that (1) can be satisfied for all primes q and that matrices Q and $\bar{Q}$ can be found for q any prime power. These facts together with lemmas 5 and 6 complete the proofs of Theorems 1 and 2.

In this section T will be used for the circulant matrix of order q given by

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

3.1 *The case of q prime*

Choose q block matrices R_i of order q^2, $0 \leqq i \leqq q-1$, thus

$$R_i = \begin{bmatrix} I & T^i & T^{2i} & \cdots & T^{(q-1)i} \\ T^{(q-1)i} & I & T^i & \cdots & T^{(q-2)i} \\ \vdots & & & & \vdots \\ T^i & T^{2i} & T^{3i} & \cdots & I \end{bmatrix} = (T^{(m-s)i})_{sm}$$

and let

$$Q = \begin{bmatrix} I & I & I & \cdots & I \\ I & T & T^2 & \cdots & T^{q-1} \\ I & T^2 & T^{2\cdot 2} & \cdots & T^{(q-1)2} \\ \vdots & & & & \vdots \\ I & T^{q-1} & T^{2(q-1)} & \cdots & T^{(q-1)(q-1)} \end{bmatrix} = (T^{(i-1)(j-1)})_{ij}$$

and

$$\bar{Q} = \begin{bmatrix} I & I & I & \cdots & I \\ I & T & T^2 & & T^{q-1} \\ I & T^2 & T^{2\cdot 2} & \cdots & T^{(q-1)2} \\ \vdots & & & & \vdots \\ I & T^{k-1} & T^{2(k-1)} & \cdots & T^{(q-1)(k-1)} \end{bmatrix}.$$

We now verify that these matrices satisfy the conditions (1). Note that $JT^i = J$ for all i, so (i), (vii) and (viii) are immediate.

(ii) $$R_i R_j^T = (\sum_{m=0}^{q-1} T^{(m-s)i}T^{(n-m)j})_{s,n}$$
$$= (\sum_{m=0}^{q-1} T^{m(i-j)+nj-si})_{s,n}$$
$$= (\sum_{r=0}^{q-1} T^r)_{s,n} = (J)_{s,n} = J\times J \qquad \text{for } i \neq j.$$

(iii) $$R_i R_i^T = (\sum_{m=0}^{q-1} T^{(m-s)i}T^{(n-m)i})_{s,n}$$
$$= (qT^{(n-s)i})_{s,n}$$
$$= qR_i;$$

$$\sum_{i=0}^{q-1} R_i = \begin{bmatrix} qI & J & \cdots & J \\ J & qI & \cdots & J \\ \vdots & & & \vdots \\ J & J & \cdots & qI \end{bmatrix} = qI\times I+(J-I)\times J,$$

so the result follows.

(v) $$QQ^T = (\sum_{m=1}^{q} T^{(i-1)(m-1)}T^{-(m-1)(j-1)})_{ij}$$
$$= (\sum_{m=1}^{q} T^{(m-1)(i-j)})_{ij}$$

then if $i = j$ we have $\sum_{m=1}^{q} I = qI$, and if $i \neq j$, we have $\sum_{m=1}^{q} T^{(m-1)(i-j)} = J$, which gives the result.

(vi) This follows since we have chosen $\bar{Q}$ as the first kq rows of Q.

3.2 *The case of q a prime power*

In this case, unless stated otherwise, I, J are of order q.

It is known that a $(q^2+q+1, q+1, 1)$ configuration exists whenever q is a prime power. If we form the incidence matrix of this configuration then we may rearrange its rows and columns until the following matrix is obtained:

$$A = \begin{bmatrix} 1 & e & 0 \\ e^T & 0 & I \times e \\ 0 & I \times e^T & N \end{bmatrix}$$

where $e = [1, 1, \cdots, 1]$ is of size $1 \times q$ and N is of size p^2.

Now $AA^T = pI_r + J_r$, where $r = p^2 + p + 1$, and

$$AA^T = \begin{bmatrix} q+1 & e & e \times e \\ e^T & qI+J & (I \times e)N^T \\ e^T \times e^T & N(I \times e^T) & I \times J + NN^T \end{bmatrix}$$

so

(a) N is of order q^2;

(b) $NN^T = qI \times I + J \times J - I \times J = qI \times I + (J-I) \times J$;

(c) $N(I \times e^T) = J'$ where J' is of size $q^2 \times q$.

This last condition implies that if N is partitioned into q^2 block matrices N_i then each block matrix N_i has exactly one element in each row and column. Now rearrange the columns of N keeping the first $q+1$ rows of A unaltered until the first row of block matrices in the partitioned N are all I_q and similarly alter the rows of N keeping the first $q+1$ columns of A unaltered until the first column of block matrices in the partitioned N are all I_q. Then this new matrix obtained from N satisfies all the conditions for the matrix Q. We again choose $\bar{Q}$ to consist of the first kq rows of Q.

3.3 *The case of q certain prime powers*

We have not been able to derive enough information from the matrix N to ensure the existence of the matrices R_i when q is a general prime power. However, as noted in the introduction, we can construct these matrices for the following value of q:

$$2^2,\ 2^3,\ 2^4,\ 3^2,\ 3^3,\ 7^2.$$

The methods used do not generalize.

4. Remarks on numerical results

The block designs given by Theorem 2 with $k > 4$ all have $r > 20$, and are outside the range of the tables in [2], [3], [4] and [6]. Consequently it is hard to check whether individual designs are new. We observe, however, that the existence of a (16,6,2) configuration yields a design with parameters (608, 96, 38, 6, 2); this is the multiple by 2 of the design (304, 96, 19, 6, 1) which is listed as unknown by Sprott [6]. Also the (11, 6, 3) configuration yields a (429, 66, 39, 6, 3) configuration, which is a multiple by 3 of a (143, 66, 13, 6, 1) design. The solution of the latter design in [4] does not appear to have arisen as one of a series of designs. We note in passing that Hall [3] mistakenly lists (143, 66, 13, 6, 1) as 'solution unknown'.

Theorem 3 yields a (34, 18, 17, 9, 8) configuration, which was previously unknown according to [6]. It also gives a (26, 14, 13, 7, 6) configuration, which was already known but was completely omitted from Hall's list, as well as a number of apparently new configurations with $r > 20$.

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