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Note

Amicable Hadamard Matrices

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If X is a symmetric Hadamard matrix, Y is a skew-Hadamard matrix, and XY^T is symmetric, then X and Y are said to be amicable Hadamard matrices. A construction for amicable Hadamard matrices is given, and then amicable Hadamard matrices are used to generalize a construction for skew-Hadamard matrices.

We refer the reader to Marshall Hall, Jr. [1] and Jennifer Wallis [3] for the definitions of *Hadamard matrix*, *skew-Hadamard matrix*, *skew-type*, *circulant* and *back-circulant*. In [3] we define *m-type matrices*, which we will henceforth call *amicable Hadamard matrices*, to be a pair of Hadamard matrices M and N of the same order such that M in skew-type, N is symmetric and

$$MN^T = NM^T;$$

where the superscript T denotes matrix transpose. Here we give another construction for amicable Hadamard matrices and generalize a theorem in [3].

We shall construct two Hadamard matrices of order $2y + 2$ of the form below with X symmetric and Y skew-type:

$$X = \begin{bmatrix} 1 & 1 & e & e \\ 1 & -1 & -e & e \\ e^T & -e^T & A & -B \\ e^T & e^T & -B & -A \end{bmatrix}, \qquad Y = \begin{bmatrix} 1 & 1 & e & e \\ -1 & 1 & e & -e \\ -e^T & -e^T & C & D \\ -e^T & e^T & -D & C \end{bmatrix},$$

where A, B, C, D are of order y, A, B, D are symmetric, and C is skew-type, and $e = [1, \ldots, 1]$ is $1 \times y$. The Hadamardness of X and Y imposes the following properties on the submatrices of X and Y:

$$eA^T = e = eB^T, \qquad AB^T = BA^T, \qquad eC^T = e = eD^T, \qquad CD^T = DC^T.$$

Then XY^T is symmetric if and only if

$$AC^T - BD^T, BC^T + AD^T$$

are symmetric.

We recall lemma 6 of [3]:

LEMMA 1. *If P is circulant and Q is back-circulant then PQ^T is symmetric.*

Let $R = (r_{ij})$ of order y be defined by $r_{i,y-i+1} = 1$ and for $j \neq y - i + 1$, $r_{ij} = 0$. Then if A, B, D are back-circulant matrices, with AR, BR and DR circulant and symmetric, and if C is circulant such that X is a symmetric Hadamard matrix, and Y is a skew-Hadamard matrix, then X and Y are amicable Hadamard matrices.

Let y be prime. Define $W = (w_{ij})$ by $w_{ii} = 0$, $w_{ij} = \chi(j - i)$ for $j \neq i$ where $\chi(b)$ is the Legendre symbol. For y (prime) $\equiv 1 \pmod 4$ $W^T = W$. Now choose $A = (I + W)R$ and $B = (I - W)R$.

In his paper [2] G. Szekeres shows how to construct twin difference sets which will yield the required C and D for q, where $(2q + 1)$ (prime power) $\equiv 3 \pmod 4$. Also with H_i, $i = 0, 1, 2, 3$, as in the proof of theorem 5 of [2] $K = H_0 \cup H_1$ and $K^* = H_0 \cup H_2$ can be used to form the required C and D for $y = 5, 13, 29, 53$. So we have

THEOREM 2. *If q is a prime such that*

(i) 5, 13, 29, 53, *or*

(ii) $2q + 1$ *is a prime, and q is odd,*

then there are amicable Hadamard matrices of order $2(q + 1)$.

Summarizing, using the proof of Lemma 8 of [3], we have amicable Hadamard matrices of the following orders:

I	2;	
II	$p^r + 1$	p^r (prime power) $\equiv 3 \pmod 4$;
III	$2(q + 1)$	q (prime) $\equiv 1 \pmod 4$ and $2q - 1$ is prime;
IV	S,	where S is a product of any of the above orders.

We note the following theorem, which is a generalization of corollary 9 of [3]. The proof is similar to that in [3].

THEOREM 3. *Let m and m' be the orders of amicable Hadamard matrices. If there is a skew-Hadamard matrix of order*

$$\text{(i)}\ \frac{(m-1)\,m'}{m}, \qquad \text{(ii)}\ \frac{(m-1)(m'-4)}{m},$$

then there is a skew-Hadamard matrix of order

$$\text{(i)}\ m'(m'-1)(m-1), \qquad \text{(ii)}\ (m'-1)(m'-4)(m-1),$$

respectively.

References

1. M. Hall, Jr., "Combinatorial Theory," Blaisdell, Waltham, Mass., 1967.
2. G. Szekeres, Tournaments and Hadamard matrices, *Enseignment Math.* **15** (1969), 269–278.
3. J. Wallis, (v, k, λ)-configurations and Hadamard matrices, *J. Austral. Math. Soc.* **11** (1970), 297–309.