Inner Product Vectors for skew-Hadamard Matrices

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Dedicated to Hadi Kharaghani on his 70th birthday

Abstract Algorithms to find new orders of skew-Hadamard matrices by complete searches are not efficient and require extensive CPU time. We consider a method relying on pre-calculation of *inner product vectors* aiming to reduce the search space. We apply our method to the algorithm of Seberry-Williamson to construct skew-Hadamard matrices. We find all possible solutions for ≤ 29. We use these results to improve analysis in order to reduce the search space.

1 Introduction

1.1 Definitions

An *Hadamard matrix*, H, of order n is a square ± 1 matrix whose rows (and columns) are pairwise orthogonal, that is $HH^\top = nI_n$. Hadamard matrices of order n are conjectured to exist for all orders $n \equiv 0 (\text{mod } 4)$. A weighing matrix, $W = W(n, k)$, of order n and weight k has elements $0, \ \pm 1$ and satisfies $WW^\top = kI_n$. If an Hadamard matrix M, can be written in the

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form $M = I + S$ where $S^\top = -S$, then M is said to be a *skew-Hadamard matrix.* Skew Hadamard matrices are also conjectured to exist for all orders $n \equiv 0 (\mathrm{mod}\ 4)$.

However, compared with the knowledge regarding the existence of Hadamard matrices very little is known regarding the existence of skew-Hadamard matrices.

In all our examples minus (”$-$”) is used to denote minus one (”-1”).

Example 1. **Hadamard matrices**

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} \; H_{4_{symmetric}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix} \; H_{4_{skew-type}} = \begin{bmatrix} 1 & - & 1 & 1 \\ 1 & - & - & - \\ - & - & 1 & - \\ 1 & 1 & 1 & - \end{bmatrix}$$

1.2 Circulant and Type 1 Matrix Basics

Because it is so important to the rest of our work we now spend a little effort to establish why the properties required for Williamson matrices are so important.

We define the *shift matrix,* T of order t by

$$T = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

So any circulant matrix, of order t and first row $x_1,\ x_2, \cdots, x_t$, that is,

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_t \\ x_t & x_1 & x_2 & \cdots & x_{t-1} \\ x_{t-1} & x_t & x_1 & \cdots & x_{t-2} \\ \vdots & & & & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{bmatrix} \qquad (1)$$

can be written as the polynomial

$$x_1 T^\top + x_2 T + x_3 T^2 \cdots x_t T^{t-1}.$$

We now note that polynomials commute, so any circulant matrices of the same order t commute.

We define the *back-diagonal matrix,* R of order t by

$$R = \begin{bmatrix} 0\,0 \cdots 1 \\ 0\,0 \cdots 0 \\ \vdots \quad\quad \vdots \\ 1\,0 \cdots 0 \end{bmatrix}.$$

We note $T^a R$ is a polynomial,for integer a, so that, similarly, any back-circulant matrix, of order t and first row $x_1,\ x_2, \cdots, x_t$, that is,

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_t \\ x_2 & x_3 & x_4 & \cdots & x_1 \\ x_3 & x_4 & x_5 & \cdots & x_2 \\ \vdots & & & & \vdots \\ x_t & x_1 & x_2 & \cdots & x_{t-1} \end{bmatrix}$$

can be written as the polynomial

$$x_1 T^\top R + x_2 T R + x_3 T^2 R \cdots x_t T^{t-1} R.$$

We now note that polynomials commute, so any two back-circulant matrices of the same order t commute.

Definition 1. A *circulant matrix* $C = (c_{ij})$ of order t is a matrix which satisfies the condition that

$$c_{ij} = c_{1,j-i+1} = c_{i+k,j+k} \tag{2}$$

where $j - i + 1$ is reduced modulo t [36]. A *back circulant* matrix $B = (b_{ij})$ order n is a matrix with property that

$$b_{ij} = b_{1,i+j-1} = b_{i+k,j-k} \tag{3}$$

where $i + j - 1$ is reduced modulo t [36].

Two matrices, X and Y of the same order t which satisfy

$$XY^\top = YX^\top \tag{4}$$

are said to be *amicable matrices.*

A back circulant matrix has transpose as the same as itself, so it is also a symmetric matrix.

Lemma 1. *If X is a back circulant matrix and Y is a circulant matrix, then X and Y are amicable matrices because $XY^\top = YX^\top$, see [36].*

Here are examples of amicable matrices of order 3. (1,-1)-matrices are used rather than other matrices because we are talking about Hadamard matrices whose elements only ones and minus ones.

$$C = \begin{bmatrix} 1 & - & 1 \\ 1 & 1 & - \\ - & 1 & 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & - & - \\ - & - & 1 \\ - & 1 & - \end{bmatrix} \quad , \quad CB^\top = BC^\top = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -3 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$

In all our definitions of circulant and back-circulant matrices we have assumed that the rows and columns have been indexed by the order, that is for order t, the rows are 1, 2, $\cdots$, t and similarly for the columns. The internal entries are then defined by the first row using a 1:1 and onto mapping. However we could have indexed the rows and columns using the elements of a group G, with elements g_1, g_2, , g_t. Loosely a type one matrix will then be defined so the (ij) element depends on a 1:1 and onto mapping of $g_j - g_i$ for type 1 matrices and on $g_j + g_i$ for type 2 matrices. We use additive notation, but that is not necessary. (Seberry) Wallis and Whiteman [37] have shown that circulant and type 1 can be used interchangeably and can the terms back-circulant and type 2. This can be used to explore similar theorems in more structured groups.

Definition 2. [Additive Property] k circulant matrices, A_1, A_2, $\cdots$ A_k, of order t with elements ± 1 only and which satisfy

$$\sum_{i=1}^{k} A_I A_i^\top = ktI_t,$$

will be said to satisfy the *additive property (for k).*

1.3 Historical Background

Hadamard matrices firstly we appeared in the literature in an 1867 paper written by J. J. Sylvester [29]. In 1893 Hadamard matrices appear, called matrices on the unit circle, they satisfied Hadamard's inequality for the determinant of matrices with entries within the unit circle [20].

Many matrices were found by Scarpis in 1898 [28].

In 1933 Paley [27] conjectured that the matrices existed for all positive integer orders divisible by 4. This has become known as the Hadamard conjecture:

Conjecture 1. Hadamard matrices exist for all orders 1, 2, $4t$, where t is a positive integer.

After Paley's work [27] the following orders less than 200: 23(4), 29(4), 39(3), 43(3), 47(4), 59(12), 65(3), 67(5), 73(7), 81(3), 89(4), 93(3), 101(10), 103(3), 107(10), 109(9), 113(8), 119 for p(3), 127(25), 133(3), 149(4), 151(5), 153(3), 163(3), 167(4), 179(8), 183(3), 189(3), 191, 193(3) were unresolved.

The number in brackets, if it is provided, indicates that one of order $2^a t$ is given in Seberry and Yamada [33]. The first unsolved cases are currently for primes $p = 167$, 179, 191 and 193.

1.4 Williamson Array

In 1944 [38] Williamson proposed using what has come known as the *Williamson array.* It can be shown, for example see [36] that if we can calculate suitable matrices, of order t, satisfying the additive property for 4, they can be plugged-in to his array to give Hadamard matrices of order $4t$. Hence

Definition 3. [Williamson matrices] Four circulant symmetric matrices, A, B, C and D, of order t with elements ± 1 only and which satisfy

$$AA^\top + BB^\top + CC^\top + DD^\top = 4tI_t,$$

will be called *Williamson matrices* of order t.

Theorem 1. [Williamson's Theorem] *Suppose A, B, C and D of order t, are Williamson matrices Then these matrices can be plugged into the* **Williamson array**

$$W_{array} = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix} \tag{5}$$

to obtain an Hadamard matrix of order $4t$.

Remark 1. Crucial part of proof. When we look at the terms of $W_{array}W_{array}^\top$ for, say the (2,3) element we obtain

$$-BC^\top + AD^\top - DA^\top + CB^\top$$

but for Williamson matrices $A = A^\top$, $B = B^\top$, $C = C^\top$ and $D = D^\top$. Further more A, B, C and D pairwise commute so

$$-BC^\top + AD^\top - DA^\top + CB^\top = -BC + AD - DA + -CB = -BC + BC - DA + DA = 0,$$

formally for all off diagonal elements of $W_{array}W_{array}^\top$. □

Example 2. **Williamson matrix of order** 4×3

$$W_{12_{Williamson}} = \left[\begin{array}{ccc|ccc|ccc|ccc}
1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 \\
- & 1 & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 \\
- & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 \\
\hline
- & 1 & 1 & 1 & - & - & - & - & - & 1 & - & - \\
1 & - & 1 & - & 1 & - & - & - & - & - & 1 & - \\
1 & 1 & - & - & - & 1 & - & - & - & - & - & 1 \\
\hline
- & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - \\
1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - \\
1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
\hline
- & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - \\
- & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
- & - & - & 1 & 1 & - & 1 & 1 & - & - & - & 1
\end{array}\right]$$

Many researchers have continued to search for Williamson and Williamson-like matrices (Williamson type, good matrices, best matrices, Goethals-Seidel type matrices, see below) which satisfy the additive property for 4.

1.4.1 Existence of Williamson matrices

In the 1960's Hadamard matrices were studied as part of a program to find the best possible error correction codes to be used to transmit data from deep space back to Earth.

Williamson did not use a computer to construct Williamson matrices but used some clever number theory. After Williamson the following orders t: 23, 29, 47, 59, 65, 67, 77, 103, 105, 107, 111, 119, 133, 143, 151, 155, 161, 163, 167, 171, 179, 183, 185, 191, 203, 207, 209, 215, 219, 221, 223, 227, 237, 245, 247, 249, 251, 253, 259, 267, 273, 283, 287, 291, 299 less than 300 were unknown.

Baumert, Golomb and Hall [2] used a computer with the Williamson construction to construct the order $92 = 4 \times 23$.

More recent results have been sporadic or the results of extensive calculations. Baumert and Hall [4] gave a very pretty construction to find the Hadamard matrix of order 156 which used what came to be called Baumert-Hall arrays. Now new methods were being discovered to find Hadamard matrices and some of these used *skew-Hadamard matrices.*

In [12] Djokovic showed that there is no Williamson matrix for $t = 35$. The computational state-of-the-art paper on Williamson matrices is [23], in which the Authors show that Williamson matrices do not exist for $t = 47, 53, 59$.

1.4.2 Seberry-Williamson Arrays

Williamson matrices are symmetric: the Hadamard matrix they form is neither symmetric nor skew-symmetric. As interest rose in the usefulness of skew-Hadamard matrices for further construction modifications of the

Williamson array were proposed to enable computer search. This led to what we will now call the *Seberry-Williamson array*, which first appeared in the PhD thesis of (Seberry) Wallis) [35].

Definition 4. [Good matrices] Four circulant matrices, of order t with elements ± 1 only, where A, is skew-symmetric $((A-I)^\top = -(A-I))$ and B, C and D, and which satisfy

$$AA^\top + BB^\top + CC^\top + DD^\top = 4tI_t,$$

will be called *good matrices* of order t.

Theorem 2. [(Seberry) Wallis's Theorem] *Suppose A, B, C and D of order t, are good matrices. Then these matrices can be plugged into the* **Seberry-Williamson array**

$$SW_{array} = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & -DR & CR \\ -CR & DR & A & -BR \\ -DR & -CR & BR & A \end{bmatrix} \tag{6}$$

to obtain a skew Hadamard matrix of order $4t$.

Remark 2. The proof is similar to that for the Williamson array after noting that circulant matrices and back-circulant matrices are amicable which pairs of back-circulant matrices commute.

These were some of the first searched for by computer. The ones given in (Seberry) Wallis' PhD thesis [35] were found on a PDP6 taking over 100 hours per week for many months in 1969. The limitation was basically the RAM memory of 4K. The matrices for $92 = 4 \times 23$ [35] were found at the University of Newcastle, NSW in 1970 using about 200 hours of CPU time. Again RAM was the largest constraint.

Later Koukouvinos et al [22] found further examples. We do not have time-space data for these calculations but give details of our own experiments.

Example 3. **Seberry-Williamson matrix of order** 4×3

The Seberry-Williamson matrix for first rows $A = 11-, B = 1--, C = 1--, D = 111$ is:

$$W_{Seberry-Williamson} = \left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 \\
1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 \\
\hline
1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - \\
1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 \\
\hline
1 & 1 & - & - & - & - & 1 & 1 & - & - & - & 1 \\
1 & - & 1 & - & - & - & - & 1 & 1 & - & 1 & - \\
- & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - \\
\hline
- & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - \\
- & - & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 \\
- & - & - & 1 & - & - & - & 1 & 1 & 1 & - & 1
\end{array}\right].$$

We note that because of the polynomial nature of back-circulants we could have also said that the $W_{Seberry-Williamson}$ skew-Hadamard matrix just described we could have used the equivalent matrix W_{SW}

$$W_{SW} = \left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 \\
- & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 \\
1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 \\
\hline
- & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - \\
1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 \\
\hline
- & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & - \\
1 & 1 & - & - & - & - & - & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & - & - & 1 & - & 1 & - & 1 & - \\
\hline
- & - & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - \\
- & - & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 \\
- & - & - & - & 1 & - & 1 & - & 1 & 1 & - & 1
\end{array}\right].$$

2 IPV Vectors

We first note

Definition 5. The *inner product* of rows i and j of any matrix, $G = (g_{xy})$, of order t is

$$\sum_{k=1}^{t} g_{ik} g_{jk}.$$

Example 4. **Inner products of matrix G_1**

$$G_1 = \begin{bmatrix} 1 & 1 & - & 1 & - \\ - & 1 & 1 & - & 1 \\ 1 & - & 1 & 1 & - \\ - & 1 & - & 1 & 1 \\ 1 & - & 1 & - & 1 \end{bmatrix}.$$

Inner product between the first row and the second row is calculated as

$$\sum_{k=1}^{5}(g_{1k} \times g_{2k}) = (1 \times -1) + (1 \times 1) + (-1 \times 1) + (1 \times -1) + (-1 \times 1) = -3.$$

□

Lemma 2. [Mirror Lemma] *Consider the inner products of rows of a circulant (back-circulant) matrices of order t. The inner product for the j th and ℓ th row is the same as the inner product for the $1st$ and $\ell - jth$ the rows so that there are at most $\frac{t-1}{2}$ distinct inner products.*

Definition 6. Let $G = (g_{xy})$ be a circulant (back-circulant) matrrix of order t. We call the $1 \times \frac{t+1}{2}$ vector with entries $(p_2, p_3, \cdots, p_{\frac{t+1}{2}})$, that is the vector has coordinates the inner products of rows 1 and 2, 1 and 3, $\cdots$, 1 and $\frac{t+1}{2}$, that is the vector

$$IPV_{x_{ij}} = (\sum_{k=1}^{t} x_{1,k}x_{1,k-j+i})$$

the inner product vector. □

Remark 3. A naive approach to finding the inner product vectors of any $n \times n$ matrix would take $\binom{n}{2} \times n^2$ calculations. Using the mirror lemma we have reduced the number of by 2.

Remark 4. Each row of a circulant ± 1 matrix can be considered as an integer, uniquely, by replacing the elements -1 by zero and converting the sequence to decimal. Thus a circulant matrix of order t can be represented by an integer of size the least integer greater than $ln_2 t$. This means any sequence we would consider in search for skew-Hadamard matrices using current technology can be represented by one word of space. This is used in the Appendices to describe the solution matrices.

Lemma 3. *We now consider circulant (or back circulant) matrices of order t, t odd, with entries ± 1 only. Then*

- *if* $t \equiv 1 (\bmod\ ) 4$ *then the entries of the* IPV *are* $\equiv 1 (\bmod\ ) 4$*;*
- *if* $t \equiv 3 (\bmod\ ) 4$ *then the entries of the* IPV *are* $\equiv 3 (\bmod\ ) 4$*;*
- *the coordinates in an* IPV *are integers between* $-(t-4)$ *and* t*;*
- *the sum of the coordinates in an* IPV *is 0.*

2.1 Sums of Squares of First Rows of Williamson matrices

We notice that for arrays which have ± 1 matrices, A, B, C, D of order t, satisfying the 4-additive property, plugged into them where

$$AA^\top + BB^\top + CC^\top + DD^\top = 4tI_t.$$

Then if $\mathbf{e}$ is the $1 \times m$ matrix of all ones and the row sums of A, B, C, D are a, b, c, and d respectively. Then

$$\mathbf{e}A = a\mathbf{e},\ \mathbf{e}B = b\mathbf{e},\ \mathbf{e}C = c\mathbf{e},\ \mathbf{e}D = a\mathbf{d},$$

and

$$\mathbf{e}(AA^\top + BB^\top + CC^\top + DD^\top) = 4mI_m = a^2\mathbf{e} + b^2\mathbf{e} + c^2\mathbf{e} + d^2\mathbf{e} = 4m\mathbf{e}.$$

2.1.1 Sums of Squares of First Rows of Good Matrices for Seberry-Williamson Matrices

Let A, B, C and D be good matrices of order t with first row sums a, b, c and d. Then using exactly the same proof as for the first rows of Williamson matrices, except that the row-sum of the skew-type matrix is $a = +1$ we have for four good matrices of order t

$$4t = 1 + b^2 + c^2 + d^2.$$

In the case of the skew-Williamson array a is always $= +1$.

3 Computational results on IPV for $t = 3, 5, 7, 9, 11, 13$

A naive algorithm to find matrices for the skew-array or good matrices was used on various machines to obtain comparison run times for $t = 1, 3, 5, \cdots, 27, 29, 31$. These show that more sophistication is needed to make the results required attainable for larger t. The results obtained for $t = 1, cdots, 13$ are given in the appendix. Further results are available from the authors for $t = 15, 17, 19, 21, 23, 25, 27, 29$. The computations were performed on SHARCNET and RQCHP clusters. This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET www.sharcnet.ca) and Compute/Calcul Canada. Computations were made on the Mammouth supercomputer managed by Calcul Québec and Compute Canada. The operation of this supercomputer is funded by the Canada Foundation for Innovation (CFI), NanoQuébec, RMGA and

the Fonds de recherche du Québec - Nature et technologies (FRQ-NT).

We consider the inner product vectors $IPVs$ for the Seberry-Williamson construction for skew-Hadamard matrices which uses the first rows of the corresponding good matrices: A (skew-type), B (symmetric), C (symmetric), and D (symmetric) in their construction. We give $\{a_1, a_2, \cdots, a_{\frac{1}{2}(t-1)}\}$, $\{b_1, b_2, \cdots, b_{\frac{1}{2}(t-1)}\}$, $\{c_1, c_2, \cdots, c_{\frac{1}{2}(t-1)}\}$, and $\{d_1, d_2, \cdots, d_{\frac{1}{2}(t-1)}\}$ for the coordinates of the IPV_A, IPV_B, IPV_C and IPV_D respectively.

In each case we note that for the coordinates in the IPV

- we have $a_i,\ b_i,\ c_i,\ d_i,\ 2 \le i \le \frac{1}{2}(t-1)$ always takes only integer values $\equiv t \pmod 4$, see Lemma 3.
- summing $a_i + b_i + c_i + d_i$ for each $i = 1,\ 2,\ \cdots,\ \frac{1}{2}(t-1)$ always gives 0 (this is required for skew-Williamson matrices);
- we have $4t = (1)^2 + (\pm b)^2 + (\pm c)^2 + (\pm d)^2$ in every case (this is required for skew-Williamson matrices).

The IPVs for t=3

There is only one solution from a complete search for $t = 3$. It involves 4 matrices A, B, C, and D which have row sums

$$(-1)(-1)(-1)(3).$$

1. the maximum coordinate that appears is $+3$ and the absolute minimum coordinate that appears is -1;
2. the sum of the row sums is $(-1) + (-1) + (-1) + 3 = 0$;
3. $4t = 12 = (1)^2 + (-1)^2 + (-1)^2 + (3)^2$;
4. the first rows of the corresponding good matrices A (skew-type), B (symmetric), C (symmetric), and D (symmetric) are

```
1 1 -1; 1 -1 -1; 1 -1 -1; 1, 1, 1;
```

5. the integers which correspond to these first rows are 6, 4, 4, 7;

The IPVs for t=5

A complete search yields two possible solutions for $t = 5$. They each involve 4 good matrices A (skew-type), B (symmetric), C (symmetric), and D (symmetric) which have row sums

$$5.1 : (1, -3)(-3, 1)(1, 1)(1, 1)$$

and

$$5.2 : (-3, 1)(1, -3)(1, 1)(1, 1)$$

1. the maximum absolute value coordinate that appears in any of the IPVs is 1 and the minimum absolute value coordinate that appears in any of the IPVs is -3;
2. summing $a_i + b_i + c_i + d_i$ for i=1, 2, 3, 4, for the IPVs gives $(-2)+(-2)+(2)+(2) = 0$ in both cases;
3. $4t = 20 = (1)^2 + (1)^2 + (-3)^2 + (-3)^2$;
4. the first rows of the corresponding good matrices A (skew-type), B (symmetric), C (symmetric), and D (symmetric) are

   ```
   1 1 1 -1 -1; 1  1 -1 -1 1; 1 -1 -1 -1 -1; 1,-1,-1 -1 -1;
   ```

   and

   ```
   1 1 -1 1 -1; 1 -1  1 1 -1; 1 -1 -1 -1 -1; 1,-1,-1 -1 -1;
   ```

5. the integers which correspond to these first rows are 30, 25, 16, and 16; and 26, 22, 16 and 16;

The IPVs for t=7

A complete search gives a total of seven solutions.

1. the IPV vectors obtained are:

   ```
   7.1 :(-5, 3, -1) (-1, -5, 3) (3, -1, -5) (3, 3, 3)
   7.2 :(-5, 3, -1) (-1, -1, 3) (3, -1, -1) (3, -1, -1)
   7.3 :(-1, -5, 3) (-5, 3, -1) (3, -1, -5) (3, 3, 3)
   7.4 :(-1, -5, 3) (-1, 3, -1) (-1, 3, -1) (3, -1, -1)
   7.5 :(-1, -1, -1) (-1, -1, 3) (-1, 3, -1) (3, -1, -1)
   7.6 :(3, -1, -5) (-5, 3, -1) (-1, -5, 3) (3, 3, 3)
   7.7 :(3, -1, -5) (-1, -1, 3) (-1, -1, 3) (-1, 3, -1)
   ```

2. the IPV values range from -5 to +3. All coordinates n the IPV are $\equiv 3 \pmod 4$;
3. the sum of the row sums of the IPV is $(-3)+(-3)+(-3)+(9) = 0$ in three cases and $(-3)+(1)+(1)+(1) = 0$, in four cases cases;
4. $4t = 28 = (1)^2 + (1)^2 + (1)^2 + (-5)^2 = (1)^2 + (3)^2 + (3)^2 + (3)^2$;
5. the integers which correspond to the first rows to make the good matrices A (skew-type), B (symmetric), C (symmetric), and D (symmetric) are:

   ```
   106 76 97 64
   106 109 115 115
   89 82 97 64
   89 94 94 115
   75 109 94 115
   ```

```
120 82 76 64
120 109 109 94
```

The IPVs for t=9

A complete search gives a total of seven solutions.

1. The IPVs obtained are:

```
9.1 :(-3, 1, -3, 1) (-3, -3, 1, 1) (5, 1, -3, -3) (1, 1, 5, 1)
9.2 :(1, -3, -3, 1) (1, -3, 1, -3) (-3, 5, -3, 1) (1, 1, 5, 1)
9.3 :(1, 1, -3, -3) (-3, 1, 1, -3) (1, -3, -3, 5) (1, 1, 5, 1)
```

2. the IPV values range from -3 to +5. All coordinates in the IPV are $\equiv$ 1 (mod 4);
3. the sum of the row sums of the IPV is $(-4) + (-4) + (0) + (0) = 0$ in all three cases;
4. $4t = 36 = (1)^2 + (1)^2 + (-3)^2 + (5)^2$;
5. the integers which correspond to the first rows to make the good matrices A (skew-type), B (symmetric), C (symmetric), and D (symmetric) are:

```
468 358 385 475
369 316 322 475
279 421 280 475
```

In future research we will seek to find the relationships between the IPVs to limit our search space and time.

The IPVs for t=11

A complete search gives a total of fifteen solutions.

1. The IPVs obtained are:

```
11.1:(-5,-1,3,-5,3)(-1,-1,-5,3,3)(7,3,-1,-5,-5)(-1,-1,3,7,-1)
11.2:(-5,-1,3,-1,-1)(-5,3,-1,-1,3)(3,-5,-1,3,-1)(7,3,-1,-1,-1)
11.3:(-5,-1,3,-1,-1)(3,-5,-5,-1,7)(3,-1,3,-1,-5)(-1,7,-1,3,-1)
11.4:(-5,3,-5,3,-1)(-5,-1,7,-5,3)(3,-5,-1,3,-1)(7,3,-1,-1,-1)
11.5:(-1,-5,-1,-1,3)(-5,3,-1,-1,3)(7,3,-1,-5,-5)(-1,-1,3,7,-1)
11.6:(-1,-5,-1,-1,3)(-1,3,3,-5,-1)(3,-5,-1,3,-1)(-1,7,-1,3,-1)
11.7:(-1,-5,3,3,-5)(-1,3,3,-5,-1)(3,-5,-5,-1,7)(-1,7,-1,3,-1)
11.8:(-1,-1,-1,3,-5)(-5,-1,7,-5,3)(-1,-1,-5,3,3)(7,3,-1,-1,-1)
11.9:(-1,-1,-1,3,-5)(-5,3,-1,-1,3)(3,-1,3,-1,-5)(3,-1,-1,-1,7)
11.10:(-1,3,-5,-1,-1)(-1,-5,3,7,-5)(-1,3,3,-5,-1)(3,-1,-1,-1,7)
```

```
11.11:(-1,3,-5,-1,-1)(-1,-1,-5,3,3)(3,-1,3,-1,-5)(-1,-1,7,-1,3)
11.12:(3,-5,-5,-1,3)(-5,7,-5,3,-1)(3,-1,3,-1,-5)(-1,-1,7,-1,3)
11.13:(3,-1,-1,-5,-1)(-5,7,-5,3,-1)(3,-5,-1,3,-1)(-1,-1,7,-1,3)
11.14:(3,-1,-1,-5,-1)(-1,-1,-5,3,3)(-1,3,3,-5,-1)(-1,-1,3,7,-1)
11.15:(3,3,-1,-5,-5)(-5,3,-1,-1,3)(-1,-5,3,7,-5)(3,-1,-1,-1,7)
```

2. the IPV values range from -5 to +7. All coordinates in the IPV are $\equiv$ 3 (mod 4);
3. the row sums of the IPV is $(-5)+(-1)+(-1)+(7)$ in all fifteen cases;
4. $4t = 44 = (1)^2 + (3)^2 + (3)^2 + (-5)^2$;
5. the integers which correspond to the first rows to make the good matrices A (skew-type), B (symmetric), C (symmetric), and D (symmetric) are:

1381	1486	1927	1096
1449	1717	1843	1537
1449	1657	1276	1282
1195	1462	1843	1537
1836	1717	1927	1096
1836	1867	1843	1282
1892	1867	1657	1282
1582	1462	1486	1537
1582	1717	1276	1072
1960	1741	1867	1072
1960	1486	1276	1156
1127	1402	1276	1156
1505	1402	1843	1156
1505	1486	1867	1096
1071	1717	1741	1072

Observations.

1. no solution for "A" contains a 7;
2. some "A" appear in more than one solution;
3. "C" is the only matrix for which the first coordinate, here $c_1 = 7$;

The IPVs for t=13

A complete search gave 36 solutions. The IPV values range from -7 to +9. All coordinates in the IPV are $\equiv$ 1 (mod 4);

1. The IPVs obtained are:

```
13.1:(-7,1,1,-3,5,-3)(5,-3,-7,-3,1,1)(-3,1,1,5,-3,5)(5,1,5,1,-3,-3)
13.2:(-7,1,1,1,-3,1)(-3,-7,1,5,1,-3)(1,1,-3,-3,5,5)(9,5,1,-3,-3,-3)
13.3:(-7,1,1,1,-3,1)(-3,1,-3,-3,1,1)(5,-3,-3,1,5,1)(5,1,5,1,-3,-3)
13.4:(-7,1,1,1,-3,1)(1,-7,-3,1,5,-3)(1,5,1,-3,-3,-7)(5,1,1,1,1,9)
```

13.5:(-3,-7,5,1,-3,1)(1,5,1,-3,-3,-7)(-3,5,-3,1,1,5)(5,-3,-3,1,5,1)
13.6:(-3,-3,-3,1,5,-3)(1,1,-7,-3,-3,5)(1,-3,5,5,-3,1)(1,5,5,-3,1,-3)
13.7:(-3,-3,1,-3,-3,5)(-7,5,-3,1,1,-3)(5,-3,-3,1,5,1)(5,1,5,1,-3,-3)
13.8:(-3,-3,5,-3,1,-3)(5,1,-3,1,-3,-7)(-3,1,1,5,-3,5)(1,1,-3,-3,5,5)
13.9:(-3,1,-3,-3,1,1)(-3,-7,1,5,1,-3)(1,5,1,-3,-3,-7)(5,1,1,1,1,9)
13.10:(-3,1,-3,-3,1,1)(-3,-3,1,1,-7,5)(5,1,-3,1,-3,-7)(1,1,5,1,9,1)
13.11:(-3,1,-3,-3,1,1)(1,-7,-3,1,5,-3)(1,-3,5,-3,-7,1)(1,9,1,5,1,1)
13.12:(-3,1,-3,5,-3,-3)(1,-3,5,-3,-7,1)(-3,5,1,-3,5,1)(5,-3,-3,1,5,1)
13.13:(-3,1,1,1,-7,1)(-3,1,5,-7,1,-3)(5,-3,-7,-3,1,1)(1,1,1,9,5,1)
13.14:(-3,1,1,1,-7,1)(1,-3,1,1,-3,-3)(-3,5,1,-3,5,1)(5,-3,-3,1,5,1)
13.15:(-3,1,1,1,-7,1)(1,1,-7,-3,-3,5)(-3,1,5,-3,9,-3)(5,-3,1,5,1,-3)
13.16:(-3,5,-7,1,-3,1)(-3,1,5,-7,1,-3)(1,-3,5,5,-3,1)(5,-3,-3,1,5,1)
13.17:(1,-7,-3,1,1,1)(-7,1,-3,5,-3,1)(-3,1,5,-7,1,-3)(9,5,1,1,1,1)
13.18:(1,-7,-3,1,1,1)(-3,-3,1,-7,5,1)(-3,9,-3,5,-3,1)(5,1,5,1,-3,-3)
13.19:(1,-7,-3,1,1,1)(1,-3,1,1,-3,-3)(-3,5,-3,1,1,5)(1,5,5,-3,1,-3)
13.20:(1,-3,-7,1,1,1)(-3,-3,1,1,-7,5)(1,5,1,-3,-3,-7)(1,1,5,1,9,1)
13.21:(1,-3,-7,1,1,1)(-3,1,-3,-3,1,1)(1,-3,5,5,-3,1)(1,5,5,-3,1,-3)
13.22:(1,-3,-7,1,1,1)(5,1,-3,1,-3,-7)(-3,-3,9,1,-3,5)(-3,5,1,-3,5,1)
13.23:(1,-3,-3,-7,1,5)(-7,1,-3,5,-3,1)(1,5,5,-3,1,-3)(5,-3,1,5,1,-3)
13.24:(1,-3,-3,5,1,-7)(-3,-3,1,1,-7,5)(1,1,-3,-3,5,5)(1,5,5,-3,1,-3)
13.25:(1,-3,1,1,-3,-3)(-7,1,-3,5,-3,1)(-3,-3,1,-7,5,1)(9,5,1,1,1,1)
13.26:(1,-3,1,1,-3,-3)(-7,5,-3,1,1,-3)(5,-3,-7,-3,1,1)(1,1,9,1,1,5)
13.27:(1,-3,1,1,-3,-3)(-3,1,5,-7,1,-3)(1,1,-7,-3,-3,5)(1,1,1,9,5,1)
13.28:(1,1,1,-7,1,-3)(-3,-3,1,1,-7,5)(1,-7,-3,1,5,-3)(1,9,1,5,1,1)
13.29:(1,1,1,-7,1,-3)(-3,1,-3,-3,1,1)(-3,1,1,5,-3,5)(5,-3,1,5,1,-3)
13.30:(1,1,1,-7,1,-3)(1,-3,5,-3,-7,1)(-3,5,-3,1,1,5)(1,-3,-3,9,5,-3)
13.31:(1,1,1,-3,1,-7)(-7,1,-3,5,-3,1)(5,-3,-7,-3,1,1)(1,1,9,1,1,5)
13.32:(1,1,1,-3,1,-7)(-7,5,-3,1,1,-3)(1,-3,5,5,-3,1)(5,-3,-3,-3,1,9)
13.33:(1,1,1,-3,1,-7)(1,-3,1,1,-3,-3)(-3,1,1,5,-3,5)(1,1,-3,-3,5,5)
13.34:(1,5,-3,-3,-3,-3)(-3,-3,1,-7,5,1)(-3,1,1,5,-3,5)(5,-3,1,5,1,-3)
13.35:(5,-3,-3,-3,-3,1)(-3,-7,1,5,1,-3)(-3,5,-3,1,1,5)(1,5,5,-3,1,-3)
13.36:(5,1,1,-3,-7,-3)(1,-7,-3,1,5,-3)(-3,1,1,5,-3,5)(-3,5,1,-3,5,1)

2. the row sums of the IPV are -6, -6, 6, 6.
3. $4t = 52 = (1)^2 + (1)^2 + (5)^2 + (5)^2 = (1)^2 + (1)^2 + (1)^2 + (-7)^2$;
4. the integers which correspond to the first rows to make the good matrices A (skew-type), B (symmetric), C (symmetric), and D (symmetric) are:

6858 7267 5998 5116
6486 5734 6046 7951
6486 5362 6649 5116
6486 5020 7435 4192
5325 7435 6901 6649
4947 5902 7579 7831
4811 6805 6649 5116
4699 4600 5998 6046

5573	5734	7435	4192
5573	6757	4600	4240
5573	5020	7315	5122
5969	7315	7531	6649
5213	4852	7267	4360
5213	6925	7531	6649
5213	5902	5878	7783
4439	4852	7579	6649
6598	5530	4852	6145
6598	6505	5626	5116
6598	6925	6901	7831
7260	6757	7435	4240
7260	5362	7579	7831
7260	4600	7021	7531
7768	5530	7831	7783
7620	6757	6046	7831
7880	5530	6505	6145
7880	6805	7267	4612
7880	4852	5902	4360
7106	6757	5020	5122
7106	5362	5998	7783
7106	7315	6901	7069
6238	5530	7267	4612
6238	6805	7579	7411
6238	6925	5998	6046
8016	6505	5998	7783
4303	5734	6901	7831
6081	5020	5998	7531

The IPVs for $t = 15, 17, 19, 21, 23, 25, 27, 29$

We summarize the results for the above values of t below. Full details are available from the authors.

- $t = 15$, a complete search gave 44 solutions. The values in the IPV range from -13 to +11.
- $t = 17$, a complete search gave 16 solutions. The IPV values range from -11 to +9.
- $t = 19$, a complete search gave 64 solutions. The IPV values range from -9 to +11.
- $t = 21$, a complete search gave 60 solutions. The IPV values range from -11 to +9.
- $t = 23$, a complete search gave 66 solutions. The values in the IPV range from -13 to +11.

- $t = 25$, a complete search gave 90 solutions. The values in the IPV range from -15 to +13.
- $t = 27$, a complete search gave 117 solutions. The values in the IPV range from -13 to +15.
- $t = 29$, a complete search gave 71 solutions. The values in the IPV range from -15 to +11.

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