

THREE-LEVEL CRETAN MATRICES OF ORDER 37

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Purpose: This note discusses three level orthogonal matrices which were first highlighted by J. J. Sylvester. Hadamard matrices, symmetric conference matrices, and weighing matrices are the best known of these matrices with entries from the unit disk. The goal of this note is to develop a theory of such matrices based on preliminary research results. **Methods:** Extreme solutions (using the determinant) have been established by minimization of the maximum of the absolute values of the elements of the matrices followed by their subsequent classification. **Results:** We give a three level Cretan(37). This is new and the first time such a matrix has been found whose order is other than a 2^k , k is an even integer. The methods given in this note may be used to construct many more Cretan matrices based on regular Hadamard matrices. **Practical relevance:** The over-riding aim is to seek Cretan(n) with absolute or relative (local) maximal determinants as they have many applications in image processing and masking. Web addresses are given for other illustrations and other matrices with similar properties. Algorithms to construct Cretan matrices have been implemented in developing software of the research program-complex.

Keywords – Hadamard Matrices, Regular Hadamard Matrices, Orthogonal Matrices, Cretan Matrices.

AMS Subject Classification: 05B20; 20B20.

Introduction

In this and further papers we use some names, definitions, notation differently than we have in the past [1]. This, we hope, will cause less confusion, bring our nomenclature closer to common usage and conform for mathematical purists. We have chosen the use of the word level, instead of value for the entries of a matrix, to conform to earlier writings. We note that the strict definition of an orthogonal matrix, \mathbf{X} , of order n , is that $\mathbf{X}^T\mathbf{X} = \mathbf{X}\mathbf{X}^T = \mathbf{I}_n$, where \mathbf{I}_n is the identity matrix of order n . In this paper we consider $\mathbf{S}^T\mathbf{S} = \mathbf{S}\mathbf{S}^T = \omega\mathbf{I}_n$ where ω is a constant. We call these orthogonal matrices [2, 3]. We refer to [2, 4, 5] for definitions not given below.

Definitions

Definition 1. A Cretan(n) (CM) matrix, \mathbf{S} , is a orthogonal matrix of order n with entries with moduli ≤ 1 , where there must be at least one 1 per row and column. The inner product of a row of CM(n) with itself is the weight ω . $\mathbf{S}^T\mathbf{S} = \mathbf{S}\mathbf{S}^T = \mathbf{I}_n$. The inner product of distinct rows of CM(n) is zero. A μ -level Cretan(n ; μ ; ω) matrix, CM(n ; μ ; ω), has μ levels or values for its entries.

Cretan(n), or CM(n) orthogonal matrices are studied in [2, 4]. In more general notation these are can be CM(order), CM(order; number of levels = τ), CM(order; number of levels = τ ; occurrences of

levels = L_1, L_2, \dots, L_τ), CM(order; number of levels = τ ; weight = ω), and CM(order; number of levels = τ ; weight; occurrences of levels in whole matrix), etc. etc. etc.

The definition of Cretan is not that each variable occurs some number of times per row and column but L_1, L_2, \dots, L_τ times in the whole matrix. So we have CM(n ; τ ; ω ; L_1, L_2, \dots, L_τ) so

$$\begin{pmatrix} -0.5 & 1 & 1 \\ 1 & -0.5 & 1 \\ 1 & 1 & -0.5 \end{pmatrix}$$

is a CM(3), a CM(3;2), a CM(3;2;2,1), a CM(3;2;2.25), a CM(3;2;2.25;6,3) depending on which numbers (in brackets) are currently of interest. We call them Cretan matrices because they were first discussed in this generality at a conference in Crete in July, 2014.

The over-riding aim is to seek CM(n) with absolute or relative (local) maximal determinants as they have many applications in image processing and masking [1, 2].

The matrix orthogonality equation $\mathbf{S}^T\mathbf{S} = \mathbf{S}\mathbf{S}^T = \omega\mathbf{I}_n$, is a set of n^2 scalar equations, giving two kinds of formulae: $g(a, b, s) = \omega$, there are n such equations, and $f(a, b, s) = 0$, there are $n^2 - n$ such equations. We concentrate on two of them: $g(a, b, s) = \omega, f(a, b, s) = 0$.

The entries in $\omega\mathbf{I}_n$ which are on the diagonal, are given by the radius equation $\omega = g(a, b, s)$, they

depend on the choice of a, b, s . At least one of the levels has value 1.

The maximal weight $\omega = n$ arises from Hadamard matrices, and for symmetric conference matrices $\omega = n - 1$. *Orthogonal* matrices can have also irrational values for the weight.

The second equation $f(a, b, s) = 0$ we call the *characteristic equation*, as it allows us to find a formulae for levels $b \leq s \leq a$.

Definition 2. A *regular Hadamard matrix*, \mathbf{H} , of order $4t$, has elements plus one and minus one only. It satisfies the *orthogonality equation* $\mathbf{H}^T \mathbf{H} = \mathbf{H} \mathbf{H}^T = 4t \mathbf{I}$. The sum of the entries in any row or column is $2\sqrt{t}$.

We first note that a computer search undertaken recently in Iran found over 31 million inequivalent regular Hadamard matrices of order 36. We use [5] to give our example, \mathbf{H} , for this note.

A new Cretan(37, 3)

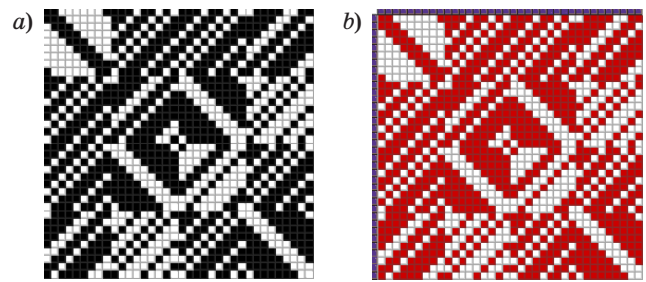
Construction. Let $\mathbf{G}(a, b)$ be the 36×36 regular Hadamard matrix [5] with more ones than minus ones in each row and column, and then the ones replaced by “ a ” and minus ones replaced by “ $-b$ ”. Let \mathbf{F} be the 37×37 matrix

$$\mathbf{F} = \begin{pmatrix} a & s & \dots & s \\ s & & & \\ \vdots & & \mathbf{G}(a,b) & \\ s & & & \end{pmatrix}.$$

Then if the radius equations are $a^2 + 36s^2 = \omega$, $15a^2 + 21b^2 + s^2 = \omega$; and the characteristic equations are $16a - 21b = 0$, $6a^2 - 18ab + 12b^2 + s^2 = 0$: \mathbf{F} is a *Cretan(37;3;27.9388;541,756,72)* using the definition (Figure, a and b).

References

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■ Orthogonal matrices: the regular Hadamard matrix $\mathbf{H}(36)$ (a) and *Cretan(37; 3)* (b)

In the Figure the white square is for element “ a ”, black and red (core) square is for element “ $-b$ ”, and blue square is for border “ s ”.

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Conclusion

The methods given in this note may be used to construct many more Cretan matrices based on regular Hadamard matrices. The matrices \mathbf{G} and \mathbf{F} both give completely new Cretan matrices $CM(4t, 2)$ and $CM(4t + 1, 3)$. They will be studied in depth in paper of J. Seberry, in preparation and add new members to the set of Cretan matrices constructed with one core and one border, observed in [2, 3, 6]. This strongly suggests we have discovered a new branch of Cretan matrices.

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