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Some new constructions of orthogonal designs

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Abstract

In this paper we construct OD$(4pq^r(q+1); pq^r, pq^r, pq^r, pq^r, pq^{r+1}, pq^{r+1}, pq^{r+1}, pq^{r+1})$ for each core order $q \equiv 3 (\mathrm{mod}\ 4)$, $r \geq 0$ or $q = 1$, p odd, $p \leq 21$ and $p \in \{25, 49\}$, and COD$(2q^r(q+1); q^r, q^r, q^{r+1}, q^{r+1})$ for any prime power $q \equiv 1 (\mathrm{mod}\ 4)$ (including $q = 1$), $r \geq 0$.

1 Introduction

An orthogonal design (OD) X of order n and type $(s_1, \ldots, s_m)$, s_i positive integers, is an $n \times n$ matrix with entries $\{0, \pm x_1, \ldots, \pm x_m\}$ (the x_i are commuting indeterminates) satisfying

$$XX^T = (\sum_{i=1}^{m} s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. This is denoted by OD$(n; s_1, \ldots, s_m)$.

Such generically orthogonal matrices have played a significant role in the construction of Hadamard matrices (see, e.g., [3], [6]) and they have been extensively used in the study of weighing matrices (e.g. [3] and [8]).

Since Baumert and Hall [9] gave the first example of Baumert-Hall arrays, or OD($4t; t, t, t, t$), and Plotkin [7] defined Plotkin arrays, or OD($8t; t, t, t, t, t, t, t, t$), to construct Hadamard matrices, many research results have been published for T-matrices that are used in the construction of Plotkin arrays (see [3], [5], [9], [10]).

Turyn [11] introduced the notion of a complex Hadamard matrix, i.e., an $n \times n$ matrix C whose entries are chosen from $\{\pm 1, \pm i\}$ and satisfy $CC^* = nI_n$ (* is conjugate transpose). He further showed how such matrices could be used to construct Hadamard matrices, and gave several examples. Further examples of such matrices are given in [3] and [4].

For a complex analogue of orthogonal designs there are several possible generalizations; we choose the one which gives real orthogonal designs as a special case.

A complex orthogonal design (COD) [4] of order n and type $(s_1, \ldots, s_m)$ (s_i positive integers) on the real commuting variables $x_1, \ldots, x_m$ is an $n \times n$ matrix X, with entries chosen from $\{\varepsilon_1 x_1, \ldots, \varepsilon_m x_m : \ \varepsilon_i$ a fourth root of $1\}$ satisfying

$$XX^* = (\sum_{i=1}^{m} s_i x_i^2) I_n.$$

For further discussion we need the following definitions from [6].

Definition 1 [**Amicable Matrices; Amicable Set**] Two square real matrices of order n, A and B, are said to be *amicable* if $AB^T - BA^T = 0$.

A set $\{A_1, \ldots, A_{2n}\}$ of square real matrices is said to be *an amicable set* if

$$\sum_{i=1}^{n} (A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T) = 0.$$

It is easy to generalize an amicable set to the case of square complex matrices. For this, we just need to replace A^T by A^*, the conjugate transpose of A.

Definition 2 [**T-matrices**] $(0, \pm 1)$ type 1 matrices T_1, T_2, T_3 and T_4 of order n are called *T-matrices* if the following conditions are satisfied:

(a) $T_i * T_j = 0, \ i \neq j, \ 1 \leq i, \ j \leq 4$, where $*$ denotes Hadamard product;

(b) $\sum_{i=1}^{4} T_i T_i^T = nI_n$.

T-matrices can be used to construct orthogonal designs (see [1]).

The following definition was first used by Holzmann and Kharaghani in [5].

Definition 3 [**Weak amicable**] The T-matrices T_1, T_2, T_3 and T_4 are said to be *weak amicable* if

$$T_1(T_3 + T_4)^T + T_2(T_3 - T_4)^T = (T_3 + T_4)T_1^T + (T_3 - T_4)T_2^T.$$

Definition 4 [**Core**] Let Q be a matrix of order n, with zero diagonal and all other elements ± 1 satisfying

$$QQ^T = nI_n - J_n,\ QJ_n = J_nQ = 0,$$

where J_n is the matrix of order n, consisting entirely of 1's. Further if $n \equiv 1 \pmod 4$, $Q^T = Q$, and if $n \equiv 3 \pmod 4$, then $Q^T = -Q$. Here Q is called the *core* and n is the *core order*.

If $H = I_n + K$ is an Hadamard matrix of order n with $K^T = -K$, we call it skew type Hadamard matrix.

Here we rewrite the following theorem as

Theorem 1 (**[12]**) *If there exists a skew type Hadamard matrix of order $q+1$, then there exists a core of order q.*

It is well-known that if $q+1 = 2^t n_1 \ldots n_s$, each n_i of the form $p^r + 1 \equiv 0 \pmod 4$, and p is prime, then q is a core order. Moreover, if $q \equiv 3 \pmod 4$ is a core order, then q^r is a core order for any odd $r \geq 1$ (see [9], p. 497).

In Section 2 we give an infinite class of OD with 8 variables. In Section 3 we construct several families of COD with 4 variables. In Section 4 we construct weak amicable T-matrices.

2 The construction of OD

The Goethals-Seidel (or Wallis-Whiteman) array has been proven to be a very useful tool for construction of orthogonal designs. Such arrays are essential for construction of orthogonal designs with more than four variables.

For convenience we need following definition:

Definition 5 [**Additive property**] A set of matrices $\{B_1, \ldots, B_m\}$ of order n with entries in $\{0, \pm x_1, \ldots, \pm x_k\}$ is said to satisfy the *additive property*, with weight $\sum_{i=1}^{k} s_i x_i^2$, if

$$\sum_{i=1}^{m} B_i B_i^T = (\sum_{i=1}^{k} s_i x_i^2) I_n. \tag{1}$$

Kharaghani [6] gave an infinite number of arrays which are suitable for any amicable set of 8 type 1 matrices. Here **suitable** means a set of matrices satisfying the **additive property**. If one substitutes the matrices in an orthogonal design, or the Goethals-Seidel array, one can get an orthogonal design. We rewrite the following theorems without proof.

Theorem 2 *([6]) There is an 8×8 array which is suitable to make an $8n \times 8n$ orthogonal matrix for any amicable set of 8 type 1 matrices of order n satisfying an additive property.*

Theorem 3 *([6]) For each prime power* $q \equiv 3 \pmod 4$ *there is an array suitable for any amicable set of eight matrices* A_i *satisfying*

$$\sum_{i=1}^{4}(A_{2i-1}A_{2i}^T + A_{2i}A_{2i-1}^T) = cI_{q+1},$$

where c *is a constant expression.*

More general results are given in [2]. As an application we give an example of such an OD.

If A is a circulant matrix of order n with the first row $(a_1, \ldots, a_n)$, we denote it by

$$A = \mathrm{circ}(a_1, \ldots, a_n).$$

Example 1 Let x_1, x_2, x_3, x_4 and x_5 be real commuting variables and

$$\begin{array}{ll} A_1 = \mathrm{circ}(x_1, x_2, x_3, x_4, -x_4, -x_3, x_2), & A_2 = \mathrm{circ}(-x_1, x_2, x_3, -x_4, x_4, -x_3, x_2), \\ A_3 = \mathrm{circ}(x_1, -x_2, x_3, -x_4, x_4, -x_3, -x_2), & A_4 = \mathrm{circ}(x_1, x_2, -x_3, -x_4, x_4, x_3, x_2), \\ A_5 = \mathrm{circ}(x_5, x_2, x_3, x_4, x_4, x_3, -x_2), & A_6 = \mathrm{circ}(-x_5, x_2, x_3, x_4, x_4, x_3, -x_2), \\ A_7 = \mathrm{circ}(x_5, -x_2, x_3, -x_4, -x_4, x_3, x_2), & A_8 = \mathrm{circ}(-x_5, -x_2, x_3, -x_4, -x_4, x_3, x_2). \end{array}$$

It is easy to verify that

$$\sum_{i=1}^{4}(A_{2i-1}A_{2i}^T - A_{2i}A_{2i-1}^T) = 0 \text{ and } \sum_{i=1}^{8} A_iA_i^T = (4(x_1^2 + x_5^2) + 16(x_2^2 + x_3^2 + x_4^2))I_7.$$

From the proof of Theorem 2, using the method in [6], one can construct an $OD(56; 4, 4, 16, 16, 16)$.

Theorem 4 *Let* $q \equiv 3 \pmod 4$ *be a core order. Then there is an* $OD(4q^r(q+1); q^r, q^r, q^r, q^r, q^{r+1}, q^{r+1}, q^{r+1}, q^{r+1})$ *for any integer* $r \geq 0$.

Proof. Let Q be a core of order q, and let $a_1, \ldots, a_8$ be real commuting variables. Set

$$A_{2i-1}(0) = a_{2i}, \quad A_{2i}(0) = a_{2i-1}, \quad i = 1, 2, 3, 4.$$

It is clear that, as $A_i(0)$ are commuting variables,

$$\begin{array}{l} A_1(0), \ldots, A_8(0) \text{ are type 1 }, \\ A_{2i-1}(0)A_{2i}^T(0) = A_{2i}(0)A_{2i-1}^T(0), \quad i = 1, 2, 3, 4, \end{array}$$

and (with $q^0 = 1$),

$$A_{2i-1}(0)A_{2i-1}^T(0) + qA_{2i}(0)A_{2i}^T(0) = q^0(qa_{2i-1}^2 + a_{2i}^2)I_{q^0}, \quad i = 1, 2, 3, 4.$$

Suppose that for $r \geq 1$ we have

$$\begin{aligned}
&A_1(r-1), \ldots, A_8(r-1) \text{ are all type 1}\\
&A_{2i-1}(r-1)A_{2i}^T(r-1) = A_{2i}(r-1)A_{2i-1}^T(r-1), \text{ and}\\
&A_{2i-1}(r-1)A_{2i-1}^T(r-1) + qA_{2i}(r-1)A_{2i}^T(r-1) = q^{r-1}(qa_{2i-1}^2 + a_{2i}^2)I_{q^{r-1}},\\
&i = 1, 2, 3, 4.
\end{aligned}$$

Write

$$A_{2i-1}(r) = J_q \times A_{2i}(r-1), \quad A_{2i}(r) = I_q \times A_{2i-1}(r-1) + Q \times A_{2i}(r-1),$$

where $\times$ is the Kronecker product. Then $A_1(r), \ldots, A_8(r)$ are type 1 of size q^r.

It is easy to verify that

$$\begin{aligned}
&A_{2i-1}(r)A_{2i}^T(r) = A_{2i}(r)A_{2i-1}^T(r),\\
&A_{2i-1}(r)A_{2i-1}^T(r) + qA_{2i}(r)A_{2i}^T(r) = q^r(qa_{2i-1}^2 + a_{2i}^2)I_{q^r}, \quad i = 1, 2, 3, 4.
\end{aligned}$$

Now let B_i of size $(q+1)q^r$ be given by

$$B_i = I_{q+1} \times A_{2i-1}(r) + K \times A_{2i}(r), \quad i = 1, 2, 3, 4, \quad K = \begin{bmatrix} 0 & e^T \\ -e & Q \end{bmatrix},$$

where $e^T = (1, \ldots, 1)$ is a row vector with q components.

Then B_1, B_2, B_3 and B_4 are of type 1 and

$$\sum_{i=1}^{4} B_i B_i^T = \sum_{i=1}^{4} q^r(qa_{2i-1}^2 + a_{2i}^2)I_{q^r(q+1)}.$$

From Theorem 3 it follows that there is an OD$(4q^r(q+1);\, q^r,\, q^r,\, q^r,\, q^r,\, q^{r+1},\, q^{r+1},\, q^{r+1},\, q^{r+1})$. □

Note that Corollary 5 of [6] is a special case of Theorem 4 with $r = 0$.

If there are type 1 T-matrices of order n, then there exist an OD$(4n; n, n, n, n)$ (see [9]). Further, from [5], weak amicable sets can be used to get the following.

Lemma 1 *For p odd, $1 \leq p \leq 21$, $p \in \{25,\ 49\}$, there exists an OD$(8p; p, p, p, p, p, p, p, p)$.*

Proof. For each $p \in \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 25, 49\}$, there exist T-matrices T_1, T_2, T_3 and T_4 of order p satisfying weak amicability.

The explicit construction of such T-matrices of these orders can be found in Table 1 of [5] and the Appendix of this paper. From Theorem 5 and Corollary 6 of [5], there exist OD$(8p; p, p, p, p, p, p, p, p)$. □

Theorem 5 *Let T_1, T_2, T_3 and T_4 be T-matrices of order p with weak amicability. Then there is an OD$(4pq^r(q+1); pq^r, pq^r, pq^r, pq^r, pq^{r+1}, pq^{r+1}, pq^{r+1}, pq^{r+1})$ for each core order $q \equiv 3 \pmod 4$ and $r \geq 0$.*

Proof. Write

$$f(a,b,c,d) = T_1a + T_2b + T_3c + T_4d.$$

Here a, b, c and d are real commuting variables. Let $A_1, \ldots, A_8$ be defined as follows:

$$\begin{array}{ll} A_1 = f(x_1,x_2,x_3,x_4), & A_2 = f(-x_8,-x_7,x_6,x_5), \\ A_3 = f(x_2,-x_1,x_4,-x_3), & A_4 = f(x_7,-x_8,-x_5,x_6), \\ A_5 = f(x_3,-x_4,-x_1,x_2), & A_6 = f(x_5,x_6,x_7,x_8), \\ A_7 = f(x_4,x_3,-x_2,-x_1), & A_8 = f(x_6,-x_5,x_8,-x_7), \end{array}$$

where $x_1, \ldots, x_8$ are real commuting variables. Set

$$A_{2i}(0) = A_{2i-1}, \quad A_{2i-1}(0) = A_{2i}, \quad i = 1,2,3,4.$$

For $r \geq 1$ let

$$\begin{aligned} A_{2i-1}(r) &= J_q \times A_{2i}(r-1), \\ A_{2i}(r) &= I_q \times A_{2i-1}(r-1) + Q \times A_{2i}(r-1), \quad i = 1,2,3,4, \end{aligned}$$

where Q is a square matrix of order q defined as in Theorem 4. Replacing

$$\begin{array}{l} A_{2i-1}(r)A_{2i}^T(r) = A_{2i}(r)A_{2i-1}^T(r), \\ A_{2i-1}(r)A_{2i-1}^T(r) + qA_{2i}(r)A_{2i}^T(r) = q^r(qa_{2i-1}^2 + a_{2i}^2)I_{q^r}, \ i = 1,2,3,4, \ r \geq 0, \end{array}$$

by

$$\begin{array}{l} \sum_{i=1}^{4}(A_{2i-1}(r)A_{2i}^T(r) - A_{2i}(r)A_{2i-1}^T(r)) = 0, \\ \sum_{i=1}^{4}(A_{2i-1}(r)A_{2i-1}^T(r) + qA_{2i}(r)A_{2i}^T(r)) = pq^r\sum_{i=1}^{4}(qx_i^2 + x_{i+4}^2)I_{pq^r}, \end{array}$$

respectively, and repeating the procedure of the proof of Theorem 4, one can obtain the theorem. □

Corollary 1 *For p odd, $1 \leq p \leq 21$ and $p \in \{25, 49\}$, there exists an $OD(8pq^r(q+1); pq^r, pq^r, pq^r, pq^r, pq^{r+1}, pq^{r+1}, pq^{r+1}, pq^{r+1})$ with each core order $q \equiv 3 \pmod 4$ and integer $r \geq 0$.*

3 The construction of COD

In this section we give several infinite classes of COD.

Theorem 6 *There exists a $COD(2q^r(q+1); q^r, q^r, q^{r+1}, q^{r+1})$ for each prime power $q \equiv 1 \pmod 4$ and $r \geq 0$.*

Proof. Let Q be the symmetric core of order $q \equiv 1 \pmod 4$.

Now let

$$A_{2i-1}(0) = a_{2i-1}, \quad A_{2i}(0) = a_{2i}, \quad i = 1, 2,$$

where a_1, a_2, a_3 and a_4 are real commuting variables. Note that $q^0 = 1$. It is clear that

$$\begin{aligned}
&A_{2i-1}(0)A_{2i}^*(0) = A_{2i}(0)A_{2i-1}^*(0),\\
&A_{2i-1}(0)A_{2i-1}^*(0) + qA_{2i}(0)A_{2i}^*(0) = q^0(a_{2i-1}^2 + qa_{2i}^2)I_{q^0}, \quad i = 1,\ 2,\\
&A_i(0)A_j(0) = A_j(0)A_i(0),\ 1 \le i,\ j \le 4.
\end{aligned}$$

Suppose that for $r \ge 1$ we have

$$\begin{aligned}
A_{2i-1}(r-1)A_{2i}^*(r-1) &= A_{2i}(r-1)A_{2i-1}^*(r-1),\\
A_{2i-1}(r-1)A_{2i-1}^*(r-1) + qA_{2i}(r-1)A_{2i}^*(r-1) &= q^{r-1}(a_{2i-1}^2 + qa_{2i}^2)I_{q^{r-1}},\\
&\qquad i = 1,\ 2,\\
A_i(r-1)A_j(r-1) &= A_j(r-1)A_i(r-1),\\
&\qquad 1 \le i,\ j \le 4.
\end{aligned}$$

Write

$$A_{2j-1}(r) = J_q \times A_{2j}(r-1), \quad A_{2j}(r) = I_q \times A_{2j-1}(r-1) + iQ \times A_{2j}(r-1),$$

$i = \sqrt{-1},\ j = 1,\ 2.$ It follows that

$$\begin{aligned}
A_{2i-1}(r)A_{2i}^*(r) &= A_{2i}(r)A_{2i-1}^*(r),\\
A_{2i-1}(r)A_{2i-1}^*(r) + qA_{2i}(r)A_{2i}^*(r) &= q^r(a_{2i-1}^2 + qa_{2i}^2)I_{q^r}, \quad i = 1,\ 2,\\
A_i(r)A_j(r) &= A_j(r)A_i(r), \qquad 1 \le i,\ j \le 4.
\end{aligned}$$

Let

$$K = \begin{bmatrix} 0 & e^T \\ e & Q \end{bmatrix}.$$

Put

$$F_j = I_{q+1} \times A_{2j-1}(r) + iK \times A_{2j}(r),\ i = \sqrt{-1},\ j = 1,\ 2.$$

We have

$$\begin{aligned}
F_jF_j^* &= q^r(a_{2j-1}^2 + qa_{2j}^2)I_{q^r(q+1)}, \quad j = 1,\ 2,\\
F_1F_2 &= F_2F_1.
\end{aligned}$$

Finally, let

$$X = \begin{pmatrix} F_1 & F_2 \\ -F_2^* & F_1^* \end{pmatrix}.$$

Then X is a $\mathrm{COD}(2q^r(q+1); q^r, q^r, q^{r+1}, q^{r+1})$, as required. □

From the proof of Theorem 6 we can obtain the following theorem.

Theorem 7 *There is a COD*$(q^r(q+1); q^r, q^{r+1})$ *for each prime power* $q \equiv 1 \pmod 4$ *and* $r \ge 0$.

4 The construction of weak amicable T-matrices

It is convenient to use the group ring $Z[G]$ of the group G of order p over the ring Z of rational integers with the addition and multiplication. Elements of $Z[G]$ are of the form

$$a_1g_1 + a_2g_2 + \cdots + a_pg_p,\ a_i \in Z,\ g_i \in G,\ 1 \le i \le p.$$

In $Z[G]$ the addition, +, is given by the rule

$$\left(\sum_g a(g)g\right) + \left(\sum_g b(g)g\right) = \sum_g (a(g) + b(g))g.$$

The multiplication in $Z[G]$ is given by the rule

$$\left(\sum_g a(g)g\right)\left(\sum_h b(h)h\right) = \sum_k \left(\sum_{gh=k} a(g)b(h)\right) k.$$

For any subset A of G, we define

$$\sum_{g\in A} g \in Z[G],$$

and by abusing the notation we will denote it by A.

Let a set $\{X_1, \ldots, X_8\}$ be a C-partition of an abelian additive group G of order p, i.e.,

$$X_i \subset G,\quad X_i \cap X_j = \emptyset,\ i \ne j,$$

and

$$\sum_{i=1}^{8} X_i = G,\ \sum_{i=1}^{8} X_iX_i^{(-1)} = p + \sum_{i=1}^{4}\left(X_iX_{i+4}^{(-1)} + X_{i+1}X_i^{(-1)}\right),$$

where the equations above hold in the group ring $Z[G]$; (see [13]).

For any $A \subset G$, set

$$I(A) = (a_{ij})_{1\le i,j\le n},\quad a_{ij} = \begin{cases} 1, & \text{if } g_j - g_i \in A,\\ 0, & \text{otherwise,}\end{cases}$$

where $g_1, \ldots, g_p$ are elements of G in any order. That is, $I(A)$ is the $(0,1)$ incidence matrix of A of type 1. Now let

$$T_i = I(X_i) - I(X_{i+4}),\quad i = 1, 2, 3, 4;$$

then T_1, T_2, T_3 and T_4 are T-matrices of order p.

Let $\sum_g a(g)g \in Z[G]$ where $a(g) \in Z$ and $g \in G$. If, for any $g \in G$, we have $a(g) = a(-g)$, then we call $\sum_g a(g)g$ symmetric in the group ring $Z[G]$.

It is clear that T-matrices T_1, T_2, T_3 and T_4 of order p satisfy weak amicability, if and only if $T_1(T_3+T_4)^T + T_2(T_3-T_4)^T$ is symmetric, and if and only if $(X_1 - X_5)(X_3^{(-1)} - X_7^{(-1)} + X_4^{(-1)} - X_8^{(-1)}) + (X_2 - X_6)(X_3^{(-1)} - X_7^{(-1)} - X_4^{(-1)} + X_8^{(-1)})$ is symmetric in the group ring $Z[G]$.

The following theorem and corollary will simplify the verification of weak amicability in some cases.

Theorem 8 *Let G be an abelian group of order n and let $\{X_1, \ldots, X_8\}$ be a C-partition of G. If both $X_1 - X_5 + X_2 - X_6$ and $X_3 - X_7 + X_4 - X_8$ are symmetric in the group ring $Z[G]$, then there exist T-matrices of order n satisfying weak amicability if and only if $(X_2 - X_6)(X_4^{(-1)} - X_8^{(-1)})$ is also symmetric in the group ring $Z[G]$.*

Using the same assumptions as in Theorem 8, we have the following corollary.

Corollary 2 *If $X_4 = X_8 = \emptyset$, then there exist T-matrices of order n satisfying weak amicability.*

Appendix

Now we give decomposition of the sum of four squares and the new sets of T-matrices which have weak amicability for $p = 9, 25, 49$. The values $1 \le p \le 21$ are given in Holtzmann and Kharaghani [5].

$$
\begin{array}{ll}
p = 9 = 3^2 + 0^2 + 0^2 + 0^2, & Q_1 = \{0, 1, x+1\},\ Q_2 = \{2\} - \{x+2\}, \\
 & Q_3 = \{2x\} - \{2x+2\},\ Q_4 = \{2x+1\} - \{x\}. \\
p = 25 = 5^2 + 0^2 + 0^2 + 0^2, & Q_1 = \{0\} - E_0 \cup E_1,\ Q_2 = E_2 - E_6,\ Q_3 = E_3 - E_7, \\
 & Q_4 = E_4 - E_5, \\
 & \text{where } E_i = \{g^{8j+i} :\ j = 0, 1, 2\},\ i = 0, \ldots, 7\}, \\
 & \text{and } g = x + 1 (\text{mod } x^2 - 3, \text{mod } 5) \text{ is a generator} \\
 & \text{of GF}(25). \\
p = 49 = 7^2 + 0^2 + 0^2 + 0^2, & Q_1 = \{0\} \cup E_0 \cup E_1 \cup E_6 \cup E_{12} - E_3 \cup E_7, \\
 & Q_2 = E_4 \cup E_{10} \cup E_{15} - E_8 \cup E_{11} \cup E_{13}, \\
 & Q_3 = E_9 - E_2,\ Q_4 = E_5 - E_{14},\ \text{where} \\
 & E_i = \{g^{16j+i} :\ j = 0, 1, 2\}, i = 0, \ldots, 15,\ \text{and} \\
 & g = x + 2\ (\text{mod } x^2 + 1,\ \text{mod } 7) \text{ is a generator} \\
 & \text{of GF}(49).
\end{array}
$$

Remark. Holzmann and Kharaghani [5] have given constructions of weak amicable T-matrices of order 9 in Z_9 and for $9 = 2^2 + 2^2 + 1^2 + 0^2$, However, our construction is given in GF(9) and for $9 = 3^2$. These constructions are different in essence.
Conjecture ([5]) There exist infinite orders of T-matrices satisfying weak amicability for all odd integers.

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