Construction of amicable orthogonal designs of quaternions

YING ZHAO JENNIFER SEBERRY TIANBING XIA

School of Software Engineering and Computer Science
University of Wollongong
Wollongong, NSW, 2522
Australia

BEATA J. WYSOCKI TADEUSZ A. WYSOCKI

School of Electrical, Computer and Telecommunications Engineering
University of Wollongong
Wollongong, NSW, 2522
Australia

Abstract

This paper introduces some construction methods for amicable orthogonal designs over the real and quaternion domain which have not been explored for code design before. The proposed construction methods generate a large number of amicable orthogonal designs of quaternions. It is also shown that amicable orthogonal designs of quaternions can be used to construct restricted quaternion orthogonal designs which can be applied as orthogonal space-time-polarization block codes for wireless communications.

1 Introduction

Space-time block codes from real and complex orthogonal designs for multiple-input multiple-output (MIMO)wireless communication systems have received considerable attention due to their inherent orthogonality, which guarantees a full transmit diversity and linear maximum-likelihood (ML) decoding [11]. Space-time block codes have been adopted in the newly proposed standard for wireless LANs IEEE, 802.11n. We expect that additional forms of diversity, namely polarization diversity and frequency diversity, could be considered together with space and time diversity to overcome multi-path fading in order to improve system performance.

It has been shown that polarization diversity, together with other forms of diversity, can add to the performance improvements offered by other diversity techniques. Isaeva and Sarytchev [8] showed that the utilization of polarization diversity with

other forms of diversity can be modelled by means of quaternions, since two orthogonal complex constellations form a quaternion. This has motivated the study of orthogonal designs over the quaternion domain for future applications in signal processing as space-time-polarization block codes [9, 4, 1].
This paper aims to use general construction techniques to generate amicable orthogonal designs of quaternions, which we believe can be used for constructing quaternion orthogonal designs, just like the applications of amicable orthogonal designs(AOD) for complex space-time codes. This paper is organized as follows: Section 2 introduces the classifications of orthogonal designs over the quaternion domain, e.g. *orthogonal design of quaternions* (ODQ), and *amicable orthogonal design of quaternions* (AODQ). In Section 3, we present several construction techniques for building AOD and AODQ, together with some examples to illustrate our methods. In Section 4, we give examples to show how to construct *restricted quaternion orthogonal design*(RQOD) from two AODQs.

2 Preliminaries

We first review the definitions of orthogonal designs and amicable designs over the real, complex domain. Then we define several types of orthogonal designs over the quaternion domain based on the research in [9].

2.1 Orthogonal designs

The concept of an *orthogonal design* was first introduced in [6, 7] and concerned only designs with real commuting variables or zero entries.

Definition 2.1. An *orthogonal design, OD*, of order n and type $(s_1, s_2, \ldots, s_u)$ in commuting real variables $x_1, x_2, \ldots, x_u$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, is an $n \times n$ matrix A with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \left(\sum_{i=1}^{u} s_i x_i^2\right) I_n,$$

where $(.)^T$ denotes the transpose of a matrix and I_n is the identity matrix of order n. This definition can be extended to include *rectangular* designs, i.e. $r \times n$ matrices which satisfy $A^T A = \left(\sum_{i=1}^{u} s_i x_i^2\right) I_n$.

Definition 2.2. Two square orthogonal designs A and B are said to be *amicable* if $A^T B = B^T A$ and $AB^T = BA^T$. We write $AOD(n; u_1, \ldots, u_s; v_1, \ldots, v_t)$ to denote that two orthogonal designs $OD(n; u_1, u_2, \ldots, u_s)$ and $OD(n; v_1, v_2, \ldots, v_t)$ are amicable.

Example 2.1. The following matrices

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2 & a_1 & 0 & a_3 \\ -a_3 & 0 & a_1 & -a_2 \\ 0 & -a_3 & a_2 & a_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -b_3 & -b_2 & -b_1 & 0 \\ -b_2 & b_3 & 0 & -b_1 \\ -b_1 & 0 & b_3 & b_2 \\ 0 & -b_1 & b_2 & -b_3 \end{bmatrix},$$

where a_1, a_2, a_3 and b_1, b_2, b_3 are real, commuting variables, are amicable orthogonal designs $AOD(4; 1, 1, 1; 1, 1, 1)$.

2.2 Complex orthogonal designs

An extension of orthogonal designs with real entries is orthogonal designs over the complex domain. There are several possible generalizations for orthogonal designs with complex entries. The first definition of such designs was given by Geramita and Geramita [5]; they treat real orthogonal designs as a special case.

Definition 2.3. A *complex orthogonal design*, *COD*, of order n and type $(s_1, s_2, \ldots, s_u)$ in real commuting variables $x_1, x_2, \ldots, x_u$, denoted $COD(n; s_1, s_2, \ldots, s_u)$, is an $n \times n$ matrix A with entries in the set $\{0, \pm x_1, \pm \mathbf{i}x_1, \pm x_2, \pm \mathbf{i}x_2, \ldots, \pm x_u, \pm \mathbf{i}x_u\}$ satisfying

$$A^H A = AA^H = \left(\sum_{h=1}^{u} s_h x_h^2\right) I_n,$$

where $(.)^H$ denotes the Hermitian transpose.

Example 2.2. The matrix $\begin{bmatrix} \mathbf{i}x_1 & x_2 \\ x_2 & \mathbf{i}x_1 \end{bmatrix}$, where x_1 and x_2 are real commuting variables, is a $COD(2; 1, 1)$.

In [15], Yuen, Guan and Tjhung defined the concept of amicable complex orthogonal designs which is a complex extension of amicable orthogonal designs.

Definition 2.4. Two complex orthogonal designs, A and B are said to be *amicable* if $AB^H = BA^H$ or $A^H B = B^H A$. We write $ACOD(n; w_1, w_2, \ldots, w_u; z_1, z_2, \ldots, z_v)$ to denote that two designs $COD(n; w_1, w_2, \ldots, w_u)$ and $COD(n; z_1, z_2, \ldots, z_v)$ are complex amicable.

Example 2.3. Let $A = \begin{bmatrix} a & b \\ -\mathbf{i}b & \mathbf{i}a \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ \mathbf{i}d & -\mathbf{i}c \end{bmatrix}$, where $a, b, c, d \in \mathbb{R}$. A and B are amicable complex orthogonal designs $ACOD(2; 1, 1; 1, 1)$.

2.3 Amicable orthogonal designs of quaternions

Definition 2.5. Given a matrix $A = (\mathbf{a}_{\ell,m})$, where $\mathbf{a}_u$ are quaternion variables or numbers, we define its *quaternion transform* by $A^Q = (\mathbf{a}^Q_{m,\ell})$.

The following definitions of orthogonal design of quaternions and restricted quaternion orthogonal design were originally given in [9].

Definition 2.6. An *orthogonal design of quaternions*, *ODQ*, of order n and type $(s_1, s_2, \ldots, s_u)$ denoted $ODQ(n; s_1, s_2, \ldots, s_u)$, on the commuting real variables $x_1, x_2, \ldots, x_u$ is a square matrix A of order n with entries from $\{0, \mathbf{q}_1 x_1, \mathbf{q}_2 x_2, \ldots, \mathbf{q}_u x_u\}$, where each $\mathbf{q}_j \in \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ such that

$$A^Q A = AA^Q = \left(\sum_{h=1}^{u} s_h x_h^2\right) I_n,$$

where $(.)^Q$ denotes quaternion transform. We can extend this definition to include *rectangular* designs that satisfy $A^Q A = (\sum_{h=1}^{u} s_h x_h^2)\, I_n$.

Example 2.4. Consider $A = \begin{bmatrix} -x_1 & x_2\mathbf{i} \\ -x_2\mathbf{j} & x_1\mathbf{k} \end{bmatrix}$, where x_1, x_2 are real, commuting variables. Then,

$$A^Q A = \begin{bmatrix} -x_1 & x_2\mathbf{j} \\ -x_2\mathbf{i} & -x_1\mathbf{k} \end{bmatrix} \begin{bmatrix} -x_1 & x_2\mathbf{i} \\ -x_2\mathbf{j} & x_1\mathbf{k} \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 & 0 \\ 0 & x_1^2 + x_2^2 \end{bmatrix},$$

so A is an $ODQ(2; 1, 1)$.

Definition 2.7. A *restricted quaternion orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$, denoted $RQOD(n; s_1, s_2, \ldots, s_u)$, on the complex variables z_1, z_2, $\ldots$, z_u, and their conjugates z_1^*, z_2^*, $\ldots$, z_u^*, is an $n \times n$ matrix A with entries from $\{0, \mathbf{q}_1 z_1, \mathbf{q}_1 z_1^*, \mathbf{q}_2 z_2, \mathbf{q}_2 z_2^*, \ldots, \mathbf{q}_u z_u, \mathbf{q}_u z_u^*\}$, where each $\mathbf{q}_p$ is a linear combination of $\{\pm\mathbf{1},\ \pm\mathbf{i},\ \pm\mathbf{j},\ \pm\mathbf{k}\}$ such that

$$A^Q A = AA^Q = (\sum_{h=1}^{u} s_h |z_h|^2) I_n.$$

This definition can be extended to include *rectangular* designs that satisfy $A^Q A = (\sum_{h=1}^{u} s_h |z_h|^2) I_n$.

Example 2.5. Consider $A = \begin{bmatrix} \mathbf{i}z_1 & \mathbf{i}z_2 \\ -\mathbf{j}z_2^* & \mathbf{j}z_1^* \end{bmatrix}$, where z_1, z_2 are complex commuting variables. Then,

$$A^Q A = \begin{bmatrix} -z_1^*\mathbf{i} & z_2\mathbf{j} \\ -z_2^*\mathbf{i} & -z_1\mathbf{j} \end{bmatrix} \begin{bmatrix} \mathbf{i}z_1 & \mathbf{i}z_2 \\ -\mathbf{j}z_2^* & \mathbf{j}z_1^* \end{bmatrix} = \begin{bmatrix} |z_1|^2 + |z_2|^2 & 0 \\ 0 & |z_1|^2 + |z_2|^2 \end{bmatrix},$$

so A is an $RQOD(2; 1, 1)$. To illustrate why this is called a *restricted* QOD, we replace complex variables in A using $z_i = x_i + y_i\mathbf{i}$, where the x_i, y_i are real variables. This gives $A = \begin{bmatrix} -y_1 + \mathbf{i}x_1 & -y_2 + \mathbf{i}x_2 \\ -\mathbf{j}x_2 - \mathbf{k}y_2 & \mathbf{j}x_1 + \mathbf{k}y_1 \end{bmatrix}$. We can see that the entries of A are quaternion variables such that certain components of the variables are *restricted* to zero.

Definition 2.8. Two orthogonal designs of quaternions, A and B, are said to be *amicable* if $AB^Q = BA^Q$ or $A^Q B = B^Q A$. We write

$$AODQ(n; w_1, w_2, \ldots, w_u; z_1, z_2, \ldots, z_v)$$

to denote that two designs $ODQ(n; w_1, w_2, \ldots, w_u)$ and $ODQ(n; z_1, z_2, \ldots, z_v)$ are amicable.

Example 2.6. Let $A = \begin{bmatrix} -x_1 & x_2\mathbf{i} \\ -x_2\mathbf{j} & x_1\mathbf{k} \end{bmatrix}$ and $B = \begin{bmatrix} y_1 & y_2\mathbf{i} \\ y_2\mathbf{j} & y_1\mathbf{k} \end{bmatrix}$, where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Here A and B are amicable orthogonal designs of quaternions of type $AODQ(2; 1,1;1,1)$.

The proof that A and B are amicable orthogonal designs of quaternions is straightforward.

Let X and Y be amicable orthogonal designs of quaternions of type $AODQ(n; u_1, \dots, u_s; v_1, \dots, v_t)$. Let us write $X = \sum_{i=1}^{s} A_i x_i$, $Y = \sum_{j=1}^{t} B_j y_j$, and we then have:

$$\begin{aligned}
&\text{i)} A_i * A_\ell = 0,\ 1 \le i \ne \ell \le s; \quad B_j * B_k = 0,\ 1 \le j \ne k \le t;\\
&\text{ii)} A_i A_i^Q = u_i I_n,\ 1 \le i \le s; \quad B_j B_j^Q = v_j I_n,\ 1 \le j \le t;\\
&\text{iii)} A_i A_\ell^Q + A_\ell A_i^Q = 0,\ 1 \le i \ne \ell \le s; \quad B_j B_k^Q + B_k B_j^Q = 0,\ 1 \le j \ne k \le t;\\
&\text{iv)} A_i B_j^Q = B_j A_i^Q,\ 1 \le i \le s,\ 1 \le j \le t,
\end{aligned} \tag{1}$$

where A_i, B_j are all $\{0, \pm 1, \pm i, \pm j, \pm k\}$ quaternion matrices. It is clear that conditions (i)–(iv) are necessary and sufficient for the existence of amicable orthogonal designs of quaternions $AODQ(n; u_1, \dots, u_s; v_1, \dots, v_t)$.

Proposition 2.1. *A necessary and sufficient condition that there exist amicable orthogonal designs of quaternions X and Y of type $AODQ(n; u_1, \dots, u_s; v_1, \dots, v_t)$ is that there exists a family of matrices of $\{A_1, \dots, A_s; B_1, \dots, B_t\}$ of order n satisfing (i)–(iv) above.*

Proof. Let X and Y be such a amicable pair and write $X = A_1 x_1 + \cdots + A_s x_s$ and $Y = B_1 y_1 + \cdots + B_t y_t$ as linear monomials in the $x_i, y_i \in \mathbb{R}$. By definition, the proof of (i) and (ii) is straightforward. Since we have

$$\begin{aligned}
XX^Q &= (A_1 x_1 + \cdots + A_s x_s)(A_1^Q x_1 + \cdots + A_s^Q x_s)\\
&= \sum_{j=1}^{s} (A_j A_j^Q x_j^2) + \sum_{j \ne k} (A_j A_k^Q + A_k A_j^Q) x_j x_k\\
&= (\sum_{j=1}^{s} u_j x_j^2) I_n,
\end{aligned}$$

conditions in (iii) are thus satisfied. Condition (iv) can be proved by comparing coefficient matrices of $XY^Q = YX^Q$ on both sides.

Conversely, if we have $\{A_1, \dots, A_s; B_1, \dots, B_t\}$ of order n satisfing (i)–(iv), then it is obvious that $X = A_1 x_1 + \cdots + A_s x_s$ and $Y = B_1 y_1 + \cdots + B_t y_t$ are an AODQ with the required type.

□

Definition 2.9. An *amicable family of quaternions*(AFQ) of type $(u_1, \dots, u_s; v_1, \dots, v_t)$ in order n is a collection of quaternion matrices $\{A_1, \dots A_s; B_1, \dots, B_t\}$ satisfying (ii), (iii), (iv) above.

The definition of amicable family of quaternions(AFQ) is analogous to the definition of amicable family of orthogonal designs given in [7].

3 Construction techniques

In this section, we present several construction techniques for building amicable orthogonal designs over the real and quaternion domain. Some methods already exist for generating real amicable orthogonal designs. We can also extend these techniques to build designs over the quaternion domain. However, due to the non-commutivity of the quaternions, we need to modify existing techniques to make them suitable for designs over the quaternion domain.

3.1 Amicable orthogonal designs

Definition 3.1. A symmetric conference matrix N of order n is a square $(0, 1, -1)$ matrix satisfying $N = N^T$ and $NN^T = (n-1)I_n$. It is shown in [3] that if such a matrix exists, one may assume it has zero diagonal.

The existence of symmetric conference matrices was discussed in [10, 13]. For example, there exist symmetric conference matrices of order n for $n = p + 1$, where $p \equiv 1 \pmod 4$, p a prime power. Here, we introduce the application of symmetric conference matrices in constructing amicable orthogonal designs.

Example 3.1. The matrix Q is a type 1 matrix [7] with properties: $QQ^T = 5I_5 - J_5$, $QJ_5 = J_5Q = 0$ and $Q = Q^T$, where J_5 is the 5×5 matrix of all ones.

$$Q = \begin{bmatrix} 0 & + & - & - & + \\ + & 0 & + & - & - \\ - & + & 0 & + & - \\ - & - & + & 0 & + \\ + & - & - & + & 0 \end{bmatrix}.$$

Then $N = \begin{bmatrix} 0 & + & \cdots & + \\ + & & & \\ \vdots & & Q & \\ + & & & \end{bmatrix}$ is a symmetric conference matrix of order $n = p+1 = 6$.

Lemma 3.1. *Let N be a symmetric conference matrix in order n and x, y real commuting variables. Then there is a complex orthogonal design $COD(n; 1, n-1)$.*

Proof. Let $Y = xI_n\mathbf{i} + yN$; then Y is easily proved to be the required COD. □

Lemma 3.2 below improves results of Theorem 2 given in [12].

Lemma 3.2. *Let N be a symmetric conference matrix in order n. Then there exist pairs of amicable orthogonal designs:*
a) $AOD(2n; n, n; n, n)$, *b)* $AOD(2n; n, n; 2, 2(n-1))$,
c) $AOD(2n; n, n; 1, n-1)$, *d)* $AOD(2n; 2, 2(n-1); 1, n-1)$.

Proof. Let a, b, c and d be real commuting variables. Then for a) the required designs are

$$\begin{bmatrix} aI_n + bN & bI_n - aN \\ bI_n - aN & -aI_n - bN \end{bmatrix} \text{ and } \begin{bmatrix} cI_n + dN & dI_n - cN \\ -dI_n + cN & cI_n + dN \end{bmatrix},$$

for b) they are

$$\begin{bmatrix} aI_n + bN & bI_n - aN \\ bI_n - aN & -aI_n - bN \end{bmatrix} \text{ and } \begin{bmatrix} cI_n + dN & cI_n - dN \\ -cI_n + dN & cI_n + dN \end{bmatrix},$$

for c) they are

$$\begin{bmatrix} aI_n + bN & bI_n - aN \\ bI_n - aN & -aI_n - bN \end{bmatrix} \text{ and } \begin{bmatrix} cI_n & dN \\ -dN & cI_n \end{bmatrix},$$

and for d) they are

$$\begin{bmatrix} aI_n + bN & aI_n - bN \\ aI_n - bN & -aI_n - bN \end{bmatrix} \text{ and } \begin{bmatrix} cI_n & dN \\ -dN & cI_n \end{bmatrix}.$$

□

Corollary 3.1. *Let n be the order of the symmetric conference matrices, where $n-1 \equiv 1 \pmod 4$ and $n-1$ is a prime power. Then the following amicable orthogonal designs of order $2n$ exist*
a)$AOD(2n; n, n; n, n)$, *b)$AOD(2n; n, n; 2, 2(n-1))$,*
c)$AOD(2n; n, n; 1, n-1)$, *d)$AOD(2n; 2, 2(n-1); 1, n-1)$.*

The following technique for constructing amicable orthogonal designs is from Theorem 1 in [12]. Here we review the original method. Let S be a circulant or type 1 $(0, 1, -1)$ matrix of order $p \equiv 3 \pmod 4$, p a prime power satisfying

$$S^T = -S, \quad SS^T = pI - J, \quad SJ = JS = 0,$$

where I is the identity matrix of order p and J the matrix of all ones. Further, let R be the back diagonal matrix of order p. Then, the matrices

$$A = \begin{bmatrix} a & b & \dots & b \\ -b & & & \\ \vdots & & aI + bS & \\ -b & & & \end{bmatrix} \text{ and } B = \begin{bmatrix} -c & d & \dots & d \\ d & & & \\ \vdots & & (cI + dS)R & \\ d & & & \end{bmatrix}$$

of order $p+1$ are defined, where a, b, c and d are real commuting variables. Then, A and B are amicable orthogonal designs of order $p+1$ and of type $(1, p; 1, p)$. Hence, we have the following lemma inferred directly from Theorem 1 in [12].

Lemma 3.3. *For $p \equiv 3 \pmod 4$ a prime power, there exists a pair of amicable orthogonal designs $AOD(p+1; 1, p; 1, p)$.*

Proof. This is an almost straightforward verification, since $aI + bS$ is type 1 and $(cI + dS)R$ is a type 2 matrix [7]. □

Remark 3.1. More amicable orthogonal designs constructed from p prime power other than $p \equiv 3 \pmod 4$ can be found in [10] by Seberry and Yamada. For example, there exists $AOD(2(p+1); 1, p; 1, p)$ for $p \equiv 1 \pmod 4$ a prime power.

Example 3.2. For $p = 3$, we define type 1 matrix $S = \begin{bmatrix} 0 & 1 & - \\ - & 0 & 1 \\ 1 & - & 0 \end{bmatrix}$ and the back diagonal matrix $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Then, we construct

$$A = \begin{bmatrix} a & b & b & b \\ -b & a & b & b \\ -b & -b & a & b \\ -b & b & -b & a \end{bmatrix} \text{ and } B = \begin{bmatrix} -c & d & d & d \\ d & c & -d & d \\ d & -d & d & c \\ d & d & c & -d \end{bmatrix}.$$

A and B is a pair of amicable orthogonal design $AOD(4; 1, 3; 1, 3)$.

3.2 Amicable orthogonal design of quaternions

Theorem 3.1. *If there exists a pair of amicable orthogonal designs of quaternions, $AODQ(n; a_1, \dots, a_s; b_1, \dots, b_t)$ and a pair of amicable orthogonal designs AOD $(m; c_1, \dots, c_u; d_1, \dots, d_v)$, then there exists a pair of amicable orthogonal designs of quaternions $AODQ(nm; b_1c_1, \dots, b_1c_{u-1}, a_1c_u, \dots, a_sc_u; b_1d_1, \dots, b_1d_v, b_2c_u, \dots, b_tc_u)$.*

Proof. Let $X = \sum_{i=1}^{s} A_i x_i$ and $Y = \sum_{j=1}^{t} B_j y_j$ be the amicable orthogonal designs of quaternions in order n and let $Z = \sum_{k=1}^{u} C_k z_k$ and $W = \sum_{l=1}^{v} D_l w_l$ are the amicable orthogonal designs in order m. Construct the matrices

$$P = \sum_{i=1}^{u-1} (B_1 \otimes C_i) p_i + \sum_{j=1}^{s} (A_j \otimes C_u) p_{j+u-1}$$

$$Q = \sum_{i=1}^{v} (B_1 \otimes D_i) q_i + \sum_{j=2}^{t} (B_j \otimes C_u) q_{j+v-1}$$

where the p_i's and q_i's are real commuting variables and $\otimes$ denotes the Kronecker product. □

The above theorem is similar to Wolfe's theorem [14] which gave a general construction method for amicable orthogonal designs. The only change in Theorem 3.1 is that X and Y are amicable orthogonal designs of quaternions (AODQ). It is important to note that Z and W must be amicable orthogonal designs over the **real** domain, otherwise the non-commutative property of quaternion can not guarantee the amicability of the results.

Example 3.3. Let $A = \begin{bmatrix} -x_1 & x_2\mathbf{i} \\ -x_2\mathbf{j} & x_1\mathbf{k} \end{bmatrix}$ and $B = \begin{bmatrix} y_1 & y_2\mathbf{i} \\ y_2\mathbf{j} & y_1\mathbf{k} \end{bmatrix}$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. A and B are amicable orthogonal designs of quaternions $AODQ(2; 1, 1; 1, 1)$. One pair of amicable orthogonal design is given as $Z = \begin{bmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{bmatrix}$ and $W = \begin{bmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{bmatrix}$, where $z_1, z_2, w_1, w_2 \in \mathbb{R}$. Theorem 3.1 gives

$$P = (B_1 \otimes C_1)p_1 + (A_1 \otimes C_2)p_2 + (A_2 \otimes C_2)p_3,$$

$$Q = (B_1 \otimes D_1)q_1 + (B_1 \otimes D_2)q_2 + (B_2 \otimes C_2)q_3.$$

Then, $P = \begin{bmatrix} p_1 & -p_2 & 0 & p_3\mathbf{i} \\ p_2 & p_1 & -p_3\mathbf{i} & 0 \\ 0 & -p_3\mathbf{j} & p_1\mathbf{k} & p_2\mathbf{k} \\ p_3\mathbf{j} & 0 & -p_2\mathbf{k} & p_1\mathbf{k} \end{bmatrix}$ and $Q = \begin{bmatrix} q_1 & q_2 & 0 & q_3\mathbf{i} \\ q_2 & -q_1 & -q_3\mathbf{i} & 0 \\ 0 & q_3\mathbf{j} & q_1\mathbf{k} & q_2\mathbf{k} \\ -q_3\mathbf{j} & 0 & q_2\mathbf{k} & -q_1\mathbf{k} \end{bmatrix}$ are amicable orthogonal designs of quaternions $AODQ(4; 1, 1, 1; 1, 1, 1)$ since they are both ODQs and satisfy $PQ^Q = QP^Q$.

Corollary 3.2. *If there exist a pair of amicable orthogonal designs of quaternions $AODQ(n; a_1, \dots, a_s; b_1, \dots, b_t)$, then there exists a pair of amicable orthogonal designs of quaternions of type*
a) $AODQ(2n; a_1, a_1, 2a_2 \dots, 2a_s; 2b_1, \dots, 2b_t)$,
b) $AODQ(2n; a_1, a_1, a_2 \dots, a_s; b_1, \dots, b_t)$,
c) $AODQ(2n; 2a_1, 2a_2 \dots, 2a_s; 2b_1, 2b_2 \dots, 2b_t)$.

Proof. Let $X = \sum_{i=1}^{s} A_i x_i$ and $Y = \sum_{j=1}^{t} B_j y_j$ be the amicable designs of quaternions in order n.
a)Let $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ be **real** weighing matrices and construct the matrices

$$P = (A_1 \otimes I_2)p_1 + (A_1 \otimes M)p_2 + \sum_{i=2}^{s}(A_i \otimes N)p_{i+1}, \quad Q = \sum_{j=1}^{t}(B_j \otimes N)q_j$$

b)Same as a), only set $N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

c)Let $C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and construct the matrices

$$P = \sum_{i=1}^{s}(A_i \otimes C)p_i, \quad Q = \sum_{j=1}^{t}(B_j \otimes C)q_j$$

It is obvious that all the quaternion matrices P_i's and Q_i's satisfy the conditions (i)–(iv) in (1) because the weighing matrices M, N and C have the following properties: $M = -M^T$, $N = N^T$; $C = C^T$ and $MN^T = NM^T$, where $(.)^T$ denotes matrix transpose. □

Example 3.4. Consider a pair of $AODQ(2;1,1;1,1)$ given in Example 2.6, and let us construct $AODQ(4;1,1,2;2,2)$ using Corollary 3.2(a):

$$P = \begin{bmatrix} -p_1 & -p_2 & p_3\mathbf{i} & p_3\mathbf{i} \\ p_2 & -p_1 & p_3\mathbf{i} & -p_3\mathbf{i} \\ -p_3\mathbf{j} & -p_3\mathbf{j} & p_1\mathbf{k} & p_2\mathbf{k} \\ -p_3\mathbf{j} & p_3\mathbf{j} & -p_2\mathbf{k} & p_1\mathbf{k} \end{bmatrix} \quad Q = \begin{bmatrix} q_1 & q_1 & q_2\mathbf{i} & q_2\mathbf{i} \\ q_1 & -q_1 & q_2\mathbf{i} & -q_2\mathbf{i} \\ q_2\mathbf{j} & q_2\mathbf{j} & q_1\mathbf{k} & q_1\mathbf{k} \\ q_2\mathbf{j} & -q_2\mathbf{j} & q_1\mathbf{k} & -q_1\mathbf{k} \end{bmatrix}$$

In Theorem 3.1, we can also replace the amicable orthogonal designs $AOD(m;c_1,\dots,c_u;d_1,\dots,d_v)$ by an amicable family to obtain more amicable orthogonal designs of quaternions.

Example 3.5. Consider a pair of $AODQ(2;1,1;1,1)$ given in Example 2.6, and let $C_1 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $D_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ be an amicable family $\{C_1,C_2;D_1,D_2\}$. We construct

$$P = (B_1 \otimes C_1)p_1 + (A_1 \otimes C_2)p_2 + (A_2 \otimes C_2)p_3,$$

$$Q = (B_1 \otimes D_1)q_1 + (B_1 \otimes D_2)q_2 + (B_2 \otimes C_2)q_3.$$

The new amicable orthogonal designs of quaternions are:

$$P = \begin{bmatrix} -p_1 - p_2 & p_1 - p_2 & p_3\mathbf{i} & p_3\mathbf{i} \\ p_1 - p_2 & p_1 + p_2 & p_3\mathbf{i} & -p_3\mathbf{i} \\ -p_3\mathbf{j} & -p_3\mathbf{j} & -p_1\mathbf{k} + p_2\mathbf{k} & p_1\mathbf{k} + p_2\mathbf{k} \\ -p_3\mathbf{j} & p_3\mathbf{j} & p_1\mathbf{k} + p_2\mathbf{k} & p_1\mathbf{k} - p_2\mathbf{k} \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 + q_2 & -q_1 + q_2 & q_3\mathbf{i} & q_3\mathbf{i} \\ q_1 - q_2 & q_1 + q_2 & q_3\mathbf{i} & -q_3\mathbf{i} \\ q_3\mathbf{j} & q_3\mathbf{j} & q_1\mathbf{k} + q_2\mathbf{k} & -q_1\mathbf{k} + q_2\mathbf{k} \\ q_3\mathbf{j} & -q_3\mathbf{j} & q_1\mathbf{k} - q_2\mathbf{k} & q_1\mathbf{k} + q_2\mathbf{k} \end{bmatrix}.$$

In this design, some entries are linear combinations of two variables which may make it unsuitable for real applications in communications. To normalize the above design, we set new variables $a_1 = p_1 + p_2$, $a_2 = p_1 - p_2$, $a_3 = p_3$, and $b_1 = q_1 + q_2$, $b_2 = q_1 - q_2$, $b_3 = q_3$, and then we obtain

$$P = \begin{bmatrix} -a_1 & a_2 & a_3\mathbf{i} & a_3\mathbf{i} \\ a_2 & a_1 & a_3\mathbf{i} & -a_3\mathbf{i} \\ -a_3\mathbf{j} & -a_3\mathbf{j} & -a_2\mathbf{k} & a_1\mathbf{k} \\ -a_3\mathbf{j} & a_3\mathbf{j} & a_1\mathbf{k} & a_2\mathbf{k} \end{bmatrix} \quad Q = \begin{bmatrix} b_1 & -b_2 & b_3\mathbf{i} & b_3\mathbf{i} \\ b_2 & b_1 & b_3\mathbf{i} & -b_3\mathbf{i} \\ b_3\mathbf{j} & b_3\mathbf{j} & b_1\mathbf{k} & -b_2\mathbf{k} \\ b_3\mathbf{j} & -b_3\mathbf{j} & b_2\mathbf{k} & b_1\mathbf{k} \end{bmatrix}.$$

This is an $AODQ(4;1,1,2;1,1,2)$ design without zero entries and with no linear processing.

In [15], Yuen, Guan and Tjhung offered a construction method for amicable complex orthogonal designs. We can also apply it in constructing amicable orthogonal designs of quaternions.

Lemma 3.4. *If there exists a pair of amicable orthogonal designs of quaternions $AODQ(n; a_1, \ldots, a_s; b_1, \ldots, b_t)$, then there exists a pair of amicable orthogonal designs of quaternions of type $AODQ(4n; a_1, a_1, a_1, b_2, \ldots, b_t; b_1, b_1, b_1, a_2, \ldots, a_s)$.*

Proof. Let $X = \sum_{i=1}^{s} A_i x_i$ and $Y = \sum_{j=1}^{t} B_j y_j$ be the amicable orthogonal designs of quaternions in order n and let us define following **real** weighing matrices:

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Let us construct the matrices

$$P = \sum_{i=1}^{3} (A_1 \otimes N_i) p_i + \sum_{j=2}^{t} (B_j \otimes I_4) p_{2+j}$$

$$Q = \sum_{i=1}^{3} (B_1 \otimes M_i) q_i + \sum_{j=2}^{s} (A_j \otimes I_4) q_{2+j}.$$

Matrices P_i's and Q_i's satisfy the conditions (i)-(iv) in (1) because the weighing matrices $\{M_i\}$ and $\{N_i\}$ are skew-symmetric and they also form an amicable family. □

Example 3.6. Let us consider a pair of $AODQ(2; 1, 1; 1, 1)$ given in Example 2.6, we apply Lemma 3.4 to construct the following $AODQ(8; 1, 1, 1, 1; 1, 1, 1, 1)$:

$$P = \begin{bmatrix} 0 & -p_1 & -p_2 & -p_3 & p_4\mathbf{i} & 0 & 0 & 0 \\ p_1 & 0 & p_3 & -p_2 & 0 & p_4\mathbf{i} & 0 & 0 \\ p_2 & -p_3 & 0 & p_1 & 0 & 0 & p_4\mathbf{i} & 0 \\ p_3 & p_2 & -p_1 & 0 & 0 & 0 & 0 & p_4\mathbf{i} \\ p_4\mathbf{j} & 0 & 0 & 0 & 0 & p_1\mathbf{k} & p_2\mathbf{k} & p_3\mathbf{k} \\ 0 & p_4\mathbf{j} & 0 & 0 & -p_1\mathbf{k} & 0 & -p_3\mathbf{k} & p_2\mathbf{k} \\ 0 & 0 & p_4\mathbf{j} & 0 & -p_2\mathbf{k} & p_3\mathbf{k} & 0 & -p_1\mathbf{k} \\ 0 & 0 & 0 & p_4\mathbf{j} & -p_3\mathbf{k} & -p_2\mathbf{k} & p_1\mathbf{k} & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & q_1 & q_2 & q_3 & q_4\mathbf{i} & 0 & 0 & 0 \\ -q_1 & 0 & q_3 & -q_2 & 0 & q_4\mathbf{i} & 0 & 0 \\ -q_2 & -q_3 & 0 & q_1 & 0 & 0 & q_4\mathbf{i} & 0 \\ -q_3 & q_2 & -q_1 & 0 & 0 & 0 & 0 & q_4\mathbf{i} \\ -q_4\mathbf{j} & 0 & 0 & 0 & 0 & q_1\mathbf{k} & q_2\mathbf{k} & q_3\mathbf{k} \\ 0 & -q_4\mathbf{j} & 0 & 0 & -q_1\mathbf{k} & 0 & q_3\mathbf{k} & -q_2\mathbf{k} \\ 0 & 0 & -q_4\mathbf{j} & 0 & -q_2\mathbf{k} & -q_3\mathbf{k} & 0 & q_1\mathbf{k} \\ 0 & 0 & 0 & -q_4\mathbf{j} & -q_3\mathbf{k} & q_2\mathbf{k} & -q_1\mathbf{k} & 0 \end{bmatrix}.$$

Although we only give examples of $AODQ$ of orders 2, 4 and 8 in this paper, there are actually many designs of orders other than the power of 2. We know that symmetric conference matrices exist for orders $n = q + 1$, $q \equiv 1 \pmod 4$ a prime power, e.g. $n = 6$. Applying Theorem 3.1 on $AODQ(2;1,1;1,1)$ and amicable orthogonal designs constructed from Corollary 3.1 gives us the following corollary.

Corollary 3.3. *Let $n \equiv 2 \pmod 4$ be the order of the symmetric conference matrices; then there exist*
a)$AODQ(4n;n,n,n;n,n,n)$, *b)$AODQ(4n;n,n,n;2,2(n-1),n)$,*
c)$AODQ(4n;n,n,n;1,n-1,n)$, *d)$AODQ(4n;2,2(n-1),2(n-1);1,n-1,2(n-1))$.*

Examples are $AODQ(24;6,6,6;6,6,6)$, $AODQ(24;6,6,6;2,10,6)$ for $n = 6$.

Corollary 3.4. *For $q \equiv 3 \pmod 4$ a prime power, there exists $AODQ(2(q+1);1,q,q;1,q,q)$.*

Proof. This corollary follows by applying Theorem 3.1 on $AODQ(2;1,1;1,1)$ and amicable orthogonal designs constructed using Lemma 3.3. □

The above corollary also gives an example of $AODQ(24;1,11,11;1,11,11)$ when $q = 11$.

4 Combined designs from amicable AODQs

In [9], Seberry et al. propose a technique named *combined quaternion orthogonal designs* from real and complex orthogonal designs. This combined design uses the property that if AB^H is a symmetric matrix, where A and B are matrices with complex entries, so that $AB^H\mathbf{q} = \mathbf{q}BA^H$ for $\mathbf{q} \in \{\pm\mathbf{j}, \pm\mathbf{k}\}$, then this will construct a new $RQOD$. If A and B are complex orthogonal designs with elements from complex commuting variables, and if $A^{\mathcal{H}}B$ is symmetric, then $A + B\mathbf{j}$ is a restricted quaternion orthogonal design. We say that $A + B\mathbf{j}$ is a *combined design* [9].
There is a connection between combined designs and amicable orthogonal designs, in that the form of $AB^{\mathcal{H}}$ is examined. For amicable orthogonal designs of quaternions, the condition that AB^Q is a symmetric matrix can be relaxed, since we have $AB^Q = BA^Q$ for A and B. In the case of combined designs from amicable orthogonal designs of quaternions, we also need to be careful about what quaternion appears as entries of AB^Q. We illustrate this with the following example:

Example 4.1. Consider the $AODQ(2;1,1;1,1)$ designs A and B from Example 2.3. We have

$$A^Q B = \begin{bmatrix} -x_1 & x_2\mathbf{j} \\ -x_2\mathbf{i} & -x_1\mathbf{k} \end{bmatrix} \begin{bmatrix} y_1 & y_2\mathbf{i} \\ y_2\mathbf{j} & y_1\mathbf{k} \end{bmatrix} = B^Q A.$$

Let $D = A + B\mathbf{q}$, $\mathbf{q} \in \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ be a new design, for which we have

$$\begin{aligned} D^Q D &= (A^Q - \mathbf{q}B^Q)(A + B\mathbf{q}) \\ &= A^Q A + A^Q B\mathbf{q} - \mathbf{q}B^Q A - \mathbf{q}B^Q B\mathbf{q} \\ &= (A^Q A + B^Q B) + (A^Q B)\mathbf{q} - \mathbf{q}(B^Q A), \end{aligned}$$

where $A^Q B = B^Q A$ for the amicability of A and B. We also notice that all entries in $A^Q B$ are either real or products with quaternion $\mathbf{i}$. Thus $A^Q B\mathbf{i} = \mathbf{i}B^Q A$, and we have $D^Q D = A^Q A + B^Q B = (x_1^2 + x_2^2 + y_1^2 + y_2^2)I_2$. The new design $D = A + B\mathbf{i}$ is of the form:

$$D = \begin{bmatrix} -x_1 + y_1\mathbf{i} & x_2\mathbf{i} - y_2 \\ -x_2\mathbf{j} - y_2\mathbf{k} & x_1\mathbf{k} + y_1\mathbf{j} \end{bmatrix}.$$

Let complex symbols $z_i = x_i + \mathbf{i}y_i$, for $1 \le i \le 2$, then we can rewrite D above as

$$D = \begin{bmatrix} -z_1^* & \mathbf{i}z_2 \\ -\mathbf{j}z_2^* & \mathbf{k}z_1 \end{bmatrix}.$$

The above design satisfies $D^Q D = (|z_1|^2 + |z_2|^2)I_2$ and is thus an $RQOD(2;1,1)$ on complex variables z_1 and z_2. The new RQOD in Example 4.1 has no zero entries, which may have practical advantages when used in wireless communication, since there is no need to switch antennas off and back on during transmission.

We now provide another example of $RQOD$ with order 4.

Example 4.2. Consider the $AODQ(4;1,1,2;1,1,2)$ designs A and B in Example 3.3 with variables a_1, a_2, a_3 and $b_1, b_2, b_3 \in \mathbb{R}$. We have $X = A^Q B$

$$\begin{aligned} &= \begin{bmatrix} -a_1 & a_2 & a_3\mathbf{j} & a_3\mathbf{j} \\ a_2 & a_1 & a_3\mathbf{j} & -a_3\mathbf{j} \\ -a_3\mathbf{i} & -a_3\mathbf{i} & a_2\mathbf{k} & -a_1\mathbf{k} \\ -a_3\mathbf{i} & a_3\mathbf{i} & -a_1\mathbf{k} & -a_2\mathbf{k} \end{bmatrix} \begin{bmatrix} b_1 & -b_2 & b_3\mathbf{i} & b_3\mathbf{i} \\ b_2 & b_1 & b_3\mathbf{i} & -b_3\mathbf{i} \\ b_3\mathbf{j} & b_3\mathbf{j} & b_1\mathbf{k} & -b_2\mathbf{k} \\ b_3\mathbf{j} & -b_3\mathbf{j} & b_2\mathbf{k} & b_1\mathbf{k} \end{bmatrix} \\ &= \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{12}^Q & X_{22} & X_{23} & X_{24} \\ X_{13}^Q & X_{23}^Q & X_{33} & X_{34} \\ X_{14}^Q & X_{24}^Q & X_{34}^Q & X_{44} \end{bmatrix} \\ &= B^Q A, \end{aligned}$$

where $X_{11} = -a_1b_1 + a_2b_2 - 2a_3b_3$, $X_{12} = a_1b_2 + a_2b_1$, $X_{13} = (-a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2)\mathbf{i}$, $X_{14} = (-a_1b_3 - a_2b_3 + a_3b_1 - a_3b_2)\mathbf{i}$, $X_{22} = a_1b_1 - a_2b_2 - 2a_3b_3$, $X_{23} = (a_1b_3 + a_2b_3 + a_3b_1 - a_3b_2)\mathbf{i}$, $X_{24} = (-a_1b_3 + a_2b_3 - a_3b_1 - a_3b_2)\mathbf{i}$, $X_{33} = a_1b_2 - a_2b_1 + 2a_3b_3$, $X_{34} = a_1b_1 + a_2b_2$ and $X_{44} = -a_1b_2 + a_2b_1 + 2a_3b_3$. Since only quaternion $\mathbf{i}$ appears in X, we then set $D = A + B\mathbf{i}$ as the new design:

$$D = \begin{bmatrix} -a_1 + b_1\mathbf{i} & a_2 - b_2\mathbf{i} & a_3\mathbf{i} - b_3 & a_3\mathbf{i} - b_3 \\ a_2 + b_2\mathbf{i} & a_1 + b_1\mathbf{i} & a_3\mathbf{i} - b_3 & -a_3\mathbf{i} + b_3 \\ -a_3\mathbf{j} - b_3\mathbf{k} & -a_3\mathbf{j} - b_3\mathbf{k} & -a_2\mathbf{k} + b_1\mathbf{j} & a_1\mathbf{k} - b_2\mathbf{j} \\ -a_3\mathbf{j} - b_3\mathbf{k} & a_3\mathbf{j} + b_3\mathbf{k} & a_1\mathbf{k} + b_2\mathbf{j} & a_2\mathbf{k} + b_1\mathbf{j} \end{bmatrix}.$$

Let complex symbols $z_i = a_i + \mathbf{i}b_i$, for $1 \le i \le 3$, then we can write D above as

$$D = \begin{bmatrix} -z_1^* & z_2^* & \mathbf{i}z_3 & \mathbf{i}z_3 \\ z_2 & z_1 & \mathbf{i}z_3 & -\mathbf{i}z_3 \\ -\mathbf{j}z_3^* & -\mathbf{j}z_3^* & -\mathbf{k}(a_2 - b_1\mathbf{i}) & \mathbf{k}(a_1 - b_2\mathbf{i}) \\ -\mathbf{j}z_3^* & \mathbf{j}z_3^* & \mathbf{k}(a_1 + b_2\mathbf{i}) & \mathbf{k}(a_2 + b_1\mathbf{i}) \end{bmatrix}.$$

This design D satisfies $D^Q D = (|z_1|^2+|z_2|^2+2|z_3|^2)I_4$ and is thus an $RQOD(4;1,1,2)$ on complex variables z_1, z_2 and z_3. Note that if an entry in the orthogonal design is a linear combination of variables from the given domain, the design is said to be **with linear processing**. Obviously, the new $RQOD$ design has the property of no zero entries but with linear processing on some entries, i.e the position (3,3) is the quaternion combination of the real part of symbol z_2 and the imaginary part of symbol z_1.

The following Lemma shows the construction of orthogonal designs of quaternions (ODQ) by using symmetric conference matrices.

Lemma 4.1. *Suppose a, b, c, d are real commuting variables. Let N be a symmetric conference matrix of order n and I the identity matrix of the same order. Then, $X = aI\mathbf{i}+bN$ and $Y = cI\mathbf{j}+dN\mathbf{k}$ are orthogonal designs of quaternions $ODQ(n;1,n-1)$, and $XY^Q+YX^Q=0$, so X and Y are $AAODQ(n;1,n-1;1,n-1)$ (anti-amicable orthogonal design of quaternions). Hence $\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}$ is an $ODQ(2n;1,1,n-1,n-1)$.*

The proof for Lemma 4.1 is straightforward.

Example 4.3. From the symmetric conference matrix N given in Example 3.1 of order 6, we construct the following matrices according to Lemma 4.1:

$$X = \begin{bmatrix} a\mathbf{i} & b & b & b & b & b \\ b & a\mathbf{i} & b & -b & -b & b \\ b & b & a\mathbf{i} & b & -b & -b \\ b & -b & b & a\mathbf{i} & b & -b \\ b & -b & -b & b & a\mathbf{i} & b \\ b & b & -b & -b & b & a\mathbf{i} \end{bmatrix} \quad Y = \begin{bmatrix} c\mathbf{j} & d\mathbf{k} & d\mathbf{k} & d\mathbf{k} & d\mathbf{k} & d\mathbf{k} \\ d\mathbf{k} & c\mathbf{j} & d\mathbf{k} & -d\mathbf{k} & -d\mathbf{k} & d\mathbf{k} \\ d\mathbf{k} & d\mathbf{k} & c\mathbf{j} & d\mathbf{k} & -d\mathbf{k} & -d\mathbf{k} \\ d\mathbf{k} & -d\mathbf{k} & d\mathbf{k} & c\mathbf{j} & d\mathbf{k} & -d\mathbf{k} \\ d\mathbf{k} & -d\mathbf{k} & -d\mathbf{k} & d\mathbf{k} & c\mathbf{j} & d\mathbf{k} \\ d\mathbf{k} & d\mathbf{k} & -d\mathbf{k} & -d\mathbf{k} & d\mathbf{k} & c\mathbf{j} \end{bmatrix}.$$

X and Y are both $ODQ(6;1,5)$, and they also form a pair of $AAODQ(6;1,5;1,5)$.

Corollary 4.1. *Let $p \equiv 1 \pmod 4$ be a prime power. Then there exist orthogonal designs of quaternions $ODQ(p+1;1,p)$ and $ODQ(2(p+1);1,p,1,p)$, and also a pair of anti-amicable orthogonal designs of quaternions $AAODQ(p+1;1,p;1,p)$.*

Corollary 4.1 follows on directly from Lemma 4.1.

Lemma 4.2. For a pair of $AODQ(n;1,n-1;1,n-1)$ X and Y given in Lemma 4.1, then $D = X+Y\mathbf{i}$ is an $RQOD(n;1,n-1)$.

Proof. We have

$$\begin{aligned} D^Q D &= (X^Q - \mathbf{i}Y^Q)(X+Y\mathbf{i}) \\ &= X^Q X + X^Q Y\mathbf{i} - \mathbf{i}Y^Q X - \mathbf{i}Y^Q Y\mathbf{i} \\ &= (X^Q X + Y^Q Y) + (X^Q Y)\mathbf{i} - \mathbf{i}(Y^Q X). \end{aligned}$$

For $X = xI\mathbf{i} + bN$ and $Y = cI\mathbf{j} + dN\mathbf{k}$, where N is a symmetric conference matrix of order n and I is the identity matrix with the same order, we have

$$\begin{aligned} X^QY &= (-aI\mathbf{i} + bN^T)(cI\mathbf{j} + dN\mathbf{k}) \\ &= -acI\mathbf{k} + adN\mathbf{j} + bcN\mathbf{j} + bdNN^T\mathbf{k} \\ &= -Y^QX, \end{aligned}$$

since only quaternions $\mathbf{k}$ and $\mathbf{j}$ appear in X^QY, we thus have $(X^QY)\mathbf{i} = \mathbf{i}(Y^QX)$. Hence, $D^QD = X^QX + Y^QY = (a^2 + (n-1)b^2 + c^2 + (n-1)d^2)I_n$, i.e. D is an $RQOD(n; 1, n-1)$.

□

Example 4.4. Consider a pair of $AODQ(6; 1, 5; 1, 5)$ given in Example 4.3, and we have the following $D = X + Y\mathbf{i}$:

$$\begin{bmatrix} \mathbf{i}(a-c\mathbf{j}) & b+d\mathbf{j} & b+d\mathbf{j} & b+d\mathbf{j} & b+d\mathbf{j} & b+d\mathbf{j} \\ b+d\mathbf{j} & \mathbf{i}(a-c\mathbf{j}) & b+d\mathbf{j} & -(b+d\mathbf{j}) & -(b+d\mathbf{j}) & b+d\mathbf{j} \\ b+d\mathbf{j} & b+d\mathbf{j} & \mathbf{i}(a-c\mathbf{j}) & b+d\mathbf{j} & -(b+d\mathbf{j}) & -(b+d\mathbf{j}) \\ b+d\mathbf{j} & -(b+d\mathbf{j}) & b+d\mathbf{j} & \mathbf{i}(a-c\mathbf{j}) & b+d\mathbf{j} & -(b+d\mathbf{j}) \\ b+d\mathbf{j} & -(b+d\mathbf{j}) & -(b+d\mathbf{j}) & b+d\mathbf{j} & \mathbf{i}(a-c\mathbf{j}) & b+d\mathbf{j} \\ b+d\mathbf{j} & b+d\mathbf{j} & -(b+d\mathbf{j}) & -(b+d\mathbf{j}) & b+d\mathbf{j} & \mathbf{i}(a-c\mathbf{j}) \end{bmatrix}.$$

In design D above, if we replace quaternion element $\mathbf{j}$ with $\mathbf{i}$, $\mathbf{i}$ with an undecided quaternion element $\mathbf{q}$, and let complex variables $z_1 = a + c\mathbf{i}$ and $z_2 = b + d\mathbf{i}$, then we can rewrite D:

$$\begin{bmatrix} \mathbf{q}z_1^* & z_2 & z_2 & z_2 & z_2 & z_2 \\ z_2 & \mathbf{q}z_1^* & z_2 & -z_2 & -z_2 & z_2 \\ z_2 & z_2 & \mathbf{q}z_1^* & z_2 & -z_2 & -z_2 \\ z_2 & -z_2 & z_2 & \mathbf{q}z_1^* & z_2 & -z_2 \\ z_2 & -z_2 & -z_2 & z_2 & \mathbf{q}z_1^* & z_2 \\ z_2 & z_2 & -z_2 & -z_2 & z_2 & \mathbf{q}z_1^* \end{bmatrix}.$$

$\mathbf{q}$ in D above can be chosen from the set $\{\pm\mathbf{k}, \pm\mathbf{j}\}$ since $\mathbf{q}z_1^*z_2^* = z_2z_1\mathbf{q}$ for any $\mathbf{q} \in \{\pm\mathbf{k}, \pm\mathbf{j}\}$. It is easy to prove $D^QD = (|z_1|^2 + 5|z_2|^2)I_6$. Hence, D is a restricted quaternion orthogonal design $RQOD(6; 1, 5)$ with no zero entries.

5 Conclusion and future work

In this paper, we have introduced some methods for building amicable orthogonal designs over the real and quaternion domain, e.g. a method of constructing amicable orthogonal designs of quaternions ($AODQ$) by using the Kronecker product with **real** amicable orthogonal designs or **real** weighing matrices from an amicable family. This construction ensures that, for any existing **real** amicable orthogonal design generated by using the Kronecker product, we can easily find an $AODQ$ with the same order and type. We have also shown that if A and B form a pair of $AODQ$, then the combined design $A + B\mathbf{q}$ for $\mathbf{q} \in \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$is an $RQOD$ by carefully choosing $\mathbf{q}$.

Our newly constructed *AODQs* and *RQODs*, especially those with no zero entries, could have applications as space-time-polarization block codes.
However, there are still some problems that need to be solved, e.g. do there exist any new amicable orthogonal designs of quaternions for which there are no such real or complex designs? Another problem is to determine the maximum number of variables in an *AODQ*. It is known that finding the maximum number of variables in an *AOD* is equivalent to finding the number of members in a *Hurwitz-Radon family* of corresponding type [7], which also implies that the so-called Clifford algebras [2] have a matrix representation of the same order. In other words, how can we find a set of anti-commuting real, complex and quaternion matrix representation to determine the maximum number of variables in an *AODQ*? We will address these problems in a future study.

References

[1] A. R. Calderbank, S. Das, N. Al-Dhahir, and S. N. Diggavi, Construction and analysis of a new quaternionic space-time code for 4 transmit antennas, *Communications in Information and Systems* 5(1) (2005), 1–26.

[2] W. K. Clifford, Applications of Grassman's extensive algebra, *Amer. J. Math.* 5 (1878), 350–358.

[3] P. Delsarte, J. M. Goethals and J. J. Seidel, Orthogonal matrices with zero diagonal, II, *Canad. J. Math.* 23 (1971), 816–832.

[4] K. Finlayson, J. Seberry, T. A. Wysocki and T. Xia, Orthogonal designs with quaternion elements, In *Proc. 8th International Symposium on Communication Theory and Applications, ISCTA'05*, pp. 270–272, Ambleside, UK, July 2005.

[5] A. V. Geramita and J. M. Geramita, Complex orthogonal designs, *J. Combin. Theory Ser. A* 25 (1978), 211–225.

[6] A. V. Geramita, J. M. Geramita, and J. Seberry Wallis, Orthogonal designs, *Linear and Multilinear Algebra*, 3 (1976), 281–306.

[7] A. V. Geramita and J. Seberry, Orthogonal Designs,Quadratic Forms and Hadamard Matrices, *Lec. Notes Pure Appl. Math.* vol.43, New York and Basel, Marcel Dekker, 1979.

[8] O. M. Isaeva and V. A. Sarytchev, Quaternion presentations polarization state, In *Proc. 2nd IEEE Topical Symposium of Combined Optical-Microwave Earth and Atmosphere Sensing*, pp. 195–196, Atlanta, GA, USA, April 1995.

[9] J. Seberry, K. Finlayson, Sarah A. Spence, T. A. Wysocki and T. Xia, The theory of orthogonal designs over the quaternion domain. Under review.

[10] J. Seberry and M. Yamada, Hadamard matrices, Sequences, and Block Designs. In *Contemporary Design Theory—A Collection of Surveys*, pp. 431–560. Eds J.H. Dinitz and D.R. Stinson, John Wiley and Sons, New York, 1992.

[11] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, Space-time block codes from orthogonal designs, *IEEE Trans. Inform. Theory*, 45(5) (1999), 1456–1467.

[12] J. S. Wallis, Constructions for amicable orthogonal designs, *Bull. Austral. Math. Soc.* 12 (1975), 179–182.

[13] W. D. Wallis, A. P. Street and J. S. Wallis, Combinatorics: Room squares, sum-free sets, Hadamard matrices. In *Lec. Notes Math.* 292, Springer-Verlag, New York, 1972.

[14] W. W. Wolfe, Orthogonal designs-amicable orthogonal designs-some algebraic and combinatorial techniques, *Ph.D. Dissertation*, Queen's University, Kingston, Ontario, 1975.

[15] C. Yuen, Y. L. Guan, and T. T. Tjhung, Amicable complex orthogonal designs. Under review.

(Received 27 Feb 2007; revised 9 Mar 2008, 21 Jan 2009)