Filtered inclusions of Path Algebras;
a combinatorial approach to Doplicher-Roberts duality

by

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Research supported by ARC Grant No. A69030956

§0 Introduction

This paper grew out of an attempt to understand the Doplicher-Roberts duality theory for compact groups, [DR1, DR2] at a combinatorial level. The fundamental object of their theory is a pair $(\mathcal{O}_\rho, \sigma)$, where $\mathcal{O}_\rho$ is a C^*-algebra canonically associated to a finite dimensional special unitary representation ρ of a compact group G on V_ρ, and σ is a very particular endomorphism of $\mathcal{O}_\rho$; they provide a mechanism for recovering $\rho(G)$ from $(\mathcal{O}_\rho, \sigma)$ in the case where $\rho(G)$ is finite or contained in $SU(V_p)$.

The point of view taken here is that $\mathcal{O}_\rho$ is (the completion of) the **path algebra** of a locally finite, pointed, directed, graph $\mathcal{G}_\rho$ canonically associated to ρ. In general, if $\mathcal{G}$ is such a graph, we may consider the **path algebra** $\mathcal{P}(\mathcal{G})$. This is the algebra spanned linearly by symbols $e(\alpha, \alpha')$, where α, β are finite paths in $\mathcal{G}$, with a common end point (but not necessarily equal lengths $|\alpha|, |\alpha'|$) emanating from the base point $\star$; the $e(\alpha, \alpha')$ are subject to the rules

$$(0.1) \qquad \begin{cases} e(\alpha, \alpha') = \sum_x e(\alpha x, \alpha' x), \quad e(\alpha, \alpha')^* = e(\alpha', \alpha) \\ e(\alpha, \alpha')\, e(\beta', \beta) = \delta_{\alpha', \beta'}\, e(\alpha, \beta), \end{cases}$$

where the sum extends over those edges x emanating from the endpoint of α (or α'), and $|\alpha'| = |\beta'|$ in the formula for the product. (See §1 for a complete description of $\mathcal{O}_\rho$ and the associated graph $\mathcal{G}_\rho$). It turns out that many other classes of C^*-algebras, such as the A.F. algebras of [B] (see also [S] and [GHJ,Chapter 2]), the Cuntz algebras $\mathcal{O}_d$ of [C], and the Cuntz-Krieger algebras $\mathcal{O}_A$ of [CK] are also completions of $\mathcal{P}(\mathcal{G})$ for suitable $\mathcal{G}$.

The algebras $\mathcal{P}(\mathcal{G})$ are filtered in the sense that if $\mathcal{P}_{m,n}(\mathcal{G}) = span\{e(\alpha, \alpha') : |\alpha| = m, |\alpha'| = n\}$, then $\mathcal{P}_{m,n}(\mathcal{G}) \subseteq \mathcal{P}_{m+1,n+1}(\mathcal{G})$ by (0.1), and $\mathcal{P}(\mathcal{G}) = span \bigcup_{m,n} \mathcal{P}_{m,n}(\mathcal{G})$. Furthermore, it turns out that endomorphism σ above is filtered in the sense that $\sigma(\mathcal{P}_{m,n}(\mathcal{G})) \subseteq \mathcal{P}_{m+1,n+1}(\mathcal{G})$.

The primary object of this note is to develop combinatorial methods for describing filtered inclusions of path algebras i.e. unital *-monomorphisms $\sigma : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{H})$ with $\sigma(\mathcal{P}_{m,n}(\mathcal{G})) \subseteq \mathcal{P}_{m,n}(\mathcal{H})$ for all $m, n \geq 0$ – the example of the Doplicher-Roberts endomorphism will be accommodated by introducing a minor modification $\overline{\mathcal{G}}_\rho$ of $\mathcal{G}_\rho$ with $\mathcal{P}_{m,n}(\mathcal{G}_\rho) = P_{m+1,n+1}(\overline{\mathcal{G}}_\rho)$. Recall that Ocneanu, [01,02], has given such a description of filtered inclusions between the 0-graded algebras $\mathcal{P}^0(\mathcal{G}) = span\{e(\alpha, \alpha') : |\alpha| = |\alpha'|\}$ and $\mathcal{P}^0(\mathcal{H})$, in terms of a "connection" $(\mathcal{T}, W)$ between $\mathcal{G}$ and $\mathcal{H}$, i.e. a directed bipartite graph $\mathcal{T}$ whose edges join vertices in $\mathcal{G}$ to vertices in $\mathcal{H}$, together with a weighting W of "cells" in $(\mathcal{G}, \mathcal{T}, \mathcal{H})$. Thus in the general case, a filtered inclusion $\sigma : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{H})$ determines a filtered inclusion $\sigma^0 : \mathcal{P}^0(\mathcal{G}) \to \mathcal{P}^0(\mathcal{H})$, and hence a connection $(\mathcal{T}, W)$. This connection in turn determines a filtered inclusion $\overline{\sigma}$ of $\mathcal{P}(\mathcal{G})$ in $\mathcal{P}(\mathcal{H})$; the only extra ingredient required is a "phase" λ which measures the twisting of σ relative to $\overline{\sigma}$.

The paper is organized as follows. In §1 we formulate the path algebras precisely, and sketch two fundamental examples ($\mathcal{O}_\rho$ and $\mathcal{O}_A$). We develop the general procedure for describing filtered inclusions in §2, and compute the description for the inclusions $\mathcal{O}_\rho \subseteq \mathcal{O}_d$ and $\sigma : \mathcal{O}_\rho \to \mathcal{O}_\rho$ in §3. This last computation provides a combinatorial version of the Doplicher-Roberts Duality Theorem for compact groups. Finally, in an appendix, we explicitly calculate the connection for a specific example.

§1 Path algebras

As in the introduction, we let $\mathcal{G}$ denote a locally finite, directed, pointed graph with marked point $\star$, vertices $V = V(\mathcal{G})$, edges $E = E(\mathcal{G})$, and denote the range and source of $x \in E$ by $r(x)$ and $s(x)$ respectively. Throughout, we assume that s is onto (so there are no "back holes") and that $V = \bigcup_{k\geq 0} (r \circ s^{-1})^k(\star)$ so that all vertices are accessible from $\star$. A (finite) path in $(\mathcal{G})$ is a sequence $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ of edges in $\mathcal{G}$ with $s(\alpha_1) = \star$ and $s(\alpha_{j+1}) = r(\alpha_j)$ for $1 \leq j \leq k-1$; we write $s(\alpha) = s(\alpha_1)$, $r(\alpha) = r(\alpha_k)$ and refer to k as the length $|\alpha|$ of α. The set of all finite paths in $\mathcal{G}$ is denoted $\Omega = \Omega(\mathcal{G})$.

For each $m, n \geq 0$, we set

$$\Gamma_{m,n} = \Gamma_{m,n}(\mathcal{G}) = \{(\alpha, \alpha') \in \Omega \times \Omega : |\alpha| = m, |\alpha'| = n \text{ and } r(\alpha) = r(\alpha')\}.$$

If $\mathcal{P}_{m,n} = \mathcal{P}_{m,n}(\mathcal{G})$ denotes the free vector space over $\Gamma_{m,n}$, with basis $\{e(\alpha, \alpha') : (\alpha, \alpha') \in \Gamma_{m,n}\}$, we have linear imbeddings $j_{m,n} : \mathcal{P}_{m,n} \to \mathcal{P}_{m+1,n+1}$, a product $\mu_{m,n,p} : \mathcal{P}_{m,n} \times \mathcal{P}_{n,p} \to \mathcal{P}_{m,p}$ and an involution $* : \mathcal{P}_{m,n} \to \mathcal{P}_{n,m}$ defined by

$$\mu_{m,n,p}\big(e(\alpha, \alpha'), e(\beta', \beta)\big) = \delta_{\alpha',\beta'}\, e(\alpha, \beta),$$

$$e(\alpha, \alpha')^* = e(\alpha', \alpha),$$

$$j_{m,n}(e(\alpha, \alpha')) = \sum_{\substack{x\in E;\\ s(x)=r(\alpha)}} e(\alpha x, \alpha' x)$$

where, if $\alpha = \alpha_1 \ldots \alpha_n$, then αx denotes the path $\alpha_1 \ldots \alpha_n x$. It is routine to check that the product and involution are compatible with the embeddings $j_{m,n}$ i.e. that

$$\mu_{m+1,n+1,p+1} \circ (j_{m,n} \times j_{n,p}) = j_{m,p} \circ \mu_{m,n,p}$$

$$j_{n,m} \circ * = * \circ j_{m,n},$$

so that if

$$\mathcal{P}^k(\mathcal{G}) = \varinjlim_m (\mathcal{P}_{m+k,m}(\mathcal{G}), j_{m+k,m})$$

and

$$\mathcal{P}(\mathcal{G}) = \oplus_{k\in\mathbb{Z}}\mathcal{P}^k(\mathcal{G}),$$

then $\mathcal{P}(\mathcal{G})$ becomes an involutive unital *-algebra.

Observe that the 0-graded algebra

$$\mathcal{P}^0(\mathcal{G}) = \lim_{\rightarrow}(\mathcal{P}_{m,m}, j_{m,m})$$

is (because of the local finiteness of $\mathcal{G}$), an inductive limit of (direct sums of full) matrix algebras i.e. a locally semi-simple algebra in the sense of [El], so that its completion in the natural C^*-norm is an A.F. algebra in the sense of [B]. (Observe also that if our graph $\mathcal{G}$ is distance oriented i.e. $r(\alpha) = r(\alpha')$ implies $|\alpha| = |\alpha'|$, then $\mathcal{P}(\mathcal{G}) = \mathcal{P}^0(\mathcal{G})$). It is also worth noting that [MRS] shows how to construct a locally compact, amenable, topological groupoid with Haar system, whose C^*-enveloping algebra is a natural C^*- completion of $\mathcal{P}(\mathcal{G})$.

Two examples will be of primary importance to us:

Example 1.1 Given a compact group G and a finite dimensional unitary representation (ρ, V) of G, we may form a graph $\mathcal{G}_\rho$ as follows:

$$V(\mathcal{G}_\rho) = \widehat{G}$$

$$E(\mathcal{G}_\rho) = \{(\pi, k, \nu) \in \widehat{G} \times \mathbb{N} \times \widehat{G} : 1 \leq k \leq \dim Hom_G(V_\pi \otimes V, V_\nu)\},$$

where $\widehat{G}$ is the set of (equivalence classes of) irreducible unitary representations of G, and for each $\pi \in \widehat{G}$, V_π is the carrier space of (a representative of) π. We view $(\pi, k, \nu) \in E(\mathcal{G}_\rho)$ as directed from π to ν, and take $\star = \iota$, the trivial representation of G.

The relevance of this example is that $\mathcal{P}(\mathcal{G}_\rho)$ may be identified with the algebra ${}^o\mathcal{O}_\rho$ of intertwiners between tensor powers of ρ as follows. For each edge $x = (\pi, k, \nu) \in E(\mathcal{G}_\rho)$, choose an isometry $E(x) : V_\nu \to V_\pi \otimes V$ intertwining the G-actions, and with

$$\sum_{s(x)=\pi} E(x)E(x)^* = 1 \text{ on } V_\pi \otimes V,$$

so that the ranges of $\{E(x) : s(x) = \pi\}$ are orthogonal. For each finite path $\alpha = x_1 \dots x_n \in \Omega(\mathcal{G}_\rho)$, we define $E(\alpha) \in Hom_G(V^{|\alpha|}, V_{r(\alpha)})$ by

$$E(\alpha) = (E(x_1) \otimes 1_V^{n-1}) \circ \cdots \circ (E(x_{n-1}) \otimes 1_V) \circ E(x_n).$$

Observe that for $(\alpha, \alpha') \in \Gamma_{m,n}$

$$E(\alpha)E(\alpha')^* \in Hom_G(V^m, V^n)$$

(where V^k is the k-fold tensor power of V) and that $\{E(\alpha)E(\alpha')^* : (\alpha, \alpha') \in \Gamma_{m,n}\}$ is a linear basis of $Hom_G(V^m, V^n)$. Furthermore, for $(\alpha, \alpha') \in \Gamma_{m,n}$,

$$\begin{aligned}\sum_{\substack{x \in E; \\ s(x)=r(\alpha)}} E(\alpha x)E(\alpha' x)^* &= \sum_{\substack{x \in E; \\ s(x)=r(\alpha)}} \Big(E(\alpha) \otimes 1_V\Big) E(x)E(x)^* \Big(E(\alpha') \otimes 1_V\Big)^* \\ &= (E(\alpha) \otimes 1_V)(1_{r(\alpha)} \otimes 1_V)(E(\alpha') \otimes 1_V)^* \\ &= E(\alpha)E(\alpha')^* \otimes 1_V.\end{aligned}$$

Thus the inductive systems $\{Hom_G(V^m, V^n), \otimes 1_V\}$ and $\{\mathcal{P}_{m,n}(\mathcal{G}), j_{m,n}\}$ are isomorphic. Since for $(\alpha, \alpha') \in \Gamma_{m,n}$, $(\beta', \beta) \in \Gamma_{n,p}$ we have

$$\begin{aligned}E(\alpha)E(\alpha')^*E(\beta')E(\beta)^* &= \delta_{\alpha',\beta'} E(\alpha)E(\beta)^*, \text{and} \\ \Big(E(\alpha)E(\alpha')^*\Big)^* &= E(\alpha')E(\alpha)^*,\end{aligned}$$

if $E(\alpha, \alpha')$ denotes the image of $E(\alpha)E(\alpha')^*$ in $\oplus_{k \in \mathbb{Z}} \varinjlim_n (Hom_G(V^{n+k}, V^n), \otimes 1_V) = {}^0\mathcal{O}_\rho$ (in the notation of [DR1, p.97]), the map

$$e(\alpha, \alpha') \in \mathcal{P}(\mathcal{G}_\rho) \mapsto E(\alpha, \alpha') \in {}^0\mathcal{O}_\rho$$

is a $*$-isomorphism. We will regard ${}^0\mathcal{O}_\rho$ as being identified with $\mathcal{P}(\mathcal{G}_\rho)$ in this way; note however that this identification is not intrinsic since at the very least the intertwiners $E(x), x \in E(\mathcal{G}_\rho)$ are determined only up to a scalar.

Example 1.2 Let A be a $d \times d$ matrix with non-negative integer entries and let $\mathcal{G} = \mathcal{G}_A$ be the directed graph on d vertices with incidence matrix A. Let $\overline{\mathcal{G}} = \overline{\mathcal{G}}_A$ be the graph obtained by adding a base-point $\star$ to $\{1 \ldots, d\}$ and, for each i, a single edge z_i directed from $\star$ to i. For each edge $x \in E(\mathcal{G}_A)$, put

$$\widetilde{F}(x) = e(z_{s(x)}x, z_{r(x)})$$

and observe that

$$\widetilde{F}(x)\widetilde{F}(x)^* = e(z_{s(x)}x, z_{s(x)}x)$$

$$\widetilde{F}(x)^*\widetilde{F}(x) = e(z_{r(x)}, z_{r(x)}).$$

Thus if $\widetilde{A}$ is the incidence matrix of the graph $\widetilde{\mathcal{G}}$ dual to $\mathcal{G}$ i.e. $V(\widetilde{\mathcal{G}}) = E(\mathcal{G}), E(\widetilde{\mathcal{G}}) = \{(x, y)$ $x, y \in V(\widetilde{\mathcal{G}})$, with $r(x) = s(y)\}$, and $\widetilde{s}(x, y) = x, \widetilde{r}(x, y) = y$, the elements $\{\widetilde{F}(x) : x \in E(\mathcal{G})\}$ satisfy the defining relations of the Cuntz-Krieger algebra $\mathcal{O}_{\widetilde{A}}$ (see [CK,p.251]) i.e.

$$\widetilde{F}(x)^*\widetilde{F}(x) \;=\; \sum_y \widetilde{A}(x, y)\widetilde{F}(y)\widetilde{F}(y)^*$$

Observe also that the elements $\{\widetilde{F}(x) : x \in E(\mathcal{G})\}$ generate $\mathcal{P}(\overline{\mathcal{G}}_A)$ – indeed for $(\alpha, \alpha') \in \widetilde{\Gamma}_{m,n}$ we have $e(\alpha, \alpha') = \widetilde{F}(\alpha_1)\widetilde{F}(\alpha_2)\cdots\widetilde{F}(\alpha_m)\widetilde{F}(\alpha'_n)^*\cdots\widetilde{F}(\alpha'_1)^*$.

Remark 1.3 The transition from A to $\widetilde{A}$ is in essence the "symbol splitting" technique of [Ev,2.6].

Remark 1.4 For a general locally finite graph $\mathcal{G}$ in which there is at most one edge joining any two vertices, we may form a variant $\mathcal{V}(\mathcal{G})$ of $\mathcal{P}(\mathcal{G})$, based on paths as finite sequences of vertices, and with generators $f(v, w)$ indexed on pairs of paths with the same final vertex, subject to the relations

$$f(v, w)^* = f(w, v),$$

$$f(u, v)f(v', w) = \delta_{v,v'}f(u, w)$$

$$f(v, w) = \sum_{u \in V(\mathcal{G})} f(vu, wu),$$

whenever v and v' have the same length. Since there is an obvious identification of paths (as sequences of edges) and paths (as sequences of vertices) in this single-bonded case, it is evident that $\mathcal{P}(\mathcal{G})$ is $*$-isomorphic to $V(\mathcal{G})$.

Note however that this provides us with a different presentation of $\mathcal{P}(\overline{\mathcal{G}}_A)$ in the case where A has values in $\{0, 1\}$. Indeed, if we put, for $1 \leq k \leq d$,

$$F(k) \;=\; f(k, \cdot) \;=\; \sum_{\ell : A(k,\ell)=1} f(k\ell, \ell),$$

then

$$F(k)F(k)^* \; = \; f(k,k)$$

and

$$F(k)^*F(k) \; = \; \sum f(\ell, k\ell)f(k\ell', \ell') \; = \sum_{\ell : A(k,\ell)=1} f(\ell,\ell)$$

so that

$$F(k)^*F(k) = \sum_{\ell} A(k,\ell)F(\ell)F(\ell)^*.$$

Thus $\{F(k) : 1 \leq k \leq d\}$ satisfy the algebraic relations characteristic of the Cuntz-Krieger algebra $\mathcal{O}_A$.

Since the matrix A satisfies condition I of [CK,p.254] if and only if $\widetilde{A}$ does, and since $\mathcal{P}(\overline{\mathcal{G}})$ is $*$-isomorphic with $\mathcal{V}(\overline{\mathcal{G}})$, we conclude that if A has $\{0,1\}$ entries and satisfies condition I, so does $\widetilde{A}$, and $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_{\widetilde{A}}$.

Remark 1.5 This last calculation raises a natural but curious question: what is the equivalence relation on square matrices over $\mathbb{N}$ defined by $A \sim B$ if $A^{\sim k} = B^{\sim \ell}$ for some $k, \ell \in \mathbb{N}$, where $A^{\sim k}$ means to take the incidence matrix of the k-fold dual of $\mathcal{G}_A$?

§2 Filtered inclusions

We consider two graphs $\mathcal{G}$ and $\mathcal{H}$ as in §1, and the associated spaces $\mathcal{P}_{m,n}(\mathcal{G})$, $\mathcal{P}_{m,n}(\mathcal{H})$ etc.

Definition 2.1 A unital $*$-monomorphism $\sigma : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{H})$ is said to be **filtered** if for all $m, n \geq 0$,

i) $\sigma(\mathcal{P}_{m,n}(\mathcal{G})) \subseteq \mathcal{P}_{m,n}(\mathcal{H})$

ii) $\sigma \circ j^{\mathcal{G}}_{m,n} = j^{\mathcal{H}}_{m,n} \circ \sigma$.

If σ is filtered, we will write $\sigma_{m,n}$ for the restriction of σ to $\mathcal{P}_{m,n}(\mathcal{G})$; for convenience we will also use $\mathcal{P}_m(\mathcal{G})$, j_m, σ_m etc. for $\mathcal{P}_{m,m}(\mathcal{G})$, $j_{m,m}, \sigma_{m,m}$ etc. Edges and paths in $\mathcal{G}$ will be denoted by x and α, and those in $\mathcal{H}$ by y and β respectively.

As remarked earlier, the algebras $\mathcal{P}^0(\mathcal{G}), \mathcal{P}^0(\mathcal{H})$ are locally semi-simple algebras. Thus by [02,p.130], any filtered inclusion $\mathcal{P}^0(\mathcal{G}) \to \mathcal{P}^0(\mathcal{H})$ is implemented by a connection $(\mathcal{T}, W)$. To facilitate subsequent computations we recall the construction:

Each of the algebras $\mathcal{P}_m(\mathcal{G})$ is a direct sum of simple algebras $\mathcal{P}^v_m(\mathcal{G}), v \in V$, where

$$\mathcal{P}^v_m(\mathcal{G}) \quad = \quad span\{e(\alpha, \alpha') : |\alpha| = |\alpha'| = m, r(\alpha) = r(\alpha') = v\}.$$

Thus the irreducible modules for $\mathcal{P}_m(\mathcal{G})$ may be taken as $\ell^2(\Omega^v_m(\mathcal{G}))$, where $\Omega^v_m(\mathcal{G}) = \{\alpha \in \Omega(\mathcal{G}) : |\alpha| = m, r(\alpha) = v\}$. We will write $\{\epsilon_\alpha\}$ for the natural orthonormal basis of $\ell^2(\Omega^v_m(\mathcal{G}))$. Note that for each edge $x \in E(\mathcal{G})$ we can define an isometry $T(x) : \ell^2(\Omega^{s(x)}_m(\mathcal{G})) \to \ell^2(\Omega^{r(x)}_{m+1}(\mathcal{G}))$ by

$$T(x)\epsilon_\alpha \quad = \quad \epsilon_{\alpha x};$$

these isometries are intertwiners in the sense that

$$T(x) \circ t \quad = \quad j^{\mathcal{G}}_m(t) \circ T(x)$$

for any $t \in \mathcal{P}_m(\mathcal{G})$. Similarly $\mathcal{P}_m(\mathcal{H}) = \oplus_{w \in E(\mathcal{H})} \mathcal{P}^w_m(\mathcal{H})$, and we have intertwining isometries $T(y), y \in E(\mathcal{H})$ with $T(y) : \ell^2(\Omega^{s(y)}_m(\mathcal{H})) \to \ell^2(\Omega^{r(y)}_{m+1}(\mathcal{H}))$. Clearly the isometries $\{T(x) : r(x) = v\}$ have orthogonal ranges which span $\ell^2(\Omega^v_m(\mathcal{G}))$.

In addition, through σ_m, we may view each of the $\mathcal{P}_m(\mathcal{H})$ modules $\ell^2(\Omega_m^w(\mathcal{H}))$ as a $\mathcal{P}_m(\mathcal{G})$ module; as such each space $\ell^2(\Omega_m^w(\mathcal{H}))$ decomposes, as a $\mathcal{P}_m(\mathcal{G})$ module, as an orthogonal direct sum of copies of $\{\ell^2(\Omega_m^v(\mathcal{G})) : v \in V(\mathcal{G})\}$. Thus for each m, we obtain a bipartite directed graph $\mathcal{T}_m$ with edges directed from $V(\mathcal{G})$ to $V(\mathcal{H})$; for each such edge p, we may choose an isometry $T(p) : \ell^2(\Omega_m^{s(p)}(\mathcal{G})) \to \ell^2(\Omega_m^{r(p)}(\mathcal{H}))$, satisfying

a) $\sigma(t)T(p) = T(p)t,\ t \in \mathcal{P}_m(\mathcal{G})$

b) $\sum\limits_{p\in E(\mathcal{T}_m),r(p)=w} T(p)T(p)^* = 1$ on $\ell^2(\Omega_m^w(\mathcal{H}))$.

We take $\mathcal{T} = \bigcup\limits_m \mathcal{T}_m$, with the obvious range and source maps.

A **contour** in $(\mathcal{G}, \mathcal{T}, \mathcal{H})$ is a quadruple (α, p, β, q) where α is a finite path in $\mathcal{G}$, β is a finite path in $\mathcal{H}$, $p \in E(\mathcal{T}_m)$ and $q \in E(\mathcal{T}_n)$ for some m, n, and $s(\alpha) = s(p)$, $r(\beta) = r(q), r(\alpha) = s(q), s(\beta) = r(p)$. As in [01,02], we will picture a contour:

$$\begin{tikzcd}
\mathcal{G}\ \bullet \arrow[r, "\alpha_m"] \arrow[d, "p"'] & \bullet \arrow[r, "\alpha_{m+1}"] & \cdots \arrow[r, "\alpha_{n-1}"] \arrow[r, phantom, "\alpha"', shift right=3] & \bullet\ \mathcal{G} \arrow[d, "q"] \\
\mathcal{H}\ \bullet \arrow[r, "\beta_m"] & \bullet \arrow[r, "\beta_{m+1}"] & \cdots \arrow[r, "\beta_{n-1}"] \arrow[r, phantom, "\beta"', shift right=3] & \bullet\ \mathcal{H}.
\end{tikzcd}$$

Observe that for the path $\alpha = \alpha_m \dots \alpha_{n-1}$ as indicated, we may define an operator $T(\alpha) : \ell^2(\Omega_m^{s(\alpha)}(\mathcal{G})) \to \ell^2(\Omega_n^{r(\alpha)}(\mathcal{G}))$ by $T(\alpha) = T(\alpha_{n-1}) \circ \cdots \circ T(\alpha_m)$, or, for $\alpha_1 \in \Omega_m^{s(\alpha)}(\mathcal{G})$, $T(\alpha)(\epsilon_{\alpha_1}) = \epsilon_{\alpha_1\alpha}$; similarly for $T(\beta) : \ell^2(\Omega_m^{s(\beta)}(\mathcal{H})) \to \ell^2(\Omega_n^{r(\beta)}(\mathcal{H}))$. In view of the fact that $\sigma \circ j_{n-1}^{\mathcal{G}} \circ \cdots \circ j_m^{\mathcal{G}} = j_{n-1}^{\mathcal{H}} \circ \cdots \circ j_m^{\mathcal{H}} \circ \sigma$, for each contour (α, p, β, q) the operator $T(p)^*T(\beta)^*T(q)T(\alpha)$ on $\ell^2(\Omega_m^{s(\alpha)}(\mathcal{G}))$ commutes with the (irreducible) action of $\mathcal{P}_m(\mathcal{G})$ and hence we have

$$T(p)^*T(\beta)^*T(q)T(\alpha) \quad = \quad W(\alpha, p, \beta, q) \text{ on } \ell^2(\Omega_m^{s(\alpha)}(\mathcal{G}))$$

for some unique scalar $W(\alpha, p, \beta, q)$. Observe also that if we consider all contours (α, p, β, q) with $s(\alpha) = u$ and $r(\beta) = v$ fixed, then

$$\sum_{\beta,q} W(\beta,r,\gamma,q)\overline{W}(\beta,p,\alpha,q) = \sum_{\beta,q}\Big(T(r)^*T(\gamma)^*T(q)T(\beta)\Big)\Big(T(\beta)^*T(q)^*T(\alpha)T(p)\Big)$$
$$= T(r)^*T(\gamma)^*T(\alpha)T(p) = \delta_{r,p}\delta_{\gamma,\alpha},$$

since the relevant isometries have orthogonal range projections adding to 1. Also, our assumption that $\sigma \circ j_k^{\mathcal{G}} = j_k^{\mathcal{H}} \circ \sigma, k \geq 1$, forces for each $u \in s(\mathcal{T}_m), v \in r(\mathcal{T}_n)$ the number of paths through $\mathcal{G}$ then $\mathcal{T}$ from u to v to agree with the number of analogous paths through $\mathcal{T}$ then $\mathcal{H}$; thus the corresponding matrix $W_{u,v}$ is unitary.

The reader should note that the notation $W_{u,v}$ is in fact ambiguous, since there may be many pairs (m,n) with $u \in s(\mathcal{T}_m)$ and $v \in r(\mathcal{T}_n)$. Note also that a contour (α,p,β,q) in which $|\alpha| = |\beta| = 1$ is called a cell in [02]; [02] also provides a procedure for calculating the values of W on contours from its values on cells.

To understand the relation of σ to W, observe that the basis element $e(\alpha,\alpha')$ of $\mathcal{P}_m(\mathcal{G})$ may be viewed as the rank one operator from $\epsilon_{\alpha'}$ to ϵ_α. As a consequence, we may compute for $(\beta,\beta') \in \Gamma_m(\mathcal{H})$ and $(\alpha,\alpha') \in \Gamma_m(\mathcal{G})$

$$\langle \sigma(e(\alpha,\alpha'))\epsilon_{\beta'}, \epsilon_\beta\rangle = \sum_{p,p',\gamma,\gamma'} \langle \sigma(e(\alpha,\alpha'))\langle \epsilon_{\beta'}, T(p')\epsilon_{\gamma'}\rangle T(p')\epsilon_{\gamma'}, \langle \epsilon_\beta, T(p)\epsilon_\gamma\rangle T(p)\epsilon_\gamma\rangle$$
$$= \sum_p \langle \epsilon_{\beta'}, T(p)\epsilon_{\alpha'}\rangle \overline{\langle \epsilon_\beta, T(p)\epsilon_\alpha\rangle}$$

where the sum is on all $p \in \mathcal{T}_m$ with $s(p) = r(\alpha)$, so that

$$\sigma(e(\alpha,\alpha')) = \sum_{p,\beta,\beta'} \langle \epsilon_{\beta'}, T(p)\epsilon_{\alpha'}\rangle \overline{\langle \epsilon_\beta, T(p)\epsilon_\alpha\rangle}\, e(\beta,\beta').$$

To compute the coefficients $\langle \epsilon_\beta, T(p)\epsilon_\alpha\rangle$, we observe that $\mathcal{P}_0(\mathcal{G}) = \mathbb{C}$, $\Omega_0(\mathcal{G}) = \star(=\star_{\mathcal{G}})$ so that $\ell^2(\Omega_0(\mathcal{G}))$ has canonical basis $\epsilon_\star$, and similarly for $\mathcal{H}$. The graph $\mathcal{T}_0$ has a single edge, also denoted $\star$ or $\star_{\mathcal{T}}$; we assume as we may that $T(\star_{\mathcal{G}})\epsilon_\star = \epsilon_\star$. We thus have

$$\langle \epsilon_\beta, T(p)\epsilon_\alpha\rangle = \langle T(\beta)T(\star)\epsilon_{\star_{\mathcal{G}}}, T(p)T(\alpha)\epsilon_{\star_{\mathcal{G}}}\rangle$$
$$= \overline{W}(\alpha,\star,\beta,p),$$

and hence

$$\sigma(e(\alpha,\alpha')) = \sum_{\beta,\beta',p} W(\alpha,\star,\beta,q)\overline{W}(\alpha',\star,\beta',p)e(\beta,\beta'),$$

as in [02,p130].

To continue the analysis, we observe that for each m and n, $\mathcal{P}_{m,n}(\mathcal{G})$ is a left $\mathcal{P}_m(\mathcal{G})$ right $\mathcal{P}_n(\mathcal{G})$ module, and hence a $\mathcal{P}_m(\mathcal{G}) \otimes \mathcal{P}_n(\mathcal{G})^o$ module, where V^o denotes the vector space opposite to V. Furthermore, we have

$$\mathcal{P}_{m,n}(\mathcal{G}) \simeq \bigoplus_{\nu} \ell^2(\Omega^{\nu}_m(\mathcal{G})) \otimes \ell^2(\Omega^{\nu}_n(\mathcal{G}))^o$$

as $\mathcal{P}_m(\mathcal{G}) \otimes \mathcal{P}_n(\mathcal{G})^o$ modules, the isomorphism being given explicitly by $e(\alpha,\beta) \leftrightarrow \epsilon_\alpha \otimes \epsilon^o_\beta$. Observe also that the summands $D^{\nu}_{m,n}(\mathcal{G}) = \ell^2(\Omega^{\nu}_m(\mathcal{G})) \otimes \ell^2(\Omega^{\nu}_n(\mathcal{G}))^o$ are irreducible for the action of $\mathcal{P}_m(\mathcal{G}) \otimes \mathcal{P}_n(\mathcal{G})^o$. Thus for $p \in E(\mathcal{T}_m), q \in E(\mathcal{T}_n)$ with $s(p) = s(q) = \nu$, $r(p) = r(q)$, the map

$$(T(p) \otimes T(q)^o)^* \circ \sigma\Big|_{D^{\nu}_{m,n}(\mathcal{G})} \in End(D^{\nu}_{m,n}(\mathcal{G})),$$

commutes with the action of $\mathcal{P}_m(\mathcal{G}) \otimes \mathcal{P}_n(\mathcal{G})^o$, and so is a scalar. We may thus define; for p, q as above

$$\lambda_{m,n}(p,q) \;=\; (T(p) \otimes T(q)^o)^* \circ \sigma\Big|_{D^{\nu}_{m,n}(\mathcal{G})} \text{ on } D^{\nu}_{m,n}(\mathcal{G}).$$

Proposition 2.2. *With notation as above, we have*

a) for each ℓ, m, n we have $\lambda_{\ell,n}(p,r) = \sum_q \lambda_{\ell,m}(p,q)\lambda_{m,n}(q,r)$

b) $\lambda_{m,m}(p,q) = \delta_{p,q}$ for all m, p, q

c) $\sum_{p,q,x} W(x,p',y,p)\,\lambda_{m+1,n+1}(p,q)\,\overline{W}(x,q',z,q) = \delta_{y,z}\,\lambda_{m,n}(p',q')$

d) $\overline{\lambda_{m,n}(p,q)} = \lambda_{n,m}(q,p)$ for all m, n, p, q.

Furthermore

e) for $(\alpha,\alpha') \in \Gamma_{m,n}(\mathcal{G})$,

$$\sigma(e(\alpha,\alpha')) = \sum_{p,p',\beta,\beta'} \lambda_{m,n}(p,p')\,W(\alpha,\star,\beta,p)\,\overline{W}(\alpha',\star,\beta',p')e(\beta,\beta').$$

Conversely, if $\lambda = \{\lambda_{m,n}(p,q)\}$ are given satisfying a), b), c), d), then the formula e) defines a family of linear maps from $\mathcal{P}_{m,n}(\mathcal{G})$ to $\mathcal{P}_{m,n}(\mathcal{H})$ which preserve the inductive, multiplicative and involutive structures, and hence a filtered inclusion of $\mathcal{P}(\mathcal{G})$ in $\mathcal{P}(\mathcal{H})$.

We will first establish a "spatial implementation" result (cf [01]) namely:

Lemma 2.3. *(Spatial Implementation)*

$$\sigma(t_{\xi,n}) \quad = \quad \sum_{p,q} \lambda_{m,n}(p,q) t_{T(p)\xi,T(q)\eta}$$

where, for $\xi \in \ell^2(\Omega_m(\mathcal{G})), \eta \in \ell^2(\Omega_n(\mathcal{G})), t_{\xi,n} \in Hom(\ell^2(\Omega_m(\mathcal{G})), \ell^2(\Omega_n(\mathcal{G}))$ is the rank one operator $t_{\xi,\eta}(\zeta) = \langle \zeta, \eta \rangle \xi \ (\xi \in \ell^2(\Omega_n(\mathcal{G})))$ so that $t_{\epsilon_\alpha,\epsilon'_\alpha} = e(\alpha, \alpha')$.

Proof. We note that by definition

$$(T(p) \otimes T(q)^o)^* \circ \sigma(t_{\xi,\eta}) = \lambda_{m,n}(p,q) t_{\xi,\eta},$$

while

$$\begin{aligned}(T(p) \otimes T(q)^o)^* \left(\sum_{u,v} \lambda_{m,n}(u,v) t_{T(u)\xi,T(v)\eta} \right) &= \sum_{u,v} \lambda_{m,n}(u,v)\, t_{T(p)^* T(u)\xi, T(q)^* T(v)\eta} \\ &= \lambda_{m,n}(p,q)\, t_{\xi,\eta}\end{aligned}$$

and our contention follows. ■

We now proceed to the proof of the Proposition:

Proof of 2.2. For a) observe that for $\xi \in \ell^2(\Omega_\ell(\mathcal{G})), \eta \in \ell^2(\Omega_m(\mathcal{G}))$, and $\zeta \in \ell^2(\Omega_{\mathrm{n}}(\mathcal{G}))$ with $\|\eta\| = 1$,

$$\sigma(t_{\xi,\zeta}) = \sum_{p,r} \lambda_{\ell,n}(p,r) t_{T(p)\xi,T(r)\zeta}$$

while

$$\begin{aligned}\sigma(t_{\xi,\eta})\sigma(t_{\eta,\zeta}) &= \sum_{p,q,q',r} \lambda_{\ell,m}(p,q)\lambda_{m,n}(q',r) t_{T(p)\xi,T(q)\eta} t_{T(q')\eta,T(r)\zeta} \\ &= \sum_{p,r} \left(\sum_q \lambda_{\ell,m}(p,q)\lambda_{m,n}(q,r) \right) t_{T(p)\xi,T(r)\zeta}\end{aligned}$$

Thus, σ defined by e) is multiplicative if and only if $\{\lambda_{m,n}(p,q)\}$ satisfies a).

Assertion b) may be proved as follows: For $\xi, \eta, \zeta \in \ell^2(\Omega_m(\mathcal{G}))$ and any m, consider the sum

$$\begin{aligned}\sum_{e\in\mathcal{T}_m} t_{T(e)\xi,T(e)\eta}\cdot\zeta &= \sum_e \langle\zeta, T(e)\eta\rangle\, T(e)\xi \\ &= \sum_e T(e)\,\langle T(e)^*\zeta,\eta\rangle\,\xi \\ &= \sum_e T(e)\, t_{\xi,\eta}\, T(e)^*\zeta \\ &= \sum_e \sigma_m(t_{\xi,\eta})\, T(e)T(e)^*\zeta \\ &= \sigma(t_{\xi,\eta})\,\zeta\end{aligned}$$

Since ζ was any element of $\ell^2(\Omega_m(\mathcal{G}))$, we have that

$$\sigma(t_{\xi,\eta}) = \sum_{e\in\mathcal{T}_m} t_{T(e)\xi,T(e)\eta}.$$

However, from the spatial implementation lemma we know that

$$\sigma(t_{\xi,\eta}) = \sum_{p,q\in\mathcal{T}_m} \lambda_{m,m}(p,q)\, t_{T(p)\xi,T(q)\eta},$$

the result then follows by comparing coefficients.

For c), we first observe that for $\alpha\in\Omega_m(\mathcal{G})$ $x\in E(\mathcal{G})$ and $p\in\mathcal{T}_{m+1}$ with $r(\alpha)=s(x), r(x)=s(\alpha)$,

$$\begin{aligned}T(p)T(x)\epsilon_\alpha &= \sum_{y,\beta}\langle T(p)T(x)\epsilon_\alpha, T(y)\epsilon_\beta\rangle\, T(y)\epsilon_\beta \\ &= \sum_{y,\beta,q}\langle T(y)^*T(p)T(x)\epsilon_\alpha, T(q)T(q)^*\epsilon_\beta\rangle\, T(y)\epsilon_\beta \\ &= \sum_{y,\beta,q} W(x,q,y,p)\,\langle T(q)\epsilon_\alpha,\epsilon_\beta\rangle\, T(y)\epsilon_\beta \\ &= \sum_{y,q} W(x,q,y,p)\, T(y)T(q)\epsilon_\alpha,\end{aligned}$$

where the final sum extends over those y,q with (x,q,y,p) a cell. We now calculate for $(\alpha,\alpha')\in\Gamma_{m,n}(\mathcal{G})$,

$$\begin{aligned}\sigma\circ j^{\mathcal{G}}_{m,n}(e(\alpha,\alpha')) &= \sum_x \sigma(e(\alpha x,\alpha' x)) \\ &= \sum_{x,p,q}\lambda_{m+1,n+1}(p,q) t_{T(p)T(x)\epsilon_\alpha, T(q)T(x)\epsilon_{\alpha'}} \\ &= \sum_{p,q,x}\lambda_{m+1,n+1}(p,q)\sum_{p',y,q',z} W(x,p',y,p)\overline{W}(x,q',z,q) t_{T(y)T(p')\epsilon_\alpha, T(z)T(q')\epsilon_{\alpha'}}\end{aligned}$$

On the other hand

$$\begin{aligned}j^{\mathcal{H}}_{m,n}\circ\sigma(e(\alpha,\alpha')) &= j^{\mathcal{H}}_{m,n}\left(\sum_{u,v}\lambda_{m,n}(u,v) t_{T(u)\epsilon_\alpha, T(v)\epsilon_{\alpha'}}\right) \\ &= \sum_{u,v,w}\lambda_{m,n}(u,v) t_{T(w)T(u)\epsilon_\alpha, T(w)T(v)\epsilon_{\alpha'}}\end{aligned}$$

Comparing coefficients, we deduce that σ defined by e) preserves the inductive structure if and only if

$$\sum_{p,q,x} W(x,p',y,p)\,\lambda_{m+1,n+1}(p,q)\,\overline{W}(x,q',z,q) \quad = \quad \delta_{y,z}\,\lambda_{m,n}(p',q')$$

as claimed.

It is routine to verify that d) is equivalent to the condition $\sigma(e(\alpha,\alpha')^*) = \sigma(e(\alpha,\alpha'))^*$.

Finally, we note that the formula e) follows routinely from the identity (for $(\alpha,\alpha') \in \Gamma_{m,n}(\mathcal{G})$),

$$\sigma(e(\alpha,\alpha')) = \sum_{p,q} \lambda_{m,n}(p,q)\, t_{T(p)\epsilon_\alpha, T(q)\epsilon_{\alpha'}},$$

and the expansion

$$T(p)\epsilon_\alpha = \sum_\beta W(\alpha,\star,\beta,p)\,\epsilon_\beta,$$

where the last sum is over all β with $(\alpha,\star,\beta,p)$ a contour. ■

Definition 2.4. *Given* $(\mathcal{G},\mathcal{T},\mathcal{H})$, *a pair* (W,λ) *satisfying*

i) W maps contours in $(\mathcal{G},\mathcal{T},\mathcal{H})$ to $\mathbb{C}$ and, for fixed $v \in s(\mathcal{T}_m), w \in r(\mathcal{T}_n)$ the matrix $\{W(\alpha,p,\beta,q) : (\alpha,p,\beta,q) \text{ a contour, } s(\alpha) = v, r(\beta) = w\}$ is unitary with respect to (α,p) and (β,q), and

ii) λ satisfies the conditions a), b), c), d) of Proposition 2.2,

will be called a **phased** (unitary) **connection** for $(\mathcal{G},\mathcal{T},\mathcal{H})$. The corresponding filtered inclusion of $\mathcal{P}(\mathcal{G})$ in $\mathcal{P}(\mathcal{H})$ will be denoted $\sigma^{(W,\lambda)}$.

In order to simplify the rest of this section, we adopt the following notation: Let $W_{m,n}$ denote the matrix $\{W_{m,n}(\alpha,p,\beta,q) : (\alpha,p,\beta,q)$ is a contour starting at level m and finishing at level $n\}$, similarly we consider $\lambda_{m,n} = \{\lambda_{m,n}(p,q)\}$. The results of Proposition 2.2 may be written, if we choose to omit the summation indices:

a) $\lambda_{\ell,n} \;=\; \lambda_{\ell,m} \cdot \lambda_{m,n}$

b) $\lambda_{m,m} \;=\; 1$

c) $W_{m,m+1}\,\lambda_{m+1,n+1}\,W^*_{n,n+1} \;=\; \lambda_{m,n}\otimes 1$

d) $\lambda^*_{m,n} \;=\; \lambda_{n,m}$

e) $\sigma \;=\; \sum W_{0,m}\,\lambda_{m,n}\,W^*_{0,n}$.

Here products such as $W_{m,m+1}\lambda_{m+1,n+1}$ are defined by summing over common indices, for example

$$W_{m,m+1}\lambda_{m+1,n+1}(\alpha,p',\beta,q) \;=\; \sum_{p\in\mathcal{T}_{m+1}} W_{m,m+1}(\alpha,p',\beta,p)\lambda_{m+1,n+1}(p,q).$$

We shall adopt this convention henceforth.

Remark 2.5 Observe that for each $z\in\mathbb{T}$, we may define an automorphism $\theta^{\mathcal{G}}_z$ of $\mathcal{P}(\mathcal{G})$ by

$$\theta^{\mathcal{G}}_z(x) = z^k x,\;\text{ for x} \in \text{span}\{\,\text{e}(\alpha,\alpha') : |\alpha| - |\alpha'| = \text{k}\,\}.$$

If $\sigma=\sigma^{(W,\lambda)}$ is a filtered inclusion of $\mathcal{P}(\mathcal{G})$ in $\mathcal{P}(\mathcal{H})$, then so is $\sigma\circ\theta^{\mathcal{G}}_z=\theta^{\mathcal{H}}_z\circ\sigma$, and we have that $\sigma^{(W,\lambda)}\circ\theta^{\mathcal{G}}_z=\sigma^{(W^z,\lambda^z)}$ where $W^z=W$ and $\lambda^z_{m,n}(p,q)=z^{m-n}\lambda_{m,n}(p,q)$.

Observe that given a filtered inclusion σ of $\mathcal{P}(\mathcal{G})$ in $\mathcal{P}(\mathcal{H})$, there will in general be many phased connections (W,λ), with $\sigma=\sigma^{(W,\lambda)}$ since the isometries used in the construction of W and λ are at best determined only up to a scalar. Slightly extending [01], the relevant equivalence relation on phased connections is:

Definition 2.6 Phased connections $(W,\lambda),(W',\lambda')$ on $(\mathcal{G},\mathcal{T},\mathcal{H})$ are **weakly equivalent** if there are matrices $u_m, m\geq 0$, indexed on pairs $(p,p')\in\mathcal{T}_m$ with $s(p)=s(p')$, and unitary when $s(p)$ is fixed, with

a) $W'_{m,n} \;=\; u_m\,W_{m,n}\,u^*_n$

b) $\lambda'_{m,n} = u_m\,\lambda_{m,n}\,u^*_n$.

By convention, we have that $u_0=[1]$; and products are defined as above.

Proposition 2.7. *If $(W,\lambda),(W',\lambda')$ are phased connections for $(\mathcal{G},\mathcal{T},\mathcal{H})$ then $\sigma^{(W,\lambda)}=\sigma^{(W',\lambda')}$ if and only if (W,λ) and (W',λ') are weakly equivalent.*

Proof. Suppose first that $(W,\lambda),(W',\lambda')$ are weakly equivalent via u, i.e. $W'_{m,n} = u_m\,W_{m,n}\,u_n^*$ and $\lambda'_{m,n} = u_m\,\lambda_{m,n}\,u_n^*$ (so that (W',λ') is a phased connection if and only if (W,λ) is). Then if $\sigma = \sigma^{(W,\lambda)}, \sigma' = \sigma^{(W',\lambda')}$ we have (formally)

$$\begin{aligned} \sigma' &= \sum W'_{0,m}\lambda'_{m,n}W'{}^{\star}_{0,n} \\ &= \sum \Big(W_{0,m}\,u_m^{\star}\Big)u_m\lambda_{m,n}u_n^{\star}\Big(u_nW_{0,n}^{\star}\Big) \\ &= \sum W_{0,m}\lambda_{m,n}W_{0,n}^{\star} \\ &= \sigma \end{aligned} .$$

– the formal manipulations may be justified by routine but generally unpleasant expansion. Thus $\sigma = \sigma'$.

Conversely, suppose that $\sigma = \sigma^{(W,\lambda)} = \sigma^{(W',\lambda')}$. For each $p \in \mathcal{T}_m$, define isometries $S(p), S'(p) : \ell^2(\Omega_m^{s(p)}(\mathcal{G})) \to \ell^2(\Omega_m^{r(p)}(\mathcal{H}))$ by

$$S(p)\epsilon_\alpha = \sum_\beta W_{0,m}(\alpha,\star,\beta,p)\,\epsilon_\beta$$

and analogously for $S'(p)$. It is routine to check that $S(p)t = \sigma(t)S(p)$, and $\mathrm{S}'(\mathrm{p})\mathrm{t} = \sigma(\mathrm{t})\mathrm{S}'(\mathrm{p})$ for $p \in \mathcal{T}_m, t \in \mathcal{P}_m(\mathcal{G})$. Furthermore we have that

$$\sum_{\substack{p\in\mathcal{T}_m,\\ r(p)=v}} S(p)S(p)^* = \sum_{\substack{p\in\mathcal{T}_m,\\ r(p)=v}} S'(p)S'(p)^* = 1$$

on $\ell^2(\Omega_m^v(\mathcal{H}))$, and, with $T(x)(x \in E(\mathcal{G})$ or $E(\mathcal{H}))$ denoting the standard isometries, we have for any contour (α,p,β,q)

$$S(q)^*T(\beta)^*S(p)T(\alpha) = W_{m,n}(\alpha,p,\beta,q)$$

$$S'(q)^*T(\beta)^*S'(p)T(\alpha) = W'_{m,n}(\alpha,p,\beta,q),$$

as a routine calculation shows. Thus we have that

$$\begin{aligned} W'_{m,n}(\alpha,p',\beta,q') &= S'(q')^*\Big(\sum_q S(q)S(q)^*\Big)T(\beta)^*\Big(\sum_p S(p)S(p)^*\Big)S'(p')S(\alpha) \\ &= \sum_{p,q}\Big(S'(q')^*S(q)\Big)S(q)^*T(\beta)^*S(p)\Big(S(p)^*S'(p')\Big)S(\alpha). \end{aligned}$$

However $S'(q')^*S(q)$ is a self-intertwiner for the action of $\mathcal{P}(\mathcal{G})$ on $\ell^2(\Omega_m^{s(q)}(\mathcal{G}))$, and hence is a scalar $u_m(q',q)$. We thus obtain

$$W'_{m,n}(\alpha, p'\beta, q') = \sum_{p,q} u_m(q', q)\, W_{m,n}(\alpha, p, \beta, q)\, \overline{u}_n(p', p)$$

and $W'_{m,n} = u_m\, W_{m,n} u_n^*$ as claimed.

We also have
$$\begin{aligned} \lambda'_{m,n}(p', q') &= (S'(p') \otimes S'(q')^o)^* \circ \sigma \\ &= \sum_{p,q} (S'(p')^* S(p) \otimes (S'(q')^* S(q))^o) \lambda_{m,n}(p, q) \\ &= \sum_{p,q} u_m(p', p) \lambda_{m,n}(p, q) \overline{u}_n(q', q), \end{aligned}$$

where we have used $\sigma = \sum_{p,q} (S(p) \otimes S(q)^o) \lambda_{m,n}(p, q)$ on $D^{\nu}_{m,n}$, c.f. [O1,O2], which concludes the proof. ■

The interested reader may continue to adapt the results of [O1,O2] on equivalence etc. to phased connections.

§3 Examples

Throughout, G is a compact group, and ρ a finite dimensional unitary representation of G on V. We use the notation $\mathcal{G}_\rho$, $E(\alpha)$ etc as developed in example 1.1.

Recall we showed there that there is a natural isomorphism of $\mathcal{P}(\mathcal{G}_\rho)$ with ${}^0\mathcal{O}_\rho$. However, ${}^0\mathcal{O}_\rho$ carries a canonical endomorphism σ (see [DR1]) constructed as follows. For each m, n the map

$$t \in Hom_G(V^m, V^n) \mapsto 1_V \otimes t \in Hom_G(V^{m+1}, V^{n+1})$$

is compatible with the maps defining the inductive limit ${}^0\mathcal{O}_\rho^k = \varinjlim_n (Hom_G(V^{n+k}, V^n), \otimes 1_V)$, and hence defines a unital $*$-endomorphism σ of ${}^0\mathcal{O}_\rho = \oplus_k {}^0\mathcal{O}_\rho^k$. We may thus view σ as an endomorphism of $\mathcal{P}(\mathcal{G}_\rho)$; evidently $\sigma(\mathcal{P}_{m,n}(\mathcal{G}_\rho)) \subseteq \mathcal{P}_{m+1,n+1}(\mathcal{G}_\rho)$, so that if $\overline{\mathcal{G}}_\rho$ is the graph obtained from $\mathcal{G}_\rho$ by adding a single vertex $\star$ (the new base point) and a single edge joining $\star$ to ι, we may view σ as a filtered unital $*$-inclusion σ of $\mathcal{P}(\overline{\mathcal{G}}_\rho)$ in $\mathcal{P}(\mathcal{G}_\rho)$. To describe a phased connection (W, λ) with $\sigma = \sigma^{(W,\lambda)}$, we let $\Phi_2 : V \otimes V \to V \otimes V$ be the flip, and record.

Proposition 3.1. *With the notation above*

a) *cells in $(\overline{\mathcal{G}}_\rho, \mathcal{T}, \mathcal{G}_\rho)$ may be viewed as quadruples (x, q, y, p) with $q, y, p, x \in E(\mathcal{G}_\rho)$*

b) $W(x, q, y, p) = E(qy)^*(1_{s(x)} \otimes \Phi_2)E(xp)$

c) $\lambda_{m,n}(p, q) = \delta_{p,q}$.

Proof Observe that if $i_{m,n}, \sigma_{m,n}$ denote the inclusions $i_{m,n}(t) = t \otimes 1_V$, $\sigma_{m,n}(t) = 1_V \otimes t$ of $Hom_G(V^m, V^n)$ in $Hom_G(V^{m+1}, V^{n+1})$, then

$$\sigma_{m,n}(t) = \Phi_{m+1}\, i_{m,n}(t)\, \Phi_{n+1}^*$$

where $\Phi_k : V^k \to V^k$ is the unitary defined by $\Phi_k(v_1 \otimes \cdots \otimes v_k) = v_k \otimes v_1 \otimes \cdots \otimes v_{k-1}$. Thus the Bratteli diagram for the $\sigma_{m,m}$ is the same as that for $i_{m,m}$. Moreover, if we define unitaries $E_m : \sum_{\nu \in \hat{G}}^{\oplus} V_\nu \otimes \ell^2(\Omega_m^\nu(\mathcal{G})) \to V^m$ by

$$E_m(v \otimes \epsilon_\alpha) = E(\alpha)v,\ v \in V_\nu, \alpha \in \Omega_m^\nu(\mathcal{G}_\rho)$$

then $E_m^* \, Hom_G(V^m, V^m) \, E_m = \oplus_{\nu \in \hat{G}} 1_\nu \otimes B(\ell^2(\Omega_m^\nu(\mathcal{G}_\rho)))$. Thus we may define actions $\kappa_m^\nu(\nu \in \hat{G})$ of $Hom_G(V^m, V^m)$ on $\ell^2(\Omega_m^\nu(\mathcal{G}_\rho))$ via the identity

$$E_m^* t E_m = \oplus_{\nu \in \hat{G}} 1_\nu \otimes \kappa_m^\nu(t), \quad t \in Hom_G(V^m, V^m).$$

With $T(x)$ as defined in §2, we have

$$T(x)\kappa_m^{s(x)}(t) = \kappa_{m+1}^{r(x)}(t \otimes 1_V) \; T(x);$$

it follows from the observation that $\sigma_{m,m} = Ad\Phi_{m+1} \circ i_{m,m}$ that, if $\overline{T}(x) = \kappa_{m+1}^{r(x)}(\Phi_{m+1})T(x)$ then we also have that

$$\overline{T}(x)\kappa_m^{s(x)}(t) = \kappa_{m+1}^{r(x)}(1_V \otimes t)\overline{T}(x)$$

for $t \in Hom_G(V^m, V^m)$.

Thus a cell in $(\overline{\mathcal{G}}_\rho, \mathcal{T}, \mathcal{G}_\rho)$ may be viewed as a quadruple (x, q, y, p) in $E(\mathcal{G}_\rho)$ with $s(x) = s(q), r(y) = r(p), r(q) = s(y), r(x) = s(p)$. Choosing $\alpha \in \Omega_m^{s(x)}(\mathcal{G}_\rho)$ with $r(\alpha) = s(x)$, we may now calculate

$$\begin{aligned} W(x,q,y,p) &= \langle \overline{T}(p)T(x)\epsilon_\alpha, T(y)\overline{T}(q)\epsilon_\alpha \rangle \\ &= \langle \kappa_{m+2}^{r(p)}(\Phi_{m+2})\epsilon_{\alpha x p} \, , \, T(y)\kappa_{m+1}^{r(q)}(\Phi_{m+1})\epsilon_{\alpha q} \rangle \\ &= \langle \kappa_{m+2}^{r(p)}(\Phi_{m+1} \otimes 1_V)^{-1}\Phi_{m+2})\epsilon_{\alpha x p} \, , \, \epsilon_{\alpha q y} \rangle \\ &= \langle \kappa_{m+2}^{r(p)}(1_V^m \otimes \Phi_2)\epsilon_{\alpha x p} \, , \, \epsilon_{\alpha q y} \rangle \end{aligned}$$

Choosing any unit vector $v_0 \in V_{r(p)}$, we find this is

$$\begin{aligned} &= \langle (1_V^m \otimes \Phi_2)E_{m+2}(v_0 \otimes \epsilon_{\alpha x p}) \, , \, E_{m+2}(v_0 \otimes \epsilon_{\alpha q y}) \rangle \\ &= \langle (1_V^m \otimes \Phi_2)E(\alpha x p)v_0 \, , \, E(\alpha q y)v_0 \rangle \\ &= \langle (1_{s(x)} \otimes \Phi_2)(E(x) \otimes 1_V)E(p)v_0 \, , \, (E(q) \otimes 1_V)E(y)v_0 \rangle \\ &= E(qy)^*(1_{s(x)} \otimes \Phi_2)E(xp) \end{aligned}$$

where we have used the structure of $E(\alpha x p)$, i.e.

$$E(\alpha x p) = (E(\alpha) \otimes 1_V^2) E(xp)$$

and the evident fact that the last line of the calculation is an intertwiner for the irreducible action of G on $V_{s(p)}$.

To compute $\lambda_{m,n}(p,q)$, observe that for appropriate α, α', p and q

$$(\overline{T}(p) \otimes \overline{T}(q)^o)(\epsilon_\alpha \otimes \epsilon^o_{\beta'}) = \kappa^{r(p)}_{m+1}(\Phi_{m+1})\epsilon_{\alpha p} \otimes (\kappa^{r(q)}_{n+1}(\Phi_{n+1})\epsilon_{\alpha' q})^o,$$

so

$$\begin{aligned}
\lambda_{m,n}(p,q) &= \langle \sigma_{m,n}(\epsilon_\alpha \otimes \epsilon^o_{\alpha'}), \kappa^{r(p)}_{m+1}(\Phi_{m+1})\epsilon_{\alpha p} \otimes (\kappa^{r(q)}_{n+1}(\Phi_{n+1})\epsilon_{\alpha' q})^o \rangle \\
&= \langle \kappa^{r(i_{m,n}(\alpha))}_{m+1}(\Phi_{m+1}) i_{m,n}(\epsilon_\alpha \otimes \epsilon^o_{\alpha'}) \kappa^{r(i_{m,n}(\alpha'))}_{n+1}(\Phi_{n+1})^* \, , \, \kappa^{r(p)}_{m+1}(\Phi_{m+1})\epsilon_{\alpha p} \otimes \kappa^{r(q)}_{n+1}(\Phi_{n+1})\epsilon^o_{\alpha' q} \rangle \\
&= \Big\langle \sum_x \kappa^{r(x)}_{m+1}(\Phi_{m+1})\epsilon_{\alpha x} \otimes \kappa^{r(x)}_{n+1}(\Phi_{n+1})\epsilon^o_{\alpha' x} \, , \, \kappa^{r(p)}_{m+1}(\Phi_{m+1})\epsilon_{\alpha p} \otimes \kappa^{r(q)}_{n+1}(\Phi_{n+1})\epsilon^o_{\alpha' q} \Big\rangle \\
&= \sum_x \langle \epsilon_{\alpha x} \otimes \epsilon^o_{\alpha' x} \, , \, \epsilon_{\alpha p} \otimes \epsilon^o_{\alpha' q} \rangle \\
&= \delta_{p,q},
\end{aligned}$$

as claimed. ■

The formulae of Proposition 3.1 have the great benefit of showing how to compute the phased connection (W, λ) entirely in terms of data associated intrinsically to the group G and its representation theory (in relation to the base representation ρ). The reader with a sufficiently large set of character tables, and sufficient patience, may now generate an unlimited number of examples, such as the one given in §4 below.

Before looking at explicit examples, we record a further general example. Recall from [DR1] that if ρ is a unitary representation of G in dimension d, then G acts canonically on the Cuntz algebra $\mathcal{O}_d$ in such a way that $\mathcal{O}_d^G = \mathcal{O}_\rho$. The inclusion $\mathcal{O}_\rho \subseteq \mathcal{O}_d$ is evidently filtered, given at the level of intertwiners by the inclusions $Hom_G(V^m, V^n) \subseteq Hom(V^m, V^n)$.

Of course, $\mathcal{O}_d \simeq \mathcal{P}(\mathcal{G}_d)$, where $\mathcal{G}_d$ is the directed graph with one vertex $\star$ and d edges. We take V^m as the only irreducible module for $Hom(V^m, V^m)$, so that the Bratteli diagram $\mathcal{T}_m$ for $Hom_G(V^m, V^m) \subseteq Hom(V^m, V^m)$ has vertices $r(\Omega_m(\mathcal{G}_\rho))$ and $\star$, with dim V_ν edges joining $\nu \in$

$r(\Omega_m(\mathcal{G}_\rho))$ to $\star$. Choosing orthonormal bases $\{v^\nu(q) : 1 \leq q \leq dim\nu\}$ for the irreducible G-modules V_ν, we obtain isometries $T^\nu(q) : \ell^2(\Omega_m^\nu(\mathcal{G})) \to V^m$ defined by $T^\nu(q)\epsilon_\alpha = E(\alpha)v^\nu(q)$, satisfying $T^\nu(q)\,t = t\,T^\nu(q)$ for $t \in Hom_G(V^m, V^m)$. Similarly, if $\{v(y) : 1 \leq y \leq d\}$ is an orthonormal basis of V, we obtain isometries $T(y) : V^m \to V^{m+1}$ given by $T(j)\,v = v \otimes v(j)$ and satisfying $T(j)\,t = (t \otimes 1)\,T(j)$ for $t \in Hom(V^m, V^m)$. Thus for any cell (x, q, y, p) in $(\mathcal{G}_\rho, \mathcal{T}, \mathcal{G}_d)$, we have (with $\alpha \in \Omega_m^{s(q)}(\mathcal{G}_\rho)$)

$$\begin{aligned} W(x,q,y,p) &= \langle T^{r(x)}(p)T(x)\epsilon_\alpha, T(y)T^{s(x)}(q)\epsilon_\alpha\rangle \\ &= \langle E(\alpha x)v^{r(x)}(p), E(\alpha)v^{s(x)}(q) \otimes v(y)\rangle \\ &= \langle E(x)v^{r(x)}(p), v^{s(x)}(q) \otimes v(y)\rangle. \end{aligned}$$

Thus, $W(\cdot,\cdot,\cdot,x)$ is the matrix of $E(x)$ with respect to appropriate bases.

To compute the phases $\lambda_{m,n}(p,p')$, observe that if we identify $Hom_G(V^m, V^n)$ with the space $\sum_\nu^\oplus \ell^2(\Omega_m^\nu(\mathcal{G}_\rho)) \otimes \ell^2(\Omega_n^\nu(\mathcal{G}_\rho))^o$, and $Hom(V^m, V^n)$ with $V^m \otimes (V^n)^o$ in the standard way, the inclusion $Hom_G(V^m, V^n) \subseteq Hom(V^m, V^n)$ is given by $\epsilon_\alpha \otimes \epsilon_{\alpha'}^o \mapsto \sum_q T^{r(\alpha)}(q)\epsilon_\alpha \otimes T^{r(\alpha')}(q)^o\epsilon_{\alpha'}$, so $\lambda_{m,n}(p,p') = \delta_{p,p'}$.

We record the conclusions as

Proposition 3.2. *With the notation above,*

a) cells in $(\mathcal{G}_\rho, \mathcal{T}, \mathcal{G}_d)$ may be viewed as quadruples (x, q, y, p), $1 \leq q \leq dims(x), 1 \leq p \leq dimr(x)$, $1 \leq y \leq d$;

b) $W(x,q,y,p) = \langle E(x)v^{r(x)}(p), v^{s(x)}(q) \otimes v(y)\rangle$ where $\{v^\nu(p) : 1 \leq p \leq dimV_\nu\}$ and $\{v(y) : 1 \leq y \leq d\}$ are orthonormal bases for V_ν and V respectively;

c) $\lambda_{m,n}(p,q) = \delta_{p,q}$.

It is of some interest to contrast Propositions 3.1 and 3.2, remembering that both $(\mathcal{O}_\rho, \sigma)$ and $\mathcal{O}_\rho \subseteq \mathcal{O}_d$ determine $\rho(G)$ (at least in the case where $\rho(G)$ is either finite or $\rho(G) \subseteq SU(d)$) Proposition 3.2 is the Tannaka-Krein Duality Theorem in the sense that giving the inclusion $\mathcal{O}_\rho \subseteq \mathcal{O}_d$ is equivalent to the spaces of intertwiners between arbitrary representations of G. Proposition

3.1 appears to be demanding considerably less – we need only know how the flip Φ_2 on $V_\nu \otimes V^2$ behaves with respect to its irreducible decompsition – and in addition manifestly displays the "permutation symmetry" possessed by σ (see [DR1,p.110]).

Example 3.3 Let $C_n = \{0, 1, ., n-1\}$ be the cyclic group, and let $G_1 = C_8 \rtimes_\alpha C_2, G_2 = C_8 \rtimes_\beta C_2$, where the actions of the non-trivial element of C_2 on C_8 are determined by $\alpha(1) = 3$ and $\beta(1) = 7$. The groups G_1 and G_2 have distinct character tables (see [L]) Nonetheless if ρ_i is the representation of G_i on $\ell^2(G_i/C_2)$ given by left translation, the corresponding graphs $\mathcal{G}_{\rho_i}$ each have 7 vertices and the same incidence matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}$$

(The first 4 entries correspond to irreducible 1-dimensional representations of G_i, and the remaining 3 to the irreducible 2-dimensional representations). We conclude that in general, the graph $\mathcal{G}_\rho$ does not determine the character table of the underlying group.

§4 Appendix: A$_4$ Example

Consider A_4 as a subgroup of the permutation group S_4, then $A_4 = C_1 \cup C_2 \cup C_3 \cup C_4$, where

$$C_1 = \{e\}, \qquad C_2 = \{(123), (142), (134), (243)\},$$

$$C_3 = \{(132), (124), (143), (234)\}, \qquad C_4 = \{(12)(34), (13)(24), (14)(23)\},$$

are the conjugacy classes. The character table, and unitary representation theory of the group A_4 are well known:

Representation	Character	C_1	C_2	C_3	C_4
π_1	χ_1	1	1	1	1
π_2	χ_2	1	ω	ω^2	1
π_3	χ_3	1	ω^2	ω	1
π_4	χ_4	3	0	0	-1

where ω is the primitive cube root of unity. (See [L,p.61] for example).

The 3-dimensional irreducible unitary representation, ρ $(= \pi_4)$, of A_4 arises from its action on the vertices of a tetrahedron in $V_\rho = \mathbb{C}^3$, whose centroid is at the origin. Since A_4 is generated by the cycles $m = (14)(23)$ and $n = (123)$, we may specify this representation by defining

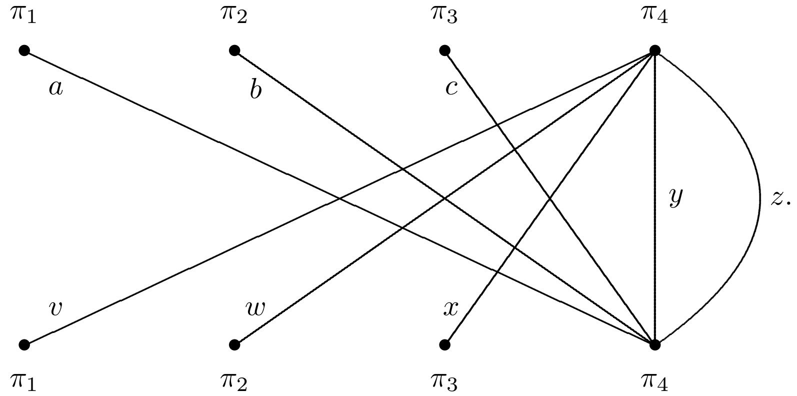
$$\rho(m) \;=\; \begin{pmatrix} \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ 0 & -1 & 0 \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{pmatrix} \qquad \text{and} \qquad \rho(\text{n}) \;=\; \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with respect to the orthonormal basis $\{e_1, e_2, e_3\}$ of $V = V_\rho$.

Examining the character table above, and using [L,p.83], we see that if $\rho = \pi_4$, we have

$$\pi_i \otimes \rho \;=\; \rho, \quad i = 1, \ldots, 3 \qquad \text{and} \qquad \pi_4 \otimes \rho \;=\; \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus 2\pi_4,$$

up to unitary equivalence. The bipartite graph associated to ρ is thus



If we identify $\mathbb{C} \otimes \mathbb{C}^3$ with $\mathbb{C}^3$ in the usual way, then we may view each of the isometries $E(a), E(b), E(c)$ as unitaries on $\mathbb{C}^3$, in fact, they give the first three of the above-mentioned unitary equivalences. Evidently $E(a) = 1_V$, whilst $E(b)$ satisfies $\chi_2(g) \cdot \rho(g) \circ E(b) = E(b) \circ \rho(g)$ for all $g \in A_4$; a similar relation holds for $E(c)$ involving χ_3. Explicitly, we may take these maps to have the following matrices

$$E(b) \;=\; \frac{1}{2}\begin{pmatrix} 1 & -i & \sqrt{2} \\ -i & -1 & i\sqrt{2} \\ \sqrt{2} & i\sqrt{2} & 0 \end{pmatrix} \quad \text{and} \quad \text{E(c)} \;=\; \frac{1}{2}\begin{pmatrix} 1 & i & \sqrt{2} \\ i & -1 & -i\sqrt{2} \\ \sqrt{2} & -i\sqrt{2} & 0 \end{pmatrix}.$$

To calculate the intertwiners for the other edges of the above graph, recall that the projection σ_i, $i = 1, \ldots, 4$ on the subspace of $V_\rho \otimes V_\rho$ supporting a multiple of π_i in $\rho^2 = \rho \otimes \rho$, is given by the formula

$$\sigma_i \;=\; \frac{\mid C_i \mid}{12} \sum_{g \in A_4} \chi_i(g) \rho^2(g) \; .$$

With $\{e_{ij} = e_i \otimes e_j \; : \; i, j = 1, \ldots, 3\}$ as an orthonormal basis for ${V_\rho}^2 = V_\rho \otimes V_\rho$, direct calculation show that the subspaces $\sigma_1 {V_\rho}^2, \sigma_2 {V_\rho}^2, \sigma_3 {V_\rho}^2$ are spanned by the vectors

$$\begin{aligned} p' &= e_{11} + e_{22} + e_{33}, \\ q' &= -ie_{11} + e_{12} - i\sqrt{2}e_{13} + e_{21} + ie_{22} - \sqrt{2}e_{23} - i\sqrt{2}e_{31} - \sqrt{2}e_{32}, \\ r' &= -ie_{11} - e_{12} - i\sqrt{2}e_{13} - e_{21} + ie_{22} + \sqrt{2}e_{23} - i\sqrt{2}e_{31} + \sqrt{2}e_{32}, \end{aligned}$$

respectively. Hence we may define the isometric intertwiners corresponding to these edges to be

$$E(v)\;\lambda \;=\; \lambda \cdot p, \qquad E(w)\;\lambda \;=\; \lambda \cdot q, \qquad E(x)\;\lambda \;=\; \lambda \cdot r,$$

where we denote by p, q, r, etc. the unit vectors in the directions of p', q', r'. The six dimensional subspace of ${V_\rho}^2$ orthogonal to $\{p,q,r\}$ supports two copies of π_4. These subspaces are spanned by the vectors

$$\begin{aligned} s_1' &= e_{23} - e_{32}, & \qquad t_1' &= -\sqrt{2}e_{11} + e_{13} + \sqrt{2}e_{22} + e_{31}, \\ s_2' &= e_{31} - e_{13}, \qquad \text{and} & \qquad t_2' &= \sqrt{2}e_{12} + \sqrt{2}e_{21} + e_{23} + e_{32}, \\ s_3' &= e_{12} - e_{21}, & \qquad t_3' &= e_{11} + e_{22} - 2e_{33}. \end{aligned}$$

These bases are easily checked to be invariant under ρ^2, and we have that

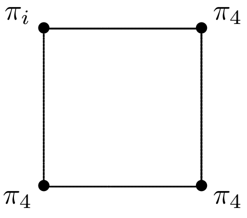
$$\rho^2(g)s_i' = \sum_{j=1}^{3} \rho_{ij}(g)s_j' \quad \text{and} \quad \rho^2(g)t_i' = \sum_{j=1}^{3} \rho_{ij}(g)t_j',$$

where $g \in A_4$, and $\rho(g) = [\rho_{ij}(g)]_{i,j=1}^3$. As above, we denote by s_i, t_j, for $i,j = 1,2,3$ the unit vectors in the directions of s_i', t_j' respectively. Thus, if we define $E(y)$ and $E(z)$ by

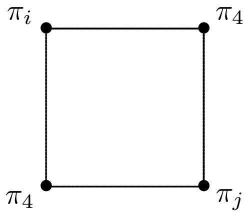
$$E(y)\left(\sum_{i=1}^{3} \alpha_i e_i\right) = \sum_{i=1}^{3} \alpha_i s_i, \quad \text{and} \quad \mathrm{E(z)}\left(\sum_{\mathrm{i=1}}^{3} \alpha_{\mathrm{i}} \mathrm{e}_{\mathrm{i}}\right) = \sum_{\mathrm{i=1}}^{3} \alpha_{\mathrm{i}} \mathrm{t}_{\mathrm{i}},$$

then we have $\rho^2(g) \circ E(y) = E(y) \circ \rho(g)$, for all $g \in A_4$, and similarly for $E(z)$.

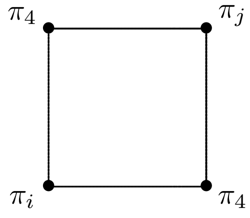
We now wish to calculate the unitary connection for the above graph; upon inspection, we note that there are only three types of cell to consider



for $i = 1, \ldots, 3$



for $i,j = 1, \ldots, 3$



for $i,j = 1, \ldots, 4$.

Examining the diagrams above reveals that for the first type, for each i there are precisely 4 such cells, for the second type for each fixed i,j there is only one cell, and for the third type there are 49 cells. Hence, labelling our unitaries by the fixed representations at the top left and bottom right corners of the cell, we have that W_{ij}, $1 \leq i,j \leq 3$ is a 1×1 matrix, W_{i4}, $1 \leq i \leq 3$ is a 2×2 matrix, and W_{44} is a 7×7 matrix.

Observe that, from Proposition 3.1b), in order to calculate the values of each unitary on any cell, we only need to evaluate the inner product

$$\langle \; \left(\; 1_{V_{s(\alpha)}} \otimes \Phi_2 \; \right) \; E(\alpha) \; v \; , \; E(\beta) \; v \; \rangle,$$

in $V_{s(\alpha)} \otimes V_\rho \otimes V_\rho$, for each pair α, β, of paths of length 2, with $s(\alpha) = s(\beta)$, $r(\alpha) = r(\beta)$ and any $v \in V_{s(\alpha)}$ of unit length. The entries of the unitary matrices W_{ij}, are then indexed by pairs (α, β) with $s(\alpha) = s(\beta) = i$ and $r(\alpha) = r(\beta) = j$, for any fixed $1 \leq i, j \leq 4$.

Tedious but straightforward calculations, using the above formulae, yield

$$W_{11} = \begin{array}{cc} & av \\ av & \left(\; 1 \; \right) \end{array}, \quad W_{12} = \begin{array}{cc} & aw \\ aw & \left(\; 1 \; \right) \end{array}, \quad W_{13} = \begin{array}{cc} & ax \\ ax & \left(\; 1 \; \right) \end{array}, \quad W_{14} = \begin{array}{cc} & \begin{array}{cc} ay & az \end{array} \\ \begin{array}{c} ay \\ az \end{array} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{array},$$

$$W_{21} = \begin{array}{cc} & bv \\ bv & \left(\; 1 \; \right) \end{array}, \quad W_{22} = \begin{array}{cc} & bw \\ bw & \left(\; 1 \; \right) \end{array}, \quad W_{23} = \begin{array}{cc} & bx \\ bx & \left(\; 1 \; \right) \end{array}, \quad W_{24} = \begin{array}{c|cc} & by & bz \\ by & \frac{1}{2} & \frac{\sqrt{3}}{2}i \\ bz & -\frac{\sqrt{3}}{2}i & -\frac{1}{2} \end{array},$$

$$W_{31} = \begin{array}{cc} & cv \\ cv & \left(\; 1 \; \right) \end{array}, \quad W_{32} = \begin{array}{cc} & cw \\ cw & \left(\; 1 \; \right) \end{array}, \quad W_{33} = \begin{array}{cc} & cx \\ cx & \left(\; 1 \; \right) \end{array}, \quad W_{34} = \begin{array}{c|cc} & cy & cz \\ cy & \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ cz & \frac{\sqrt{3}}{2}i & -\frac{1}{2} \end{array},$$

$$W_{41} = \begin{array}{c|cc} & yv & zv \\ yv & -1 & 0 \\ zv & 0 & 1 \end{array}, \quad W_{42} = \begin{array}{c|cc} & yw & zw \\ yw & \frac{1}{2} & \frac{\sqrt{3}}{2}i \\ zw & -\frac{\sqrt{3}}{2}i & -\frac{1}{2} \end{array}, \quad W_{43} = \begin{array}{c|cc} & yx & zx \\ yx & \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ zx & \frac{\sqrt{3}}{2}i & -\frac{1}{2} \end{array},$$

$$W_{44} = \begin{array}{c|ccccccc} & va & wb & xc & yy & yz & zy & zz \\ va & \frac{1}{3} & \frac{1}{3}i & \frac{1}{3}i & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} \\ wb & -\frac{1}{3}i & \frac{1}{3} & \frac{1}{3} & \frac{1}{2\sqrt{3}}i & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2\sqrt{3}}i \\ xc & -\frac{1}{3}i & \frac{1}{3} & \frac{1}{3} & \frac{1}{2\sqrt{3}}i & \frac{1}{2} & \frac{1}{2} & \frac{1}{2\sqrt{3}}i \\ yy & \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}}i & -\frac{1}{2\sqrt{3}}i & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ yz & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ zy & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ zz & \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}}i & -\frac{1}{2\sqrt{3}}i & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}.$$

Here the left border labels correspond to the indices α mentioned above and the upper border labels correspond to the indices β in the same way.

It is worth noting that while the Doplicher-Roberts theory guarantees that the graph $\mathcal{G}_\rho$, and the matrices W_{ij} given above determine $\rho(A_4)$ (and hence A_4) uniquely, it is far from clear how to recover A_4 without going through massive $C^\star$-algebraic manipulations (see [DR2]).

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